An Expansion for Laplacian Integrals in Terms of Incomplete Gamma Functions, and Some Applications*

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Introduction

LAPLACE has given us, in the Théorie Analytique des Probabilités, Book I, Part II, Chapter I, a method of approximating by means of series to the value of a definite integral of the type

$$I = \int_x^{x_2} y_1^{\theta_1} y_2^{\theta_2} \cdots y_n^{\theta_n} \phi dx,$$

where $y_1, y_2, \cdots y_n$ are functions of x whose exponents $\theta_1, \theta_2, \cdots \theta_n$ are of the same order of magnitude as a large number θ. The last function in the integrand, ϕ, embraces all factors whose exponents are of low order of magnitude compared with θ. The integral here considered must not be confused with the well known "Laplacian Transform" integral which is also embodied in the Théorie Analytique.

When the function $\phi(x)$ is other than a mere constant, the Laplacian method as presented in the Théorie Analytique does not, in certain cases, give us a series which reduces to its first term as θ approaches infinity. The object of this paper is to present a modification of the method which gives the desired result and to present two applications of considerable practical importance. The modification consists in divorcing the function ϕ from the factors of the integrand raised to high powers and associating ϕ with the factor (dx/dt) which makes its appearance with the Laplacian change of variable from x to t.

Deduction of Modified Expansion

Setting $\theta_s/\theta = r_s$ we have

$$y_1^{\theta_1} y_2^{\theta_2} \cdots y_n^{\theta_n} = (y_1^{r_1} y_2^{r_2} \cdots y_n^{r_n})^\theta = y^\theta,$$

$$I = \int_x^{x_2} [y(x)]^\theta \phi(x) dx, \tag{1}$$

which we shall write in the form

$$I = \int_x^x [y(x)^N \phi_w(x) dx,]$$

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where $N = \theta + w$, $\phi_w(x) = [y(x)]^{-w}\phi(x)$. Introducing the parameter w does not constitute an essential modification of the procedure given by Laplace, but we shall find that w plays an important part when we come to applications of the expansion presented in this paper.

In what follows it is assumed that $y(x)$ is a positive monotonically increasing or decreasing function in the range x_1, x_2; the extension to a range of integration divisible into subranges within which $y(x)$ is monotonic will be obvious.

Let X be that one of the two limits x_1, x_2 for which $y(x)$ has its greatest value; set $y(X) = Y$. Assume with Laplace that

$$\text{(2)} \qquad [\log Y - \log y(x)]^{\frac{1}{u+1}} = (x - X)g(x, X),$$

u being a positive number or zero and g a function of which $(x - X)$ is not a factor. Set

$$\text{(3)} \qquad y = Ye^{-t^{u+1}},$$

$$\text{(4)} \qquad \phi_w(x)(dx/dt) = A_0 + A_1\frac{t}{1!} + A_2\frac{t^2}{2!} + A_3\frac{t^3}{3!} + \cdots.$$

These two equations give (certain well-known conditions being fulfilled)

$$\text{(5)} \qquad I = Y^N \sum_{s=0} A_s \int_{t_1}^{t_2} \frac{t^s e^{-Nt^{u+1}} dt}{s!}.$$

Finally set $Nt^{u+1} = T$ and we obtain the *modified Laplacian expansion*

$$\text{(6)} \qquad I = Y^N \sum_{s=0} \left(\frac{1}{N}\right)^{H_s} B_s[P(H_s, T_2) - P(H_s, T_1)]$$

wherein

$$H_s = \frac{s+1}{u+1},$$

$$B_s = \left(\frac{A_s}{u+1}\right)\frac{\Gamma(H_s)}{s!}, \qquad = A_s \quad \text{for} \quad u = 0;$$

$$T_1 = N[\log Y - \log y(x_1)], \qquad = 0 \quad \text{if} \quad X = x_1;$$

$$T_2 = N[\log Y - \log y(x_2)], \qquad = 0 \quad \text{if} \quad X = x_2;$$

$$P(H_s, T') = \frac{\int_0^{T'} T^{H_s - 1} e^{-T} dT}{\Gamma(H_s)},$$

$$T' = T_1 \text{ or } T_2.$$

For the computation of $P(H_s, T')$ recourse may be had to the extensive "Incomplete Gamma Function" Tables edited by Karl Pearson which give the ratio of the incomplete to the complete function. When H_s is a positive integer we have

