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The Interconnection of Telephone Systems—Graded Multiples

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The general problem of subscriber interconnection is stated here, while some of the economic and service factors in the selection of trunking systems are briefly considered. The characteristic manner in which telephone calls fall upon ordinary straight trunk groups is presented from both common sense and theoretical standpoints.

One of the widely used trunk rearrangements by which an improved capacity may be achieved under certain conditions is known as graded multiple. A theoretical analysis of this scheme is given, from which are constructed curves for common probabilities of loss. Illustrative examples are included to make clear their use.

A detailed comparison between theory and observation is made with considerable attention paid to critically examining the validity of the assumptions underlying the theory. It is concluded that the present graded multiple engineering tables are based upon a proper modification of the theoretical formula.

Introduction

WITH the completion of the third commercial telephone instrument some fifty odd years ago was born the problem of interconnection. And as the system has grown so has the demand for a universal service. The present complexity of our communication

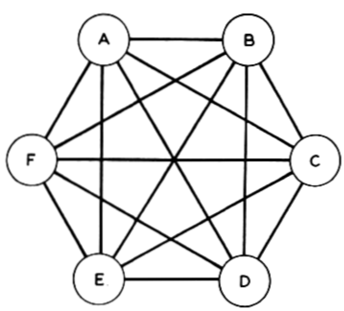

Fig. 1—Direct interconnection of subscribers.

network makes it difficult to appreciate that in those early days it was a comparatively simple thing to provide a direct line from each subscriber to every other subscriber, as shown in the schematic interconnection of the six telephone stations in Fig. 1. In such schemes there is no possibility of the line being busy; the only faults a subscriber may properly find with the service (over and above transmission troubles) are that "the called party is busy" on another line, and "he does not answer," conditions beyond the control of the interconnecting system.

For a very small number of subscribers all in close proximity such a scheme would and does serve admirably. As soon as the number of subscribers, "n," is increased, however, the number of lines, which equals $\frac{n(n-1)}{2}$, goes up at an enormous rate, almost as the square of the number of subscribers. Since a major element of the cost is in direct variation with the number of lines this plan, even on a modest scale, is quickly prohibited.

Nevertheless, if a truly universal service is to be furnished it must not only be *possible* for any subscriber to communicate directly with any other, but it must be *easily* possible. This problem was solved by the development of the central office plan of interconnection.

Elementary Studies

We may represent a simplified central office exchange system by the line diagram of Fig. 2. Two sets of 100 subscribers, A and B, may make calls to the opposite set via the 10 "trunks" C.[1] A line which connects an individual subscriber to his central office exchange is unique in that it will never be used except when *he* is talking. The lines C, however, which are provided for establishing connections between the telephones in one office and those in the other may well carry calls originated by a large number of subscribers. Thereupon it is readily seen that one interoffice line (or trunk, as we shall hereafter call it) may easily attain a very much higher efficiency, as measured by the per cent time it is in use, than an individual subscriber's line. When one subscriber is not using a particular trunk it is available for use by another; thus we make one trunk do the work for which two or more lines were required in the original arrangement of Fig. 1.

To obtain this increased call carrying capacity per trunk and the consequent savings due to reduction in the total number of trunks, it is necessary to forego one particular advantage: we cannot be absolutely sure that there will be an idle trunk available when each subscriber desires to place his various calls. For it is possible, although very improbable, that all of the subscribers might want to call one another simultaneously, and having far fewer trunks than subscribers in either office many would fail to get immediate service. The

[1] For our purpose it is unnecessary to consider how two subscribers within the set A, or the set B, may be interconnected.

analyses necessary to determine the theoretical probability (or the proportion of times in the long run) that a particular subscriber will find all of a group of trunks busy are well known; and tables and curves are available showing for wide ranges of loads and numbers of trunks the probability of a particular subscriber finding them all in use.[2]

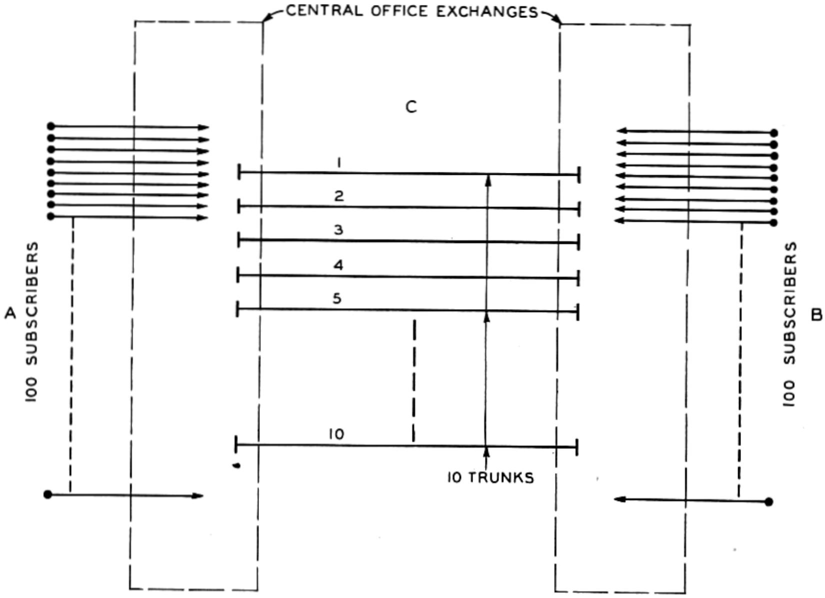

Fig. 2—A simplified central office interconnection system.

In Fig. 3 is shown the load, a, in average simultaneous calls which theoretically may be submitted to c trunks so that, on the average, one one-hundredth ($P = .01$), or one one-thousandth ($P = .001$), of all the calls submitted will find no idle trunk available. By replotting Fig. 3 to show as in Fig. 4 the average load carried per trunk (efficiency) we see that a large group of trunks is relatively much more efficient than a smaller one.

For example, to carry a load of $a = 41$ at $P = .01$ we should provide a single group of 57 trunks, while if we are required to carry the same total load over groups of $c = 16$ trunks, we shall need five such groups or a total of 80 trunks. That this should be so may become clearer by considering, say 20 trunks, first as a complete group and then as two split groups or subgroups of 10 trunks, each carrying one-half of

[2] "The Theory of Probabilities Applied to Telephone Trunking Problems," by E. C. Molina, *Bell System Technical Journal*, November, 1922.

the total load as pictured in Fig. 5. A trunk is represented by each horizontal line, and the load submitted by the vertical arrow, as though it were starting at the bottom of the group to hunt for the first idle trunk. We first observe that the proportion of calls lost on the

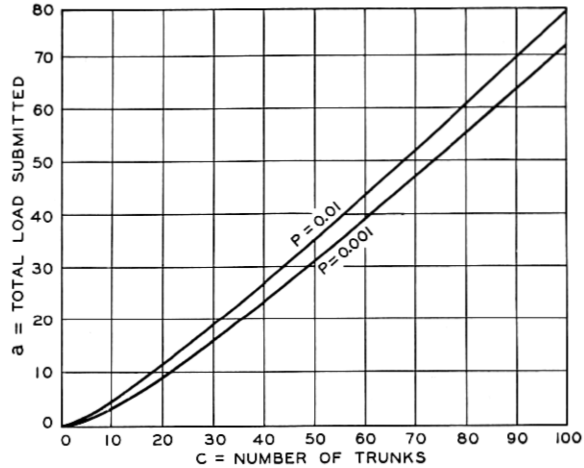

Fig. 3—Load carried by a trunk group at a constant loss.

average in the split groups cannot be less than in the complete group since when all the trunks are full in either case the calls coming in will be delayed or lost altogether, and only when *all 20 trunks* are occupied in the case of the complete group will calls be lost. On the other hand,

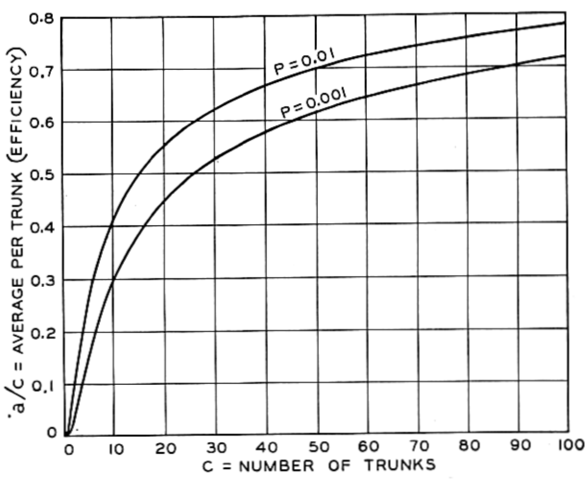

Fig. 4—Load carried per trunk at a constant loss.

in the split group calls may be lost when as few as 10 trunks are busy provided they are all in the group to which the calls at the moment are being originated. Thus the splitting of the group and the consequent reduction of the access may prevent a call in one subgroup, upon finding its 10 trunks busy, from continuing on over the remaining

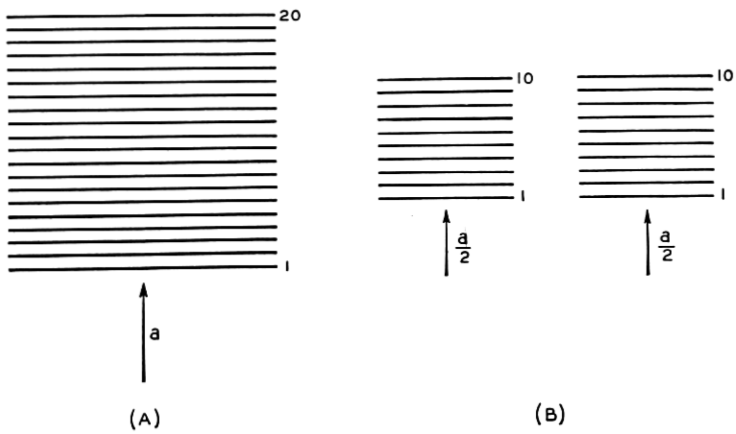

Fig. 5—Comparison of arrangements of twenty trunks.

trunks, one of which might have been idle. We conclude then that, other things being equal, a given load may be more economically carried over one large group than over two or more smaller groups.

Switching Limitations

Unfortunately, "other things" are decidedly not equal. Three major considerations may be pointed out which quite definitely tend to limit the number of trunks to which a particular source may be given access:

1. The high cost of switches having a large number of contacts.
2. The undesirable long hunts which occur in trunk groups of large size.
3. The double connections which increase directly with the load carried.

Of these the first two are usually the more instrumental in regulating the practicable upper limits to the access or hunt. When considered with the efficiency of the trunk groups in a system they comprise, in general, the fundamental data for determining the appropriate arrangement which may be economically employed to handle any given amount of traffic.

We are then faced with the problem of obtaining the maximum efficiency (maximum average load per trunk) with a given hunt or access. We have seen that the ultimate efficiency is obtained when the total trunks are arranged in a complete or straight grouping. We have also noted how greatly the efficiency is reduced by splitting the total trunks into distinct individual subgroups. It may well be that by certain rearrangements we shall be able to increase this low efficiency without increasing the access.

Graded Multiple Theory

The "graded multiple" means for improving the efficiency of trunking multiples requiring more channels than a single switch can profitably hunt over was proposed in 1905 by E. A. Gray.[3] We may gain an insight into the manner of working of graded multiple by considering a very simple example.

Suppose we have two 10-trunk subgroups as in Fig. 6(A), each carrying an average load corresponding to some predetermined probability of loss, say $P = .001$. Then the approximate average load carried by each individual trunk, *provided all calls hunt over the trunks in the same order*, is shown in Table I.[4] The important point to observe here is that the first trunk is busy a goodly proportion of the time [$a(1) = .748$], the second trunk a somewhat shorter time, and so on down to the last trunks which are comparatively lightly loaded, the tenth trunk being busy only about one-half of one per cent of the time.

The same distribution is approximately maintained in both subgroups of 10 trunks, each of which on the average presents all of its 10 trunks as busy to one out of each thousand calls submitted to it; but it is quite unlikely that this busy condition would occur simultaneously on the two groups. Hence, if on those occasions when a call is being lost in one group it could be allowed to hunt over, say, the last half of the other group, in many cases it would find an idle trunk

[3] E. A. Gray, assignor to the American Telephone and Telegraph Company, filed application July 30, 1907. The patent, No. 1002388, was granted September 5, 1911, for "A Method of and Means for Connecting Telephone Apparatus."

[4] By Erlang's statistical equilibrium method, upon the assumption that "lost" calls are immediately cleared and do not reenter the system, we find the average carried on the rth trunk is

$$a(r) = a[B(r-1, a) - B(r, a)],$$

where $B(x, a)$ is the proportion of traffic passing beyond the xth trunk and may be expressed,

$$B(x, a) = \frac{\dfrac{a^x}{\lfloor x}}{1 + a + \dfrac{a^2}{\lfloor 2} + \dfrac{a^3}{\lfloor 3} + \cdots + \dfrac{a^x}{\lfloor x}}.$$

In all cases "a" refers to the average number of simultaneous calls being submitted to an individual set of trunks.

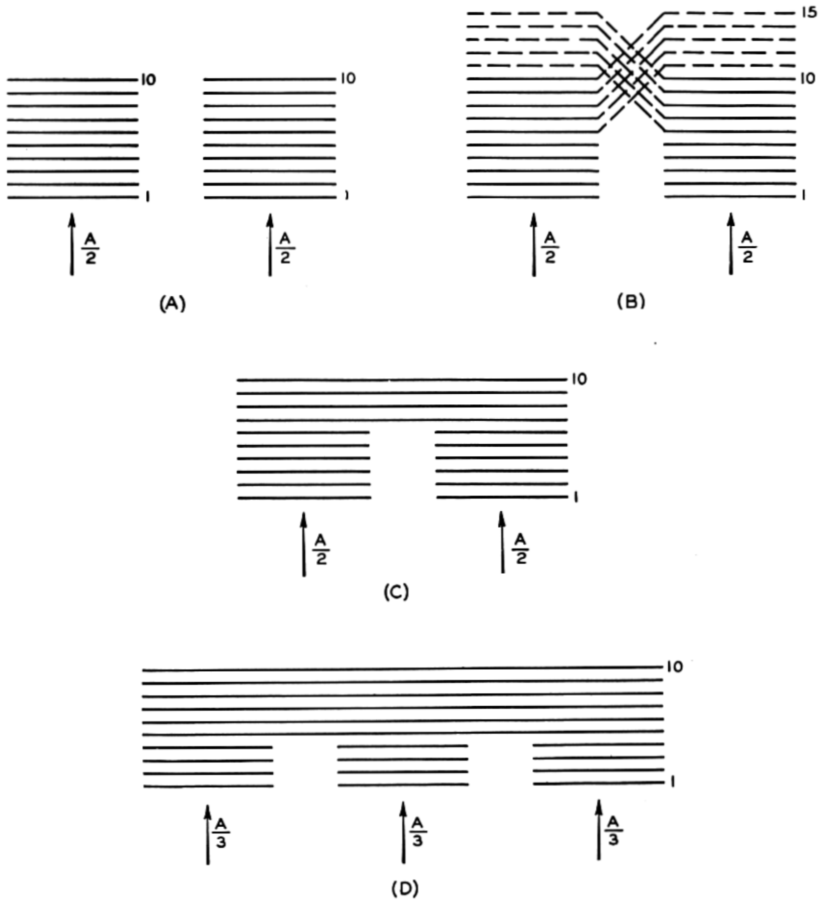

Fig. 6—Genesis of a graded multiple.

available. That is to say, we should be able to capitalize the possibilities of teamwork between the more lightly loaded parts of the groups.

TABLE I

Load Carried by Each Trunk of a Straight Multiple

Average Submitted = a = 2.96

	Number of Trunk in Order Hunted Over									
	1	2	3	4	5	6	7	8	9	10
Load carried on each trunk. .	.748	.658	.544	.413	.279	.170	.087	.039	.015	.005

This may be done by arranging the two groups as in Fig. 6(B) so that the cross-connection of the last choice trunks provides an opportunity for their mutual use by the calls from either subgroup. In this particular case, however, the hunt or access has been increased from 10 to 15 terminals. If we wish to retain the operation of the system on 10-point switching equipment we must compress the trunks into some such form as indicated in Fig. 6(*C*). In so doing we have very likely increased the split group efficiency of the trunks but, at the same time, on account of their fewer total number the load originally submitted may not be adequately served. Hence, a remedy such as shown in Fig. 6(*D*) may perhaps be devised: that is, the addition of more subgroups of the restricted-availability trunks. The study of the actual carrying capacities of these various arrangements is reserved for a later point in this paper.

In general, then, we may represent any such plan of trunking by the schematics of Figs. 7(*A*) and 7(*B*). These are called "simple

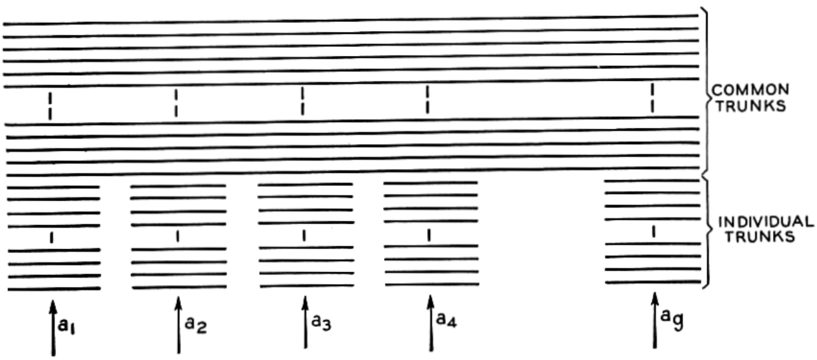

Fig. 7(*A*)—Simple graded multiple with *g* subgroups.

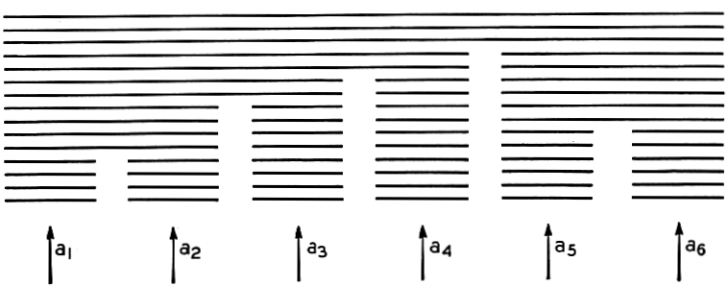

Fig. 7(*B*)—Example of a non-symmetrical progressive graded multiple.

graded multiples" and "progressive graded multiples," respectively, and by varying the number and placement of the trunks composing them a very large variety of arrangements indeed may be obtained. Here all calls hunt from the "bottom" of the group over a particular one of several sets of "individual" trunks called a subgroup, and if none are idle they continue on up to the "common" or "partially common" trunks which are made accessible to all of the switches that appear before the contributing subgroups. The essential characteristics of the graded multiple are, then: the trunks hunted over first serve a minimum number of switches while those trunks hunted over last are multipled before more or all of the switches. Provided our preliminary analysis is correct, the teamwork obtained between the latter portions of all of the subgroups should result in an increased average per trunk carrying capacity throughout the whole multiple.