$$P(H_s, T') = \sum_{k=H}^{\infty} \frac{T'^k e^{-T'}}{k!},$$

the well known Poisson Exponential Binomial Limit for which short tables will be found in Pearson's "Tables For Statisticians And Biometricians" and in T. C. Fry's "Probability and Its Engineering Uses."

Applications

I

Consider the incomplete Beta Function

$$I(p) = \int_0^p (1 - x)^{n-c} x^{c-1} dx$$

for positive integral values of n and c such that n is large compared with c. Its *modified Laplacian expansion* is

$$I(p) = (c - 1)!(n + w)^{-c} \sum_{m=0} (n + w)^{-m} A(m, c - 1) P(c + m, T_2),$$

where $T_2 = -(n + w) \log (1 - p)$ and, setting $w = (d - c + 1)/2$,[1]

$$A(0, c - 1) = 1,$$

$$A(1, c - 1) = \binom{c}{1} \frac{d}{2},$$

$$A(2, c - 1) = \binom{c+1}{2} \left[\frac{(c - 1) + 3d^2}{12} \right],$$

$$A(3, c - 1) = \binom{c+2}{3} \left[\frac{(c - 1)d + d^3}{8} \right],$$

$$A(4, c - 1) = \binom{c+3}{4} \left[\frac{5(c - 1)^2 - (2 - 30d^2)(c - 1) + 15d^4}{240} \right].$$

As many more coefficients as one desires can be obtained by means of the equations

$$A(m, c - 1) = \sum_{k=0}^{m} Q(m, k) \binom{c - 1 + m}{m + k},$$

$$Q(m + 1, k) = (m + k)Q(m, k - 1) + (w + k)Q(m, k),$$

$$Q(o, o) = 1.$$

[1] See Appendix II for the details of the analyses.

The expansion given above for $I(p)$ is of great practical value in connection with the evaluation of the incomplete binomial summation. We know [2] that

$$P(c; n, p) = \sum_{x=c}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$
$$= \frac{n!}{(c-1)!(n-c)!} \int_0^p x^{c-1}(1-x)^{n-c} dx;$$

therefore, we have

$$P(c; n, p) = \frac{n!}{(n+w)^c (n-c)!} \sum_{m=0} \left(\frac{1}{n+w}\right)^m A(m, c-1) P(c+m, T_2).$$

This approximation formula is submitted for ranges of values of c and n such that c is of the order of magnitude of $\sqrt{n}$ or less. But for values of $P(c; n, p)$ which are of most interest, say,

$$.0001 < P < .9999,$$

c is of the same order of magnitude as

$$a = np$$

when n is a large number. Therefore, the formula is particularly valuable for values of p which are of the same order of magnitude as $1/\sqrt{n}$ or less.

The first table at the end of this paper indicates the degree of accuracy obtainable by taking $d = 0$ and using only one or two terms of the approximation for $P(c; n, p)$. The table also indicates the result of taking d such that $T_2 = a = np$, and using only one, two, three or four terms of the approximating series.

I am indebted to Miss Elizabeth McCusker and Miss Catherine Lennon of the Department of Development and Research of the American Telephone and Telegraph Company for the computation work involved in the construction of the tables at the end of this paper.