Efficiency of Graded Multiples

Several theories for calculating the grade of service of graded multiples with any given submitted load per subgroup have been advanced. All of these involve certain assumptions or empirical approximations.

Dr. Fritz Lubberger[5] of the Siemens and Halske Manufacturing Company, Berlin, in collaboration with Dr. G. Rückle,[6] has presented a universal scheme, partly theoretical, partly empirical, for estimating the load carried by any trunk in a split or graded, or a combination of the two, multiples. That remaining load which is not carried then constitutes the overflow ("Verkehrsreste") from which the proportion "lost" may immediately be determined. The nub of Lubberger's method consists in the application of so-called "Zuschlagsfaktors" to correct the submitted load for the loss when splitting, and the gain when grading. This modified load, on the assumption that it has been reduced to the appropriate equivalent load, is then used to enter a chart constructed for a straight multiple. Combined with this procedure is the assumption that the busy hour loads in each subgroup may not occur simultaneously, thus giving a still freer opportunity for a cooperative usage of the common trunks.

A second plan for estimating the probability of loss of a graded multiple was proposed by the late Dr. M. Merker of Antwerp.[7] He developed a very complicated formula which involves a consideration of the various ways in which a graded multiple might accommodate a given number of calls.

The British Post Office has made some interesting and valuable

[5] F. Lubberger: "Die Wirtschaftlichkeit der Fernsprechanlagen für Ortsverkehr." R. Oldenbourg, Munchen und Berlin, 1927.

[6] G. Rückle und F. Lubberger: "Der Fernsprechverkehr als Massenerscheinung mit starken Schwankungen." Julius Springer, Berlin, 1924.

[7] M. Merker: "Some Notes on the Use of the Probability Theory to Determine the Number of Switches in an Automatic Telephone Exchange." *The Post Office Electrical Engineers' Journal*, vol. 17, Part I, April, 1924.

empirical investigations of the graded multiple problem.[8] These have included successively or progressively-graded multiples as well as those involving only a single set of subgroups feeding into a simple group of common trunks. They conclude that for more than two subgroups the successive grades are somewhat advantageous and the highest efficiencies are to be gotten when there is "a smooth progression from individuals to commons." That is, the number of trunks in each subgroup of individuals, pairs, fours, eights and commons, should be very nearly equal, and in general "no grading should be used, if it can be avoided, in which the actual number of circuits required to the next rank exceeds half the maximum possible number.[9] An 18-group grading, for example, should not be used when more than 90 circuits are required. . . . These specific data are, of course, with reference only to the gradings of 10-contact switches."

Finally, a very interesting field of study has been opened up through the design of an artificial traffic machine by Messrs. E. A. Elliman and R. W. Fraser of the Standard Telephones and Cables, Ltd.[10] In the particular machine constructed two group gradings with all arrangements of individuals and commons up to hunts of 25 could be simulated. In the single examples given of straight and graded groups the results showing the load per trunk appear to be closely in accordance with their expected values. If a more flexible mechanism could be devised and tested to insure concordance with practice a most valuable contribution to the art of trunking would be made.

In the Mathematical Appendix I of this paper Mr. E. C. Molina sets forth the analytical theory for simple (single-stage) symmetrical graded multiples as originated and practiced (with certain modifications to be mentioned later) by the Bell System. The present method of estimating the probability of loss, knowing the arrangement of trunks and the average load submitted per subgroup, is the natural outgrowth of several preliminary formulas each of which was closely studied and compared with the actual conditions to be met in operation.

The four governing assumptions which need careful scrutiny in the final formula presented in this paper are:

1. The holding time of all calls is assumed to be constant.

2. A call not receiving immediate service is held in waiting for its normal holding time period, and if an available trunk becomes idle it will occupy it till this period is completed. This is usually referred to as the "lost calls held" assumption.

[8] G. F. O'Dell: "An Outline of the Trunking Aspect of Automatic Telephones." *The Journal of the Institution of Electrical Engineers*, vol. 65, February, 1927.

[9] By "next rank" is meant, for instance, second selectors following first selectors.

[10] Elliman and Fraser: "An Artificial Traffic Machine for Automatic Telephone Studies." *Electrical Communication*, October, 1928.

3. The distribution of the load submitted to each subgroup follows the Poisson Law and each subgroup carries the same average load as every other subgroup.

4. At no time shall a call be occupying a common trunk if an idle trunk exists in the group of individual trunks assigned to the subgroup of calling sources or switches from which the call under consideration originated. In other words, it is assumed that calls which seized idle common trunks because, at the time they originated, idle individual trunks were not available, shall be immediately transferred (by some fictitious redistributing apparatus) back to their individual trunks as soon as these become idle. This assumption will be referred to below as the assumption of "no-holes-in-the-multiple."

Whether these are admissible assumptions must be decided by a comparison of what actually results in practice with the formula which they give rise to. Their concordance will be discussed in a following section.

In order to put this graded formula in a usable form for engineering study curves or tables are needed showing how much load any given arrangement of trunks will be able to carry at any specified grade of service. Such charts have been constructed for the two more commonly used probabilities of loss, $P = .01$ and $P = .001$, comprehending all possible arrangements of trunks in simple symmetrical graded multiples having an access or assignment of 10, 20, 30 and 40 terminals and subgroups from two to seven in number. These are designated as Figs. 8 to 15, 8 to 11 corresponding to $P = .01$, and 12 to 15 to $P = .001$; the four at each probability cover the ranges of access or assignment, 10, 20, 30 and 40, respectively. These distinguishing parameters are noted in the upper right-hand corner of each chart. In order to simplify the necessary descriptive terms the number of trunks in each individual subgroup is called "x," the number of common trunks is called "y," and the number of subgroups, "g." Thus the access equals $x + y$, and the total number of trunks equals $gx + y$.

For the sake of compactness and brevity both the abscissa and ordinate scales of these charts are plotted in terms of ratios; the former gives $gx + y$ or total trunks in terms of the access, $x + y$, and the latter the per cent gain in efficiency (per cent increase in average load per trunk) over the efficiency of a simple straight multiple of $x + y$ trunks. A single one of the seven semi-circular curves on each figure then yields the load information for any number of trunks having a particular number of subgroups, a designated access, and a specified grade of service. The dotted curve on each figure is included to show the "per cent gain over the efficiency of $x + y$ trunks" if the total $gx + y$ trunks are placed in a single complete-access group. A few problems will make clear the exact meaning and use of the curves.

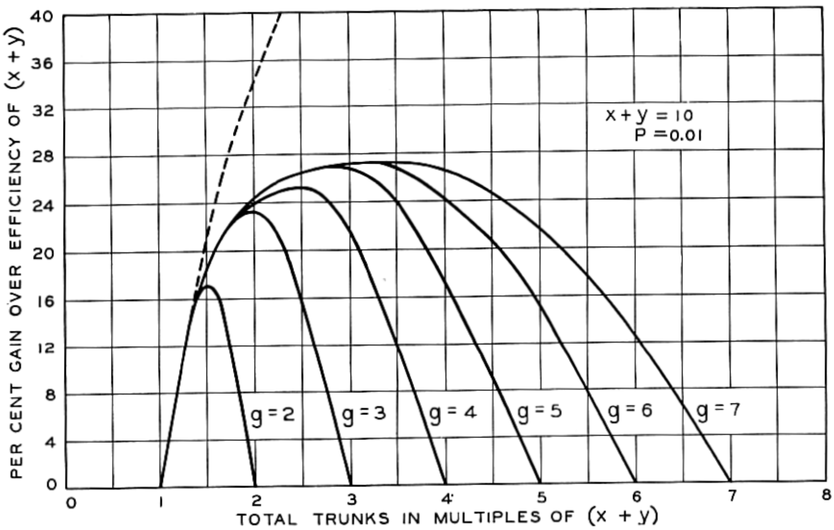

Fig. 8—Graded multiple efficiency. $x + y = 10$. $P = 0.01$.

Suppose we wish to know how much load may be submitted to each subgroup of a symmetrical graded multiple of 105 trunks having five commons out of an access of 30, such that, on the average, one call in

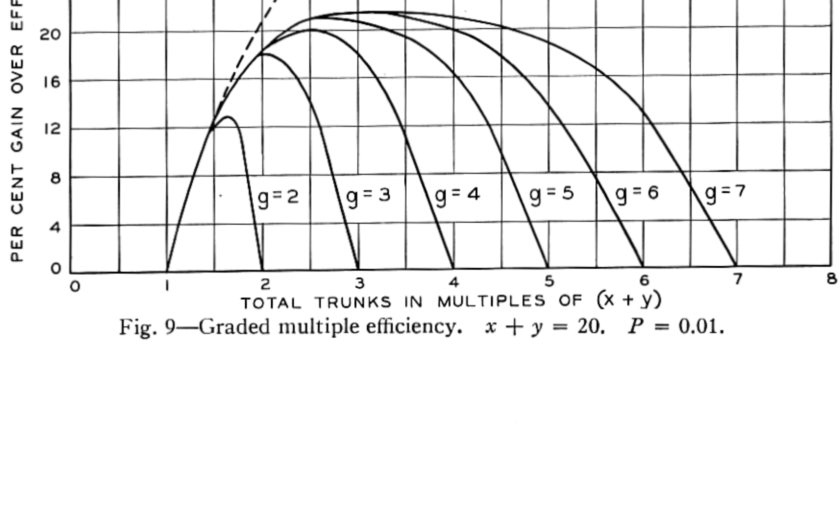

Fig. 9—Graded multiple efficiency. $x + y = 20$. $P = 0.01$.

one thousand will be lost. Evidently,

$$gx + y = 105, \quad x + y = 30, \quad x = 25, \quad y = 5, \quad P = .001;$$

from which, solving the first three equations simultaneously, we find g,

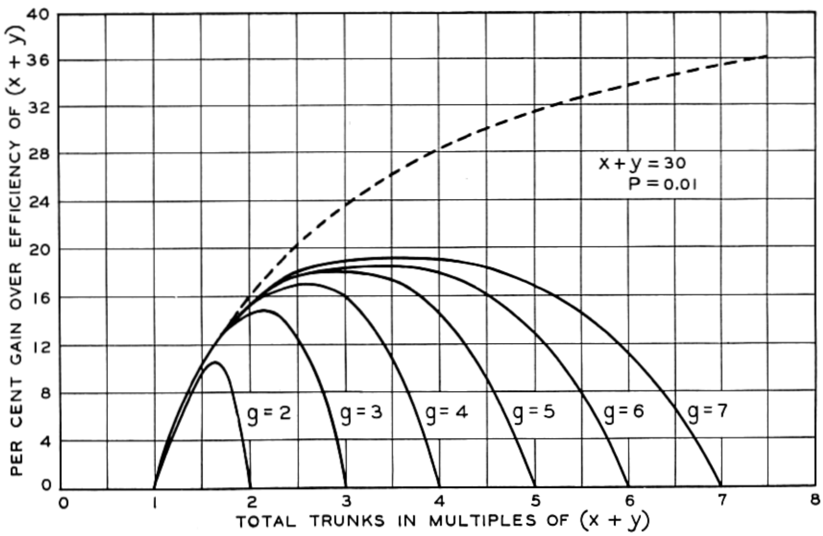

Fig. 10—Graded multiple efficiency. $x + y = 30$. $P = 0.01$.

the number of subgroups, equal to 4. To enter the appropriate chart (Fig. 14) we must determine the ratio $\frac{gx + y}{x + y}$, which equals $\frac{105}{30} = 3.5$. The $g = 4$ curve at this abscissa of 3.5 gives us the gain over the

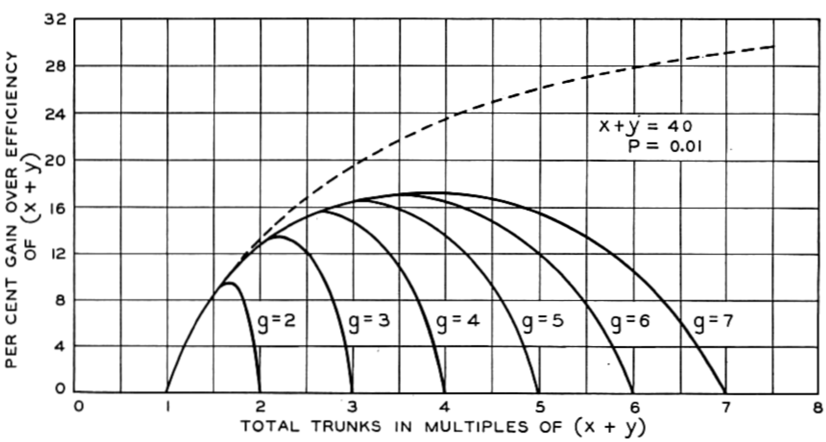

Fig. 11—Graded multiple efficiency. $x + y = 40$. $P = 0.01$.

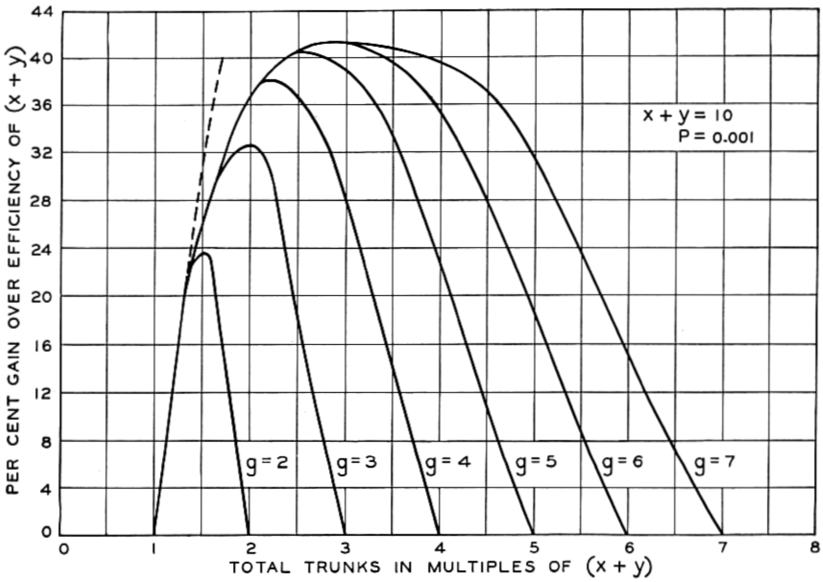

Fig. 12—Graded multiple efficiency. $x + y = 10$. $P = 0.001$.

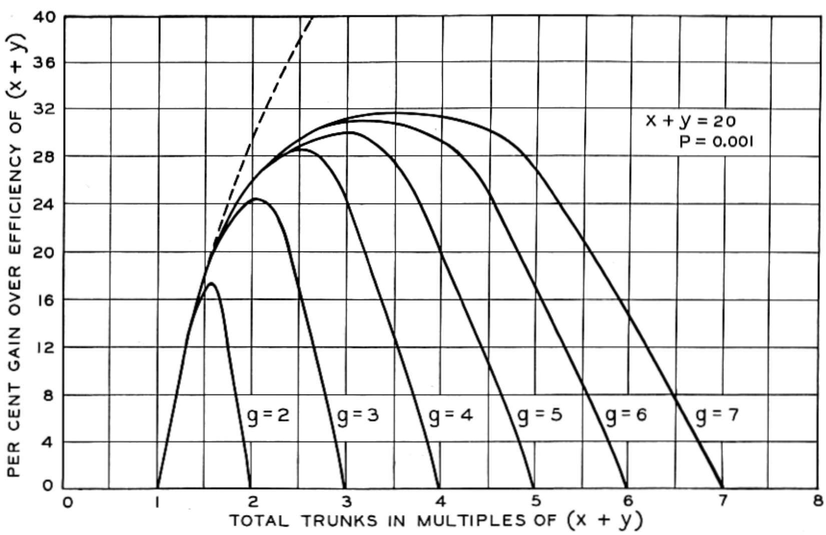

Fig. 13—Graded multiple efficiency. $x + y = 20$. $P = 0.001$.

efficiency of $x + y$ trunks as 13 per cent. This means that the average load per trunk in the graded multiple is 1.13 times that in a straight group of $x + y = 30$ trunks at the same probability of loss. From

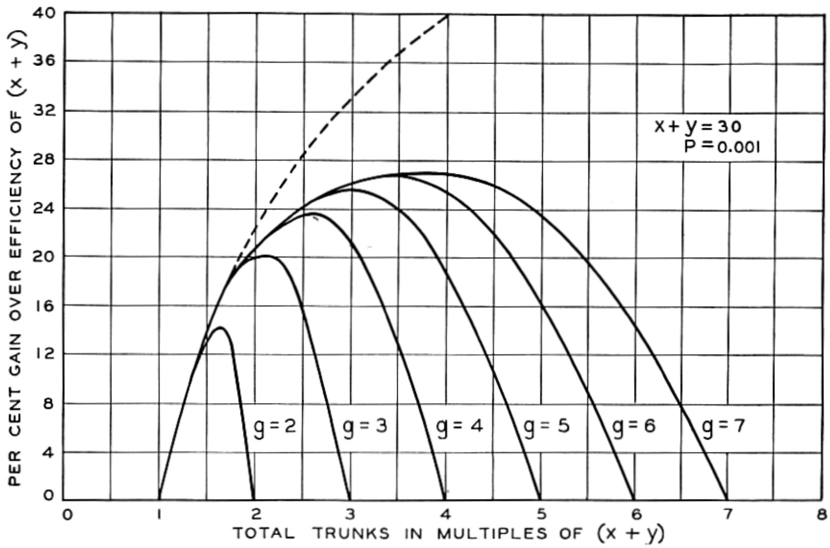

Fig. 14—Graded multiple efficiency. $x + y = 30$. $P = 0.001$.

Fig. 4 we read the average carried per trunk on a straight group of 30 as .529. Hence the total load which may be submitted [11] to the graded

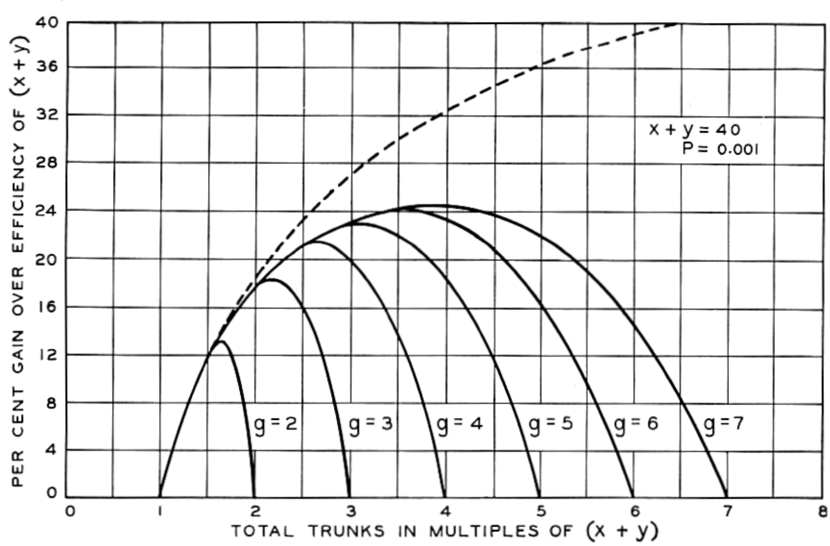

Fig. 15—Graded multiple efficiency. $x + y = 40$. $P = 0.001$.