II

Consider the Bessel Function

$$I_n(b) = \frac{(\frac{1}{2}b)^n}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})} \int_0^\pi (\sin x)^{2n} e^{b(\cos x)} dx,$$

[2] See Laplace, "*Théorie Analytique des Probabilités,*" 1st edition, p. 151.

for b large compared with n, n being a positive integer. Writing

$$e^{-b}I_n(b) = \frac{(2b)^n}{\sqrt{\pi}\Gamma(n+\frac{1}{2})}\int_0^{\pi}(\tfrac{1}{2}\,\mathrm{sin}x)^{2n}[e^{-(\frac{1}{2}-\frac{1}{2}\cos x)}]^{2b}dx,$$

leads us to the *modified Laplacian expansion*

$$e^{-b}I_n(b) = \frac{\left(\dfrac{2b}{2b+w}\right)^n}{\sqrt{\pi(2b+w)}}P(n+\tfrac{1}{2},\,2b+w)S_n,$$

where

$$S_n = \sum_{m=0}\left(\frac{1}{8b+4w}\right)^m \frac{A_{2n+2m}}{2(2n)!}\,\frac{P(n+m+\frac{1}{2},\,2b+w)}{P(n+\frac{1}{2},\,2b+w)}\,\frac{n!}{(n+m)!}.\text{[3]}$$

To determine the coefficients A_{2n+2m} we proceed as follows: For the integral now under consideration

$$N = 2b+w, \qquad x_1 = 0, \qquad x_2 = \pi,$$
$$y(x) = e^{-(\frac{1}{2}-\frac{1}{2}\cos x)}, \qquad X = 0, \qquad Y = 1,$$
$$(\log Y - \log y)^{\frac{1}{2}} = (\tfrac{1}{2}-\tfrac{1}{2}\cos x)^{\frac{1}{2}} = xg(x),$$
$$g(x) = \frac{1}{x}\sin\frac{x}{2} = \sum_{k=0}\frac{(-1)^k x^{2k}}{2^{2k+1}(2k+1)!},$$

so that

$$u = 1, \qquad y(x) = e^{-t^2}, \qquad T_1 = 0, \qquad T_2 = 2b+w,$$
$$\phi_w(x) = (\tfrac{1}{2}\,\mathrm{sin}x)^{2n}e^{w(xg)^2},$$
$$B_{2n+2m} = [n!\Gamma(n+\tfrac{1}{2})/(n+m)!(2n)!2^{2m+1}]A_{2n+2m}.$$

The relations above give

$$\phi_w(x)(dx/dt) = 2t^{2n}(1-t^2)^{n-\frac{1}{2}}e^{wt^2}.$$

Let

$$\sum_{s=0}A'_{2n+2s}t^{2n+2s}/(2n+2s)! = 2t^{2n}(1-t^2)^{n-\frac{1}{2}},$$

so that

$$A'_{2n} = 2(2n)!,$$
$$A'_{2n+2s} = \frac{(-1)^s(2n-1)(2n-3)\cdots(2n-2s+1)(2n+2s)!}{2^{s-1}s!};$$

then

$$\phi_w(x)(dx/dt) = e^{wt^2}\sum_{s=0}A'_{2n+2s}\frac{t^{2n+2s}}{(2n+2s)!}$$
$$= \sum_{k=0}\sum_{s=0}t^{2n+2s+2k}A'_{2n+2s}w^k/k!(2n+2s)!$$

[3] An expansion of this type for Bessel functions was derived many years ago by Hadamard; see Watson, "A Treatise on the Theory of Bessel Functions," pp. 204 and 205.

and

$$A_{2n+2m}/(2n + 2m)! = \sum_{k=0}^{m}(w^k/k!)A'_{2n+2m-2k}/(2n + 2m - 2k)!$$

We must now assign a value to w. One which suggests itself is to take w such that the second term in our expansion for $I_n(b)$ vanishes; in other words, such that A_{2n+2} shall be zero. This merely requires that we set $w = n - 1/2$, giving

$$A_{2n} = 2(2n)!, \qquad A_{2n+2} = 0$$
$$A_{2n+4} = -(2n - 1)(2n + 4)!/2, \qquad A_{2n+6} = -(2n - 1)(2n + 6)!/3$$
$$A_{2n+8} = (2n - 1)(2n - 5)(2n + 8)!/16, \text{ etc.}$$

For $n = 1$ and $w = 1/2$ we have

$$A_2 = 4, \qquad A_4 = 0, \qquad A_6 = -360, \qquad A_8 = -13440, \cdots$$

$$e^{-b}I_1(b) = \left[\left(\frac{2b}{2b + .5}\right) P(3/2, 2b + .5)/\sqrt{\pi(2b + .5)}\right] S_1$$

wherein

$$S_1 = 1 - 15(8b + 2)^{-2}\frac{P(7/2, 2b + .5)}{P(3/2, 2b + .5)} - 140(8b + 2)^{-3}\frac{P(9/2, 2b + .5)}{P(3/2, 2b + .5)} - \cdots$$

The degree of accuracy obtainable when only one, two or three terms of S_1 are made use of is indicated by the second table at the end of this paper. For some values of b, ranging from 6 to 16, comparison is made between the successive approximations for $e^{-b}I_1(b)$ and its true value as given by Watson in his Theory of Bessel Functions. The last three columns of Table II give the figures for $w = 0$ instead of $w = 1/2$.

Appendix I

When the expansion of $\phi_w(x)(dx/dt)$ is not obvious we may proceed as does Laplace in expanding (dx/dt).

Equations (2) and (3) give

$$x = X + tg^{-1}. \tag{7}$$

This last equation, together with the Lagrange-Laplace expansion theorem, gives for a function of x the expansion

$$f(x) = f(X) + \sum_{s=1}\frac{t^s}{s!}\left[\frac{d^{s-1}}{dx^{s-1}}\left(\frac{1}{g}\right)^s f'(x)\right]_{x=X}.$$

But $df(x)/dt = f'(x)(dx/dt)$. Therefore, taking $f(x)$ such that $f'(x) = \phi_w(x)$,

$$\phi_w(x)\frac{dx}{dt} = \sum_{s=0}\frac{t^s}{s!}A_s,$$

$$A_s = [D_x^s g^{-(s+1)}\phi_w(x)]_{x=X}, \qquad D_x^s = \frac{d^s}{dx^s},$$

$$= Y^{-w}\sum_{v=0}\frac{w^v}{v!}[D_x^s(x-X)^{(u+1)v}g^{-(s+1)+(u+1)v}\phi(x)]_{x=X},$$

or

$$\frac{A_s}{s!} = Y^{-w}\sum_{v=0}\frac{w^v}{v!}\frac{[D_x^{s-(u+1)v}\phi(x)g^{-(s+1)+(u+1)v}]_{x=X}}{[s-(u+1)v]!}. \tag{8}$$

Beta Function

For the incomplete Beta Function we have,

$$X = 0, \qquad u = 0, \qquad Y = 1, \qquad g(0) = 1,$$

$$\phi(x) = x^{c-1}(1-x)^{-c} = \sum_{r=0}\binom{c-1+r}{r}x^{c-1+r}.$$

Therefore,

$$\frac{A_s}{s!} = \sum_{v=0}^{s-(c-1)}\frac{w^v}{v!}\sum_{r=0}\binom{c-1+r}{r}\frac{[D_x^{s-v}x^{c-1+r}g^{-(s+1-v)}]_{x=0}}{(s-v)!},$$

or

$$\frac{A_s}{s!} = \sum_{v=0}^{s-(c-1)}\frac{w^v}{v!}\sum_{r=0}^{s-(c-1)-v}\binom{c-1+r}{r}\frac{[D_x^{s-(c-1)-v-r}g^{-(s+1-v)}]_{x=0}}{[s-(c-1)-v-r]!}.$$

Since $A_s = 0$ for $s < c-1$, set $s = c-1+m$; then interchanging v and $m-v$ we obtain

$$\frac{A_{c-1+m}}{(c-1+m)!} = \sum_{v=0}^{m}\frac{w^{m-v}}{(m-v)!}\sum_{r=0}^{v}\binom{c-1+r}{c-1}\frac{[D_x^{v-r}g^{-(c+v)}]_{x=0}}{(v-r)!}. \tag{9}$$

To evaluate (9) we may have recourse to the formula [4]

$$D_x^M g^{-S} = S\binom{S+M}{M}\sum_{K=1}^{M}\left[\frac{(-1)^K\binom{M}{K}D_x^M g^K}{(S+K)g^{S+K}}\right], \tag{10}$$

taking

$$M = v - r,$$

$$S = c + v.$$

[4] See formula 25, page 15, of Schlömilchs Höheren Analysis, Band II, Dritte Auflage, 1879. I am indebted to Mr. J. Riordan of the Department of Development and Research, American Telephone and Telegraph Company, for calling my attention to this formula.