[11] We may use the terms "average submitted" and "average carried" interchangeably here since at losses of $P = .01$ or less the difference is negligible.

multiple is

$$A = (1.13)(.529)(105) = 62.77.$$

To obtain the desired load per subgroup we need only divide by 4 giving $a = \frac{62.77}{4} = 15.69$. In terms of one-hundred-second-calls this average per subgroup then equals $36a = 565$ calls per hour.

As a second example we may set the question: Given 58 trunks to grade on an access of $x + y = 20$ with either three or six subgroups, which arrangement is to be preferred? We shall have to assume, since we have no additional information, that the decision rests merely upon the relative efficiencies of the two schemes. From the given data, $gx + y = 58$, $x + y = 20$, and $g = 3$ or $g = 6$. We read opposite $\frac{gx + y}{x + y} = \frac{58}{20} = 2.90$ for $g = 3$ and $g = 6$ on both the $P = .01$ and the $P = .001$ charts (since the probability was not specified) and construct the following table (Table II):

TABLE II

No. of Subgroups g	No. of Individual Trunks per Subgroup x	No. of Common Trunks y	$\frac{gx+y}{x+y}$	Per Cent Gain Over Efficiency of $(x+y)$	
				$P = .01$ Fig. 9	$P = .001$ Fig. 13
3	19	1	2.90	3.0	3.5
6	7.6	12.4	2.90	21.2	31.0

It is clear then, that as far as efficiency of arrangement goes, the six-subgroup plan is in the order of 20 per cent superior to the three-subgroup plan for ordinary grades of service. There is one difficulty here, however, and that is that in the symmetrical multiple of six subgroups the calculated number of individual trunks in each subgroup and the number of commons are not integers. We may get around this trouble by either unbalancing the grade slightly or changing the total number of trunks. For instance, we could use four subgroups of seven trunks, each pair of subgroups feeding into a single trunk common to them, before reaching the 12 through-commons into which the remaining two subgroups of eight trunks would work directly. The total of 58 trunks would then be disposed of at an efficiency gain probably not differing markedly from that estimated in our table above.

Secondly, we could have reduced the total trunks to 55, making thereby a symmetrical arrangement of $x = 7$ and $y = 13$. Had we done this we should have been able, at $P = .01$, to carry a total load (reading the gain of 21.3 per cent at $\frac{gx + y}{x + y} = 2.75$ on Fig. 9) of $A_6 = (1.213)(.554)(55) = 36.96$. At the same time the load we could have carried on 58 trunks with three subgroups is only $A_3 = (1.03) \times (.554)(58) = 33.10$. Hence, we could reduce the total number of trunks by three in the six-subgroup case and still carry more load than if we were to use the three-subgroup arrangement and the original number of trunks. This decided advantage in favor of the larger number of subgroups is even more pronounced if the $P = .001$ comparison is made ($A_3 = 26.9$ vs. $A_6 = 32.2$).

The secret of the large gains in certain cases is easily found. After a short study of the curve charts it will readily be verified that, in general, the modal or maximum gain point on any curve comes very nearly at the midpoint of the range between $x + y$ and $g(x + y)$. This midpoint is reached by always setting $x = y$ or $x = \frac{1}{2}(x + y)$, that is, by making the individuals compose one-half of the access or assignment.

Comparisons of Theory and Practice

It is, of course, eminently desirable to know whether the formula just described for the probability of loss of any simple arrangement is consistent with the grades of service which it will actually render in practice. This we could ascertain only after a prolonged and careful study of typical graded groups already at work in the Bell System. Accordingly, a set of tests lasting over a period of six months, in the latter part of 1927, was made in Chicago by the Department of Operation and Engineering of the American Telephone and Telegraph Company in cooperation with the Illinois Bell Telephone Company.

The tests were performed on district multiples, believed to be representative, by connecting holding time recorders to the groups to indicate the load being carried by each trunk. Then through the use of overflow and peg count (number of calls) registers the proportion of calls being delayed (or lost) was readily found for any particular busy hour load.

In the First Division of these tests two groups of interoffice trunks, State to Dearborn and State to Wabash, were selected as typical cases of the kinds of fluctuating busy hour loads to be found in ordinary panel graded practice. No attempt was made here to regulate the load being submitted in any busy hour or to any subgroup since what was particularly desired was not what *would* happen *if* such and such conditions obtained, but rather what *does* happen under the fluctuating load conditions which *actually occur* in the busy hours from day to day.

TABLE III

Trunk Arrangements					Observational Data					Theory			
I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV
										No. Trunks Required at the Observed Probability of Loss			
Total Trks. in Graded Group	Access or Assign-ment	No. of Sub-groups	No. of Individ-uals	No. of Commons	Average Load Sub-mitted to Graded Group	Average Proportion of Calls Lost	Maximum Proportion Lost in Any One Hour	Per Cent of Total Busy Hours Having Overflows	No. of Busy Hours Included	Full Group Efficiency	Full Graded Gain	Half Graded Gain	No Gain
$gx+y$	$x+y$	g	x	y	A								
State to Dearborn													
28	20	5	2	18	19.50	.0573	.235	100.	13	26.9	27.4	28.1	28.9
32	20	5	3	17	18 97	.0106	.054	73.4	30	30.0	30.65	32.1	34.0
36	20	5	4	16	20.83	.0090	.075	69.1	81	32.8	33.2	35.2	38.0
40	20	5	5	15	20.77	.0010	.004	25.0	4	36.5	37.2	40.9	46.4
State to Wabash													
40	20	5	5	15	28.19	.0600	.169	100.	9	37.2	37.85	39.5	41.5
44	20	5	6	14	29.08	.0187	.041	100.	13	41.4	42.2	45.2	49.2
48	20	5	7	13	32.56	.0242	.110	100.	11	44.9	46.0	49.3	53.5
52	20	5	8	12	32.75	.0199	.111	84.8	66	45.5	46.9	50.5	55.0
56	20	5	9	11	35.25	.0136	.042	90.0	10	49.6	51.5	56.1	61.7

This will account then for the rather decided non-uniformity of the loads supplied to the two groups for various numbers of trunks. In Table III are recorded in the left-hand division the nine different trunking arrangements tested. All cases consisted of grades of five subgroups ($g = 5$) in a multiple having "x" individual trunks in each subgroup placed before "y" common trunks such that the access $(x + y)$ remained constant and equal to 20. The sizes of the resulting groups were then varied from 28 to 56 trunks.

In the central division of Table III, designated "Observational Data," are shown the various loads carried by these trunk arrangements with the corresponding proportions of calls lost during the period of each test. The supporting data of columns VIII, IX and X give one an idea as to the fluctuations which may be expected in the lost calls in a limited number of week-day busy hours.

The last division of Table III, "Theory," shows in its four columns the number of trunks that would theoretically be specified, on various bases of engineering, to carry the load actually observed in each run at a probability of loss equal to the observed proportion of calls lost. Column XI gives the number of trunks which would be required in each case could the trunks all be placed in a single straight group. Since this is the most efficient arrangement possible, a minimum of trunks need be supplied. At the other extreme we have in column XIV the number of trunks which would be required if each trunk operated at the efficiency of a group of 20 ($= x + y$) trunks. This could actually be realized, of course, only when the total number of trunks required was an exact multiple of 20. As shown, from 2 to 12 more trunks are required with this decreased efficiency than when a full group is being considered.

The two other columns, XII and XIII, show the number of trunks that would be needed in a graded multiple of five subgroups having an access of 20 trunks, upon two different assumptions. The column headed "Full Gain" is obtained from curves similar to Figs. 8 to 15, but appropriate to the observed probabilities of loss, to give the "per cent gain over the efficiency of $x + y$ trunks." Knowing the total load to be carried and the enhanced efficiency of $x + y$ trunks in each case the number of trunks required is readily determined. The "Half Gain" column is arrived at in precisely the same way with the exception that only one-half of the indicated "per cent gain over the efficiency of $x + y$ trunks" is utilized.