Bessel Function

For the Bessel integral with b large compared with n we have

$$X = 0, \qquad u = 1, \qquad g(0) = 1/2, \qquad Y = 1,$$

$$\phi(x) = \left\{\left[\sin\left(\frac{x}{2}\right)\right]^2\left[1 - \sin^2\left(\frac{x}{2}\right)\right]\right\}^n = [(xg)^2 - (xg)^4]^n$$

$$= \sum_{r=0}^{n}\binom{n}{r}(-1)^r(xg)^{2n+2r}.$$

Substituting in (8) we have

$$\frac{A_s}{s!} = \sum_{v=0}\frac{w^v}{v!}\sum_{r=0}^{R}(-1)^r\binom{n}{r}\frac{[D_x^{s-2v}x^{2n+2r}g^{-(s-2n-2v-2r+1)}]_{x=0}}{(s-2v)!},$$

where $2R \not> s - 2n - 2v$; note that if $R > n$, then $\binom{n}{r} = 0$.

Since $A_s = 0$ for $s < 2n$ and g is an even function of x, set $s = 2n + 2m$. Then Leibnitz's theorem and interchanging v with $m - v$ give

$$(11) \qquad \frac{A_{2n+2m}}{(2n+2m)!} = \sum_{v=0}^{m}\frac{w^{m-v}}{(m-v)!}\sum_{r=0}^{v}(-1)^r\binom{n}{r}\frac{[D_x^{2v-2r}g^{-(2v-2r+1)}]_{x=0}}{(2v-2r)!},$$

where again we may have recourse to (10) for the differentiations of negative powers of $g(x)$.

Appendix II

Writing the incomplete Beta Function in the form

$$I(p) = \int_0^p (1-x)^{n+w}x^{c-1}(1-x)^{-(c+w)}dx,$$

we now have

$$N = n + w, \qquad x_1 = 0, \qquad x_2 = p,$$
$$y(x) = (1 - x), \qquad Y = 1, \qquad X = 0,$$
$$\phi_w(x) = x^{c-1}(1-x)^{-(c+w)},$$
$$\log Y - \log y = -\log(1 - x),$$
$$= xg(x), \qquad g(x) = \left(1 + \frac{x}{2} + \frac{x^2}{3}\cdots\right),$$
$$u = 0,$$
$$y(x) = e^{-t},$$
$$T_1 = 0,$$
$$T_2 = -(n + w)\log(1 - p).$$

From these equations we derive

$$\begin{aligned}\phi_w(x)(dx/dt) &= e^{wt}(e^t - 1)^{c-1} \\ &= \sum_{k=0}^{c-1} (-1)^k \binom{c-1}{k} e^{(w+c-1-k)t} \\ &= \sum_{m=0} \left(\frac{t^s}{s!}\right)\left[\sum_{k=0}^{c-1} (-1)^k \binom{c-1}{k} (w+c-1-k)^s\right], \\ & \qquad s = c-1+m.\end{aligned}$$

Therefore (see any work on finite differences),

$$A_{c-1+m} = \Delta^{c-1}(w)^{c-1+m} = (-1)^m \Delta^{c-1}(-w-c+1)^{c-1+m}.$$

Now

$$\begin{aligned}\Delta^{c-1}(w)^{c-1+m+1} &= w\Delta^{c-1}(w)^{c-1+m} + (c-1)\Delta^{c-2}(w+1)^{c-1+m} \\ &= (w+c-1)\Delta^{c-1}(w)^{c-1+m} + (c-1)\Delta^{c-2}(w)^{c-2+m+1}.\end{aligned}$$

Increasing c by 1, decreasing m by 1 and setting

$$A_{c-1+m} = (c-1)!A(m, c-1),$$

we have

$$A(m, c) - A(m, c-1) = (w+c)A(m-1, c)$$

or

$$\Delta_c A(m, c-1) = (w+c)A(m-1, c)$$

where the subscript c implies that Δ_c operates on c. Assume that

$$A(m, c-1) = \sum_{k=0}^{m} Q(m, k) \binom{c-1+m}{m+k},$$

where the coefficients $Q(m, k)$ are functions of w; then

$$\begin{aligned}\Delta_c A(m+1, c-1) &= \sum_{k=0}^{m} Q(m, k)(w+c)\binom{c+m}{m+k} \\ &= \sum_{k=0}^{m+1} Q(m, k)\left[(m+k+1)\binom{c+m}{m+k+1} + (w+k)\binom{c+m}{m+k}\right] \\ &= \sum_{k=0}^{m+1} [(m+k)Q(m, k-1) + (w+k)Q(m, k)]\binom{c+m}{m+k},\end{aligned}$$

from which, by finite integration with reference to c, we obtain

$$A(m+1, c-1) = \sum_{k=0}^{m+1} [(m+k)Q(m, k-1) + (w+k)Q(m, k)]\binom{c-1+m+1}{m+1+k}$$

so that

$$\begin{aligned}Q(m+1, 0) &= wQ(m, 0), \\ Q(m+1, k) &= (m+k)Q(m, k-1) + (w+k)Q(m, k), \\ & \qquad 0 < k < m+1, \\ Q(m+1, m+1) &= (2m+1)Q(m, m).\end{aligned}$$

Since

$$A(0, c-1) = 1 = \binom{c-1}{0},$$

we know that $Q(0, 0) = 1$. Hence

$$A(1, c-1) = w\binom{c}{1} + \binom{c}{2},$$

$$A(2, c-1) = w^2\binom{c+1}{2} + (3w+1)\binom{c+1}{3} + 3\binom{c+1}{4},$$

$$A(3, c-1) = w^3\binom{c+2}{3} + (6w^2+4w+1)\binom{c+2}{4} + (15w+10)\binom{c+2}{5} + 15\binom{c+2}{6}.$$

$$A(4, c-1) = w^4\binom{c+3}{4} + (10w^3+10w^2+5w+1)\binom{c+3}{5} + (45w^2+60w+25)\binom{c+3}{6} + (105w+105)\binom{c+3}{7} + 105\binom{c+3}{8},$$

etc.

As yet the value of w has not been specified. Since when m is an odd number and $w = -(c-1)/2$, the equation

$$\Delta^{c-1}(w)^{c-1+m} = (-1)^m\Delta^{c-1}(-w-c+1)^{c-1+m}$$

gives us

$$\Delta^{c-1}\left(-\frac{c-1}{2}\right)^{c-1+m} = 0,$$

let us write $w = (d/2) - (c-1)/2$, leaving d arbitrary; we obtain

$$A(0, c-1) = 1,$$

$$A(1, c-1) = \binom{c}{1}\frac{d}{2},$$

$$A(2, c-1) = \binom{c+1}{2}\left[\frac{(c-1)+3d^2}{12}\right],$$

$$A(3, c-1) = \binom{c+2}{3}\left[\frac{(c-1)d+d^3}{8}\right],$$

$$A(4, c-1) = \binom{c+3}{4}\left[\frac{5(c-1)^2-(2-30d^2)(c-1)+15d^4}{240}\right],$$

etc. Finally, for $d = 0$, or $w = -(c-1)/2$,

$$A(0, c-1) = 1, \qquad A(1, c-1) = 0,$$

$$A(2, c-1) = \binom{c+1}{3}\frac{1}{4}, \qquad A(3, c-1) = 0,$$

$$A(4, c-1) = \binom{c+3}{5}\left(\frac{5c-7}{48}\right), \qquad A(5, c-1) = 0$$

etc.