To facilitate the interpretation of these results they have been shown graphically in Fig. 16. Above each point on the abscissa at which a run with a known number of trunks was made is recorded

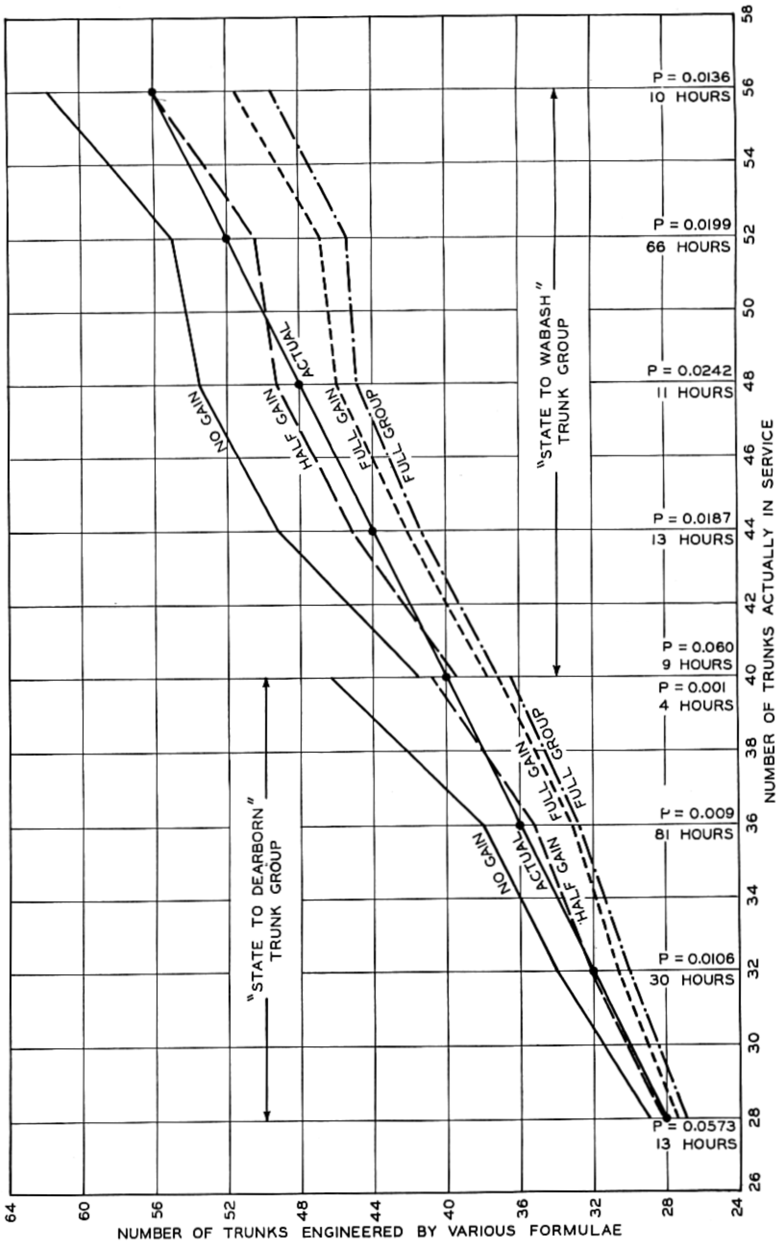

Fig. 16—Comparison of formulas with graded multiple busy hour tests, Chicago, July–December 1927.

the corresponding number of trunks which would be provided on each of the bases considered above. The actual number in service is also shown as a straight line through the shaded points for comparison with the various theoretical schedules. As noted on the figure the studies having 28 to 40 trunks were made on the State to Dearborn group and those having 40 to 56 trunks on the State to Wabash group.

As may readily be seen either from Table III or Figure 16 the "Half Gain" schedule coincides especially well with the observed data, while the other engineering plans fall consistently too high or too low on the scale. It may be remarked that the "Full Gain" values approach the limiting full group figures very closely and that it should prove of considerable interest to determine the cause of divergence between the field observations and these large theoretically possible graded loads.

Critical Inspection of Assumptions in Graded Theory

It has been noted that the number of "Full Gain" trunks specified is well below the number really required. This confirms the possible suspicion that not all of the four assumptions fundamental to the

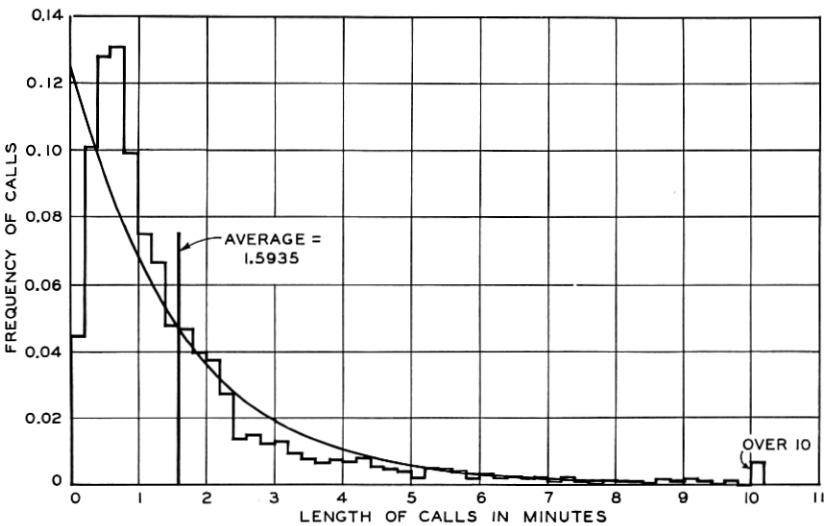

Fig. 17—Distribution of interoffice trunk holding times.

graded multiple formula presented in this paper are entirely satisfied. We shall therefore examine, in order, the accuracy of these assumptions.

We know indeed, without investigation, that calls are of widely differing lengths, and in addition are not "held" in the manner assumed. In Fig. 17 is shown a typical holding time distribution for calls carried over the State to Dearborn group, with a suggested exponential fitting frequency curve superposed on it. From other studies carried out on non-graded groups we are led to believe that as far as the probability of loss is concerned it is almost independent of the change in holding time from a constant to an exponential form, the advantage, if any, favoring the varying case. Likewise the manner of "holding" delayed calls is of negligible importance as long as losses of .01 or .02 are not greatly exceeded.

The first part of the third assumption regarding the incoming calls being distributed according to the Poisson Law was not checked in these particular tests but a wide study of results under similar conditions readily leads us to believe that calls originating from a large number of independent sources will exhibit this form of frequency distribution.

The third assumption also necessitates a study of the variations among the loads submitted to the subgroups of a graded multiple. This brings us to the Second Division of the tests made in Chicago. At the same time the first division was in progress on the busy hours of each day for the cases of 36 trunks in service on the State to Dearborn group and 52 trunks on the State to Wabash group, all of the hours of the day from 9 till 5 were observed by one-half hour periods (to minimize the error due to trends) for the number of call-seconds on each trunk, the number of calls carried over the group and the number lost. Thus an extended range of load conditions was obtained for study. For estimating the effects of subgroup load variations with a given total submitted load, these short pieces of data were combined so that the half hours having an average load in trunk hours per hour within approximately one unit of range were thrown together. The various analyses were then made on these narrow total load classifications to discover, if possible, whether the observed subgroup variation when used in the theoretical formula would cause the latter's "Full Gain" probability of loss to approach more closely the observed losses.

First, the proportion of lost calls was determined for each of these approximate unit intervals of load. The results are shown graphically in Figs. 18 and 19 for State to Dearborn and State to Wabash, respectively. On these same figures have been superposed the theoretical curves for the losses to be gotten using "Full," "Half" and "No Gain" efficiencies in the graded formula described above. These theoretical computations have assumed that equal average loads are submitted to each subgroup at all times. The observed data indicate that the correct descriptive curves lie in both cases somewhere between those for the "Half" and "Full Gain" efficiency theories. This seeming improvement of the observed data over its position in Fig. 16 is reasonable since here we have sorted out and combined only hours of like average loads while before all the busy hours, high and low, of a given period were included. It should be noted that the abscissa for the observations here is *load carried* while for theory it is *load submitted*. The comparison error is doubtless negligible for losses of,

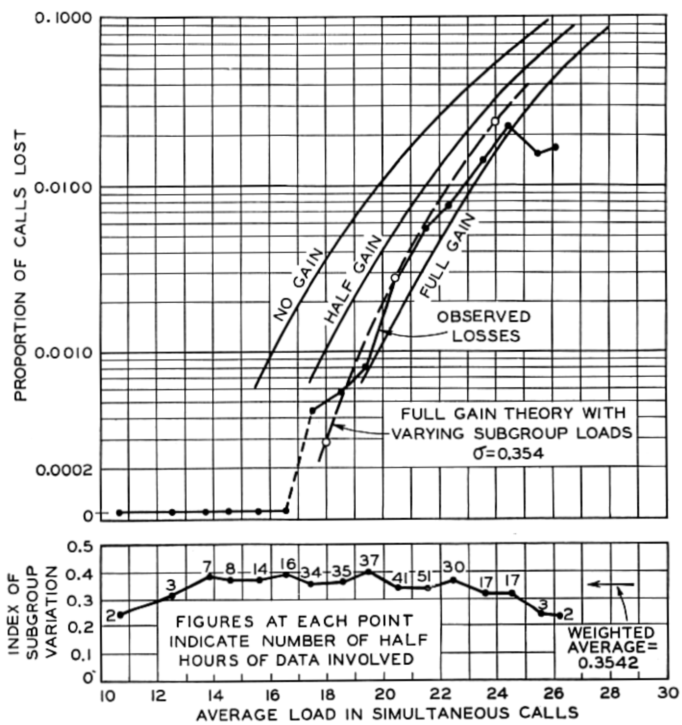

Fig. 18—Comparison of theoretical and observed graded losses, "State-Dearborn" group. $x = 4$, $y = 16$, $g = 5$.

say, $P = .03$ or less. The data beyond this figure are rather too meager for useful correction.[12]

Next the "Full Gain" formula was generalized by the author to comprehend the submission of different averages to each of the various subgroups. (See Appendix II.) Then, selecting typical loads, for instance $A = 18.00$ for the State to Dearborn tests and $A = 33.80$ for the State to Wabash tests, they were each divided up arbitrarily

[12] The advisability of correction here, were the data plentiful, may well be doubted since it would necessitate an assumption as to the manner in which calls were "held," a procedure especially precarious at high losses.

into five different magnitudes for use in this theoretical calculation. As a measure of the variation from equal loads submitted to the subgroups, the standard deviation of the estimated subgroup loads carried taken about their average value was used in each case. To estimate the load, "l," which would be carried by any subgroup to which the

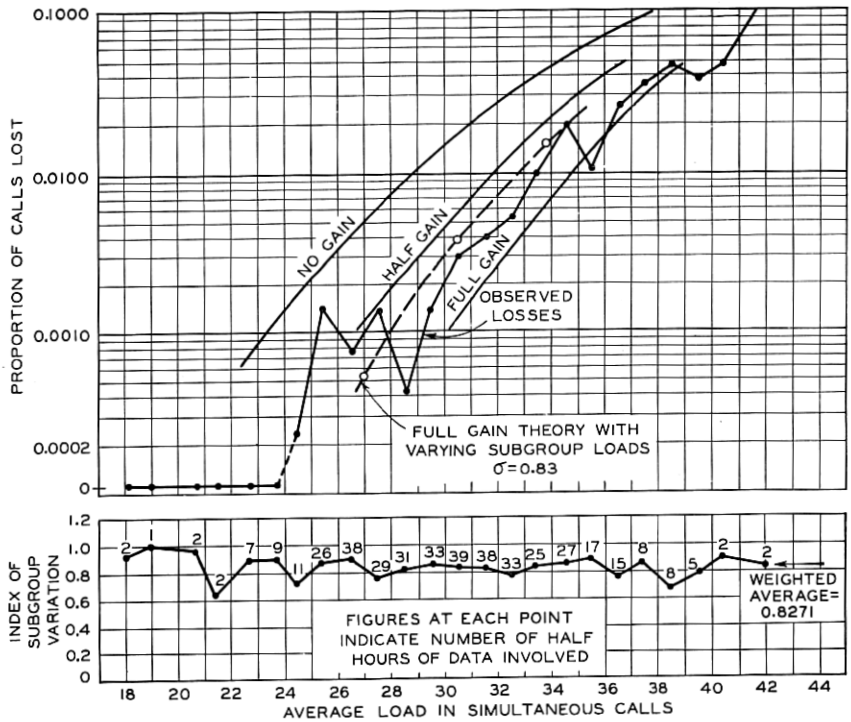

Fig. 19—Comparison of theoretical and observed graded losses, "State-Wabash" group. $x = 8$, $y = 12$, $g = 5$.

average, "a," is submitted, Erlang's formula, as set down earlier in this paper, was utilized in the following relationship:

$$l = a[1 - B(x, a)],$$

wherein, as before, $B(x, a)$ connotes the proportion of the submitted load which goes beyond the x individual trunks. The reason for working with the theoretical load carried by a subgroup rather than the load submitted to it will be clear when it is recalled that all the observed data available for comparison yielded the former values only. We shall now examine briefly the result of assuming subgroup variations when the above typical total loads are substituted in the generalized graded formula.

TABLE IV

THE EFFECT OF SUBGROUP LOAD VARIATIONS ON THE GRADE OF SERVICE—36 TRUNKS

Total Load Submitted = A = 18, x = 4, y = 16, g = 5.

Case	Loads Submitted to Subgroups					Standard Deviation of Subgroup Loads Carried	Overall Probability of Loss
	a_1	a_2	a_3	a_4	a_5		
Theoretical No. 1....	3.60	3.60	3.60	3.60	3.60	0	.000189
Theoretical No. 2....	3.00	3.00	4.00	4.00	4.00	.18	.000223
Theoretical No. 3....	2.57	2.57	3.60	4.63	4.63	.35	.000289
Theoretical No. 4....	2.20	2.20	3.60	5.00	5.00	.48	.000531
Theoretical No. 5....	2.20	2.20	2.20	5.70	5.70	.58	.000957
Theoretical No. 6....	2.00	2.00	2.00	6.00	6.00	.67	.001493
Observed...........	—	—	—	—	—	.36	.00050

TABLE V

THE EFFECT OF SUBGROUP LOAD VARIATIONS ON THE GRADE OF SERVICE—52 TRUNKS

Total Load Submitted = A = 33.80, x = 8, y = 12, g = 5.

Case	Loads Submitted to Subgroups					Standard Deviation of Subgroup Loads Carried	Overall Probability of Loss
	a_1	a_2	a_3	a_4	a_5		
Theoretical No. 1....	6.76	6.76	6.76	6.76	6.76	0	.00619
Theoretical No. 2....	5.50	5.50	7.60	7.60	7.60	.45	.00925
Theoretical No. 3....	4.40	6.40	7.40	7.40	8.20	.69	.0116
Theoretical No. 4....	4.60	4.60	8.20	8.20	8.20	.89	.0181
Theoretical No. 5....	4.00	5.00	7.00	8.20	9.60	.99	.0214
Observed...........	—	—	—	—	—	.84	.013

In the last two columns of Tables IV and V are indicated the measures of subgroup load variation and the expected grades of service, respectively, for various theoretical unbalances studied on these two symmetrical graded multiples. Figs. 20 and 21 indicate the rapidity with which the overall probability of loss on the generalized "Full Gain" formula basis may be expected to rise with increases in the load unbalances in the subgroups. Reading off on the abscissa of Fig. 21 a measure of subgroup unbalance of .45, for example, indicates that for a total load of A = 33.80 being submitted to 52 trunks arranged in a grade of five subgroups and an access of 20, the correct probability of loss is not the "Full Gain" efficiency value of .00619 but rather the more conservative figure of about .0090.

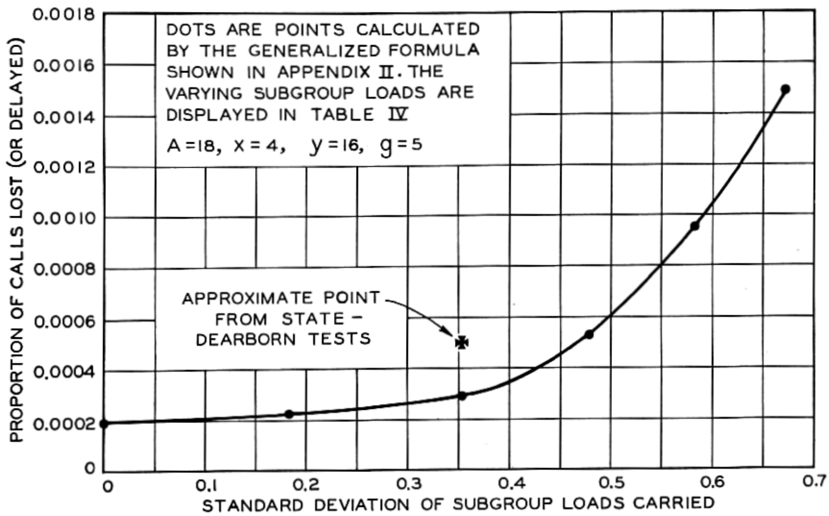

Fig.—20 Effect of subgroup load variation on probability of loss—36 graded trunks

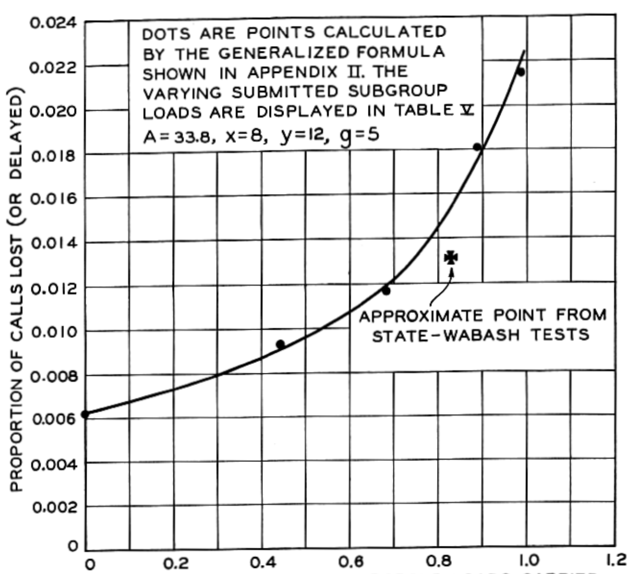

Fig. 21—Effect of subgroup load variation on probability of loss—52 graded trunks.

The corresponding subgroup variation obtaining throughout the Second Division of the tests was then determined and appears in the Figs. 18 and 19 as an auxiliary broken line at the bottom of the charts. Strangely enough, in both tests, the subgroup variation maintains an almost constant magnitude whatever the load. In fact, as mentioned in a later paragraph this criterion of variation seems rather to depend in practice upon the particular trunking arrangement being considered. Entering Fig. 20 with the observed average subgroup variation of .3542 for the 36 trunks in the State to Dearborn group, we should expect the proportion of calls lost at a total load of 18.00 submitted to be about .000289. Likewise, we should find that this same variation occurring in total loads of 20.5 and 24.0 gives us losses of .00278 and .0239, respectively. The dotted curve on Fig. 18 is drawn through these three points and represents the theoretical probability-load curve for this arrangement of trunks when the index of variation in loads from subgroup to subgroup is equal to the average observed. This schedule falls slightly above the observed losses over a considerable portion of the important range of loads although at the lower losses it seems quite likely to coincide fairly well with a curve drawn by eye through the data represented by the irregular line.

Similarly, in the case of the 52 trunks between State and Wabash, we may construct a schedule of losses based on a subgroup load variation having a measure equal to the observed value, .8271. Such a curve is shown dotted in Fig. 19 and as before indicates that upon taking into approximate consideration the variation in loads being submitted to the subgroups the resultant discrepancy between a curve fitted to the observed losses and the "Full Gain" theoretical grades of service is, in the main, slightly more than accounted for. In Figs. 20 and 21 the observed points fall one above and one below the theoretical schedules. These deviations do not appear to be of any significance except to illustrate the many chance elements which enter into telephone traffic problems.

A priori one might expect also that there would be some correlation between the proportion of lost calls and the subgroup variation for half hours in any given unit interval of total load. The number of calls lost in these tests, however, is so small that plotting by half hours the large natural fluctuation due to other causes seems to completely mask any such small effects which might be predicted.