TABLE I

Incomplete Binomial Summation

$$P(c;\, n,\, p) = \frac{n!}{(c-1)!(n-c)!}\int_0^p x^{c-1}(1-x)^{n-c}dx$$

np	n	p	c	$P(c;\, n,\, p)10^6$	$w = -(c-1)/2$		w Such that $T_2 = np$			
					$(P-P_1)10^6$	$(P-P_2)10^6$	$(P-P_1)10^6$	$(P-P_2)10^6$	$(P-P_3)10^6$	$(P-P_4)10^6$
5	50	.1	2	966,214	78	0	− 77,687	4,340	− 198	8
			4	749,706	428	0	− 51,250	2,399	− 90	3
			6	383,877	518	0	− 802	520	− 2	0
			8	122,145	273	0	9,911	676	34	2
			10	24,538	77	0	4,058	426	34	2
			12	3,220	13	0	789	117	13	1
	100	.05	2	962,919	19	0	− 36,837	986	− 22	0
			4	742,161	101	0	− 23,907	535	− 10	0
			6	384,001	123	0	− 190	123	0	0
			8	127,960	68	0	5,209	175	4	0
			10	28,188	21	0	2,378	124	5	0
			12	4,274	4	0	547	40	2	0
10	100	.1	2	999,678	25	0	− 98,756	7,193	− 460	27
			4	992,164	244	0	− 155,547	14,675	− 1,085	68
			6	942,423	739	0	− 152,886	14,185	− 997	58
			8	793,949	1,268	1	− 94,502	7,232	− 430	22
			10	548,710	1,399	2	− 25,277	2,065	− 83	4
			12	296,967	1,054	2	11,472	1,244	43	3
			14	123,877	563	1	15,423	1,511	112	7
			16	39,890	221	1	8,242	1,102	112	10
			18	10,007	65	0	2,832	495	65	7
			20	1,979	15	0	699	152	24	3

TABLE I—*Continued*

np	n	p	c	$P(c; n, p)10^6$	$w = -(c-1)/2$		w Such that $T_2 = np$			
					$(P - P_1)10^6$	$(P - P_2)10^6$	$(P - P_1)10^6$	$(P - P_2)10^6$	$(P - P_3)10^6$	$(P - P_4)10^6$
	200	.05	2	999,596	6	0	− 47,020	1,644	− 51	1
			4	990,951	59	0	− 72,566	3,261	− 116	3
			6	937,657	177	0	− 70,458	3,100	− 104	3
			8	786,695	300	0	− 43,386	1,575	− 45	1
			10	545,290	330	0	− 11,460	462	− 9	0
			12	300,244	252	0	6,002	309	5	0
			14	129,892	140	0	8,241	402	15	0
			16	44,355	58	0	4,750	318	16	0
			18	12,089	18	0	1,811	160	10	0
			20	2,664	4	0	509	56	4	0

$P - P_m$ = error incurred by using only the first m terms of the series for $P(c; n, p)$ presented in this paper.

TABLE II

Bessel Function

$e^{-b}I_1(b)$

b	P	$w = n - 1/2 = 1/2$ for $n = 1$			$w = 0$		
		$\left(\frac{P - P_1}{P}\right)$	$\left(\frac{P - P_2}{P}\right)$	$\left(\frac{P - P_3}{P}\right)$	$\left(\frac{P - P_1}{P}\right)$	$\left(\frac{P - P_2}{P}\right)$	$\left(\frac{P - P_3}{P}\right)$
6	.152052	−.007497	−.001457	−.000331	−.071134	−.004188	−.000701
8	.134143	−.004043	−.000587	−.000098	−.051474	−.002187	−.000261
10	.121263	−.002529	−.000293	−.000038	−.040358	−.001344	−.000125
12	.111464	−.001731	−.000167	−.000018	−.033198	−.000911	−.000070
14	.103698	−.001259	−.000104	−.000010	−.028198	−.000658	−.000042
16	.097350	−.000958	−.000070	−.000006	−.024510	−.000497	−.000029

P = value of $e^{-b}I_1(b)$ from Table II of Watson's Theory of Bessel Functions.

$\left(\frac{P - P_m}{P}\right)$ = proportional error incurred by using the first m terms of the series for $e^{-b}I_1(b)$ presented in this paper.