To further study the manner and amount of this subgroup variation in carried loads, similar calculations were performed by half-hour units on the First Division of the tests (busy hours only) wherein the restriction of unit range of average was not present. The results shown by the heavy dots in Fig. 22, as might be expected, give slightly higher values of variation for the non-restricted load values at $gx + y = 36$ and 52 trunks than for the restricted cases belonging to the second division of the tests. The remarkable point here, however, is that the phenomenon for the ranges studied exhibits practically a straight

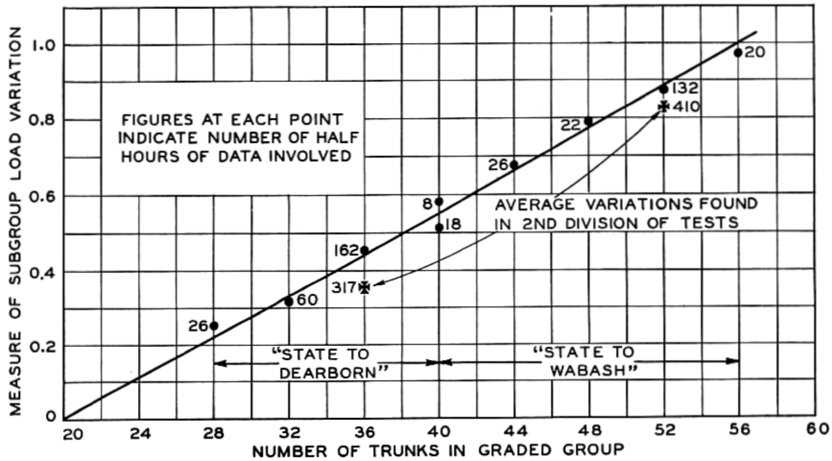

Fig. 22—Change in subgroup load variation with number of trunks in group.

line relationship between variation of subgroup loads and the number of trunks per subgroup, a difference of four in total trunks meaning an increase of one in each subgroup. An added variation for the larger number of trunks seems only natural, however, since as the subgroup size is increased part of the fluctuations previously borne by the commons is transferred to each subgroup itself. That this natural increase in subgroup variability does not affect the grade of service of the larger groups for busy hour measurements seems to be amply demonstrated by the consistency of the "Half Gain" formula in fitting the observed number of trunks in Fig. 16. We conclude, then, that a formula based on the third assumption (equality of subgroup loads), is considerably at variance with actual results; by modifying this assumption, to approximate loading differences, the graded loss formula appears to describe the observed losses quite satisfactorily.

Concerning the last assumption underlying the graded formula derived here, that of "no-holes-in-the-multiple," somewhat less is known. That the "holes" do exist is self-evident. It is suggested by these Chicago half-hourly observations, however, that under ordinary conditions the reaction of holes-in-the-multiple upon the grade of service is negligibly small in comparison with the effect of subgroup load variations.

Practical Graded Multiple Engineering

In the Bell System, engineering of equipment and lines is done on the basis of the grade of service desired in the busy hours over a considerable period of time. The theoretical formula used, then, is the one giving, for those ranges in which satisfactory service is being rendered, an approximate relationship between the average total load carried and the proportion of lost calls expected over a number of busy hours. This, rather than some less conservative plan which would simulate the losses encountered only in a particular busy hour. The use of the "Half Gain" efficiency curves, therefore, is justified from a consideration of Fig. 16, which illustrates how closely that theoretical expression

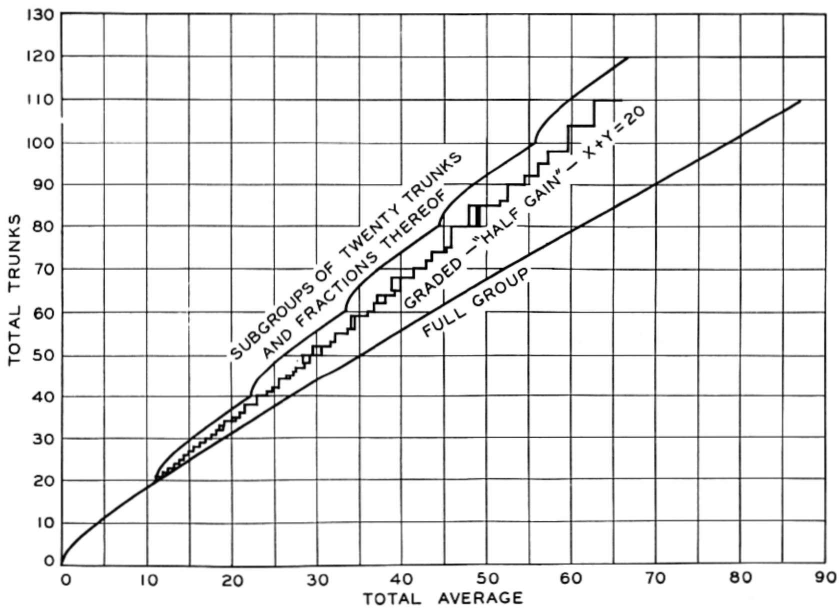

Fig. 23—Multiple capacity curves. $P = 0.01$.

approximates the actual conditions which maintain over a considerable range of trunk combinations and load values in two typical graded multiple installations.[13]

In practice, of course, the load to be carried rarely comes out exactly equal to that which a given symmetrical trunk arrangement

[13] Some five years ago it was appreciated that "Full Gain" was not likely to be attained. The value of "Half Gain" was then arbitrarily selected as the graded engineering basis in the Bell System until field studies of the efficiency could be completed. The merit of this estimate is attested by the material presented in this paper.

TABLE VI

GRADED MULTIPLE TRUNK CAPACITY TABLES

P = .01—Terminal Assignment = 20

Equivalent 100″ Calls	Number of Ind. Groups	Total Trunks	Ind. Trunks	Common Trunks	Equivalent 100″ Calls	Number of Ind. Groups	Total Trunks	Ind. Trunks	Common Trunks
423	2	21	1	19	954	3	44	12	8
448	2	22	2	18	961	4	44	8	12
448	3	22	1	19	961	5	44	6	14
470	2	23	3	17	961	7	44	4	16
470	4	23	1	19	987	6	45	5	15
492	2	24	4	16	992	3	46	13	7
492	3	24	2	18	1026	3	48	14	6
492	5	24	1	19	1030	4	47	9	11
515	2	25	5	15	1050	5	48	7	13
515	6	25	1	19	1065	3	50	15	5
540	2	26	6	14	1099	4	50	10	10
540	3	26	3	17	1099	6	50	6	14
540	4	26	2	18	1099	7	50	5	15
540	7	26	1	19	1140	5	52	8	12
562	2	27	7	13	1160	4	53	11	9
584	2	28	8	12	1215	6	55	7	13
584	3	28	4	16	1225	4	56	12	8
584	5	28	2	18	1236	5	56	9	11
609	2	29	9	11	1236	7	56	6	14
609	4	29	3	17	1300	4	59	13	7
634	2	30	10	10	1324	5	60	10	10
634	3	30	5	15	1324	6	60	8	12
634	6	30	2	18	1340	4	62	14	6
652	2	31	11	9	1370	7	62	7	13
670	2	32	12	8	1395	4	65	15	5
684	3	32	6	14	1410	5	64	11	9
684	4	32	4	16	1435	6	65	9	11
684	5	32	3	17	1494	5	68	12	8
684	7	32	2	18	1498	7	68	8	12
692	2	33	13	7	1544	6	70	10	10
717	2	34	14	6	1570	5	72	13	7
734	3	34	7	13	1627	7	74	9	11
740	2	35	15	5	1650	6	75	11	9
756	4	35	5	15	1650	5	76	14	6
756	6	35	3	17	1730	5	80	15	5
774	3	36	8	12	1757	6	80	12	8
774	5	36	4	16	1765	7	80	10	10
824	3	38	9	11	1857	6	85	13	7
824	4	38	6	14	1890	7	86	11	9
824	7	38	3	17	1958	6	90	14	6
872	3	40	10	10	2015	7	92	12	8
872	5	40	5	15	2055	6	95	15	5
872	6	40	4	16	2140	7	98	13	7
892	4	41	7	13	2258	7	104	14	6
918	3	42	11	9	2372	7	110	15	5

can carry at the allowable probability of loss. The next higher number of symmetrical trunks is then ordinarily specified. Thus, Fig. 23 shows the load-trunk curve for the "Half Gain" formula as a broken line representing the loads which may be submitted to the more important arrangements of $gx + y$ trunks for a terminal assignment of $x + y = 20$, g varying from two to seven subgroups, at a probability of loss of $P = .01$. This same figure portrays vividly the relative inefficiency of a straight subgrouped multiple compared with a like graded multiple, and again, the eminent superiority of the complete or full-group multiple over both of these. Table VI shows the same information as Fig. 23, recorded in the more familiar tabular form ready for engineering use.

Summary

We have sketched briefly some of the general principles underlying the furnishing of an adequate and economical telephone exchange service. One of the several practical means of interconnection is through the employment of special trunking arrangements of the type known as graded multiples. The common-sense theory of this plan has been discussed in some detail after which an approximate mathematical formula is presented.

A group of graded multiple tests run in Chicago in 1927 serves to indicate what modifications should be made in this theoretical trunking schedule before it is used for engineering purposes. A typical table of the loads which, on this basis, may properly be submitted to attain a specified grade of service over a wide variety of arrangements and numbers of trunks, is shown as an example of what appear as the most satisfactory graded multiple capacity figures for Bell System practice at the present time.

As a more detailed and accurate knowledge is acquired concerning the behavior of telephone traffic over increasingly complex and non-symmetrical graded trunking arrangements, some slight modification in the conclusions we have reached here may be expected. There is ample opportunity, then, for additional theoretical analysis of the graded multiple problem (as well as of many other multiple problems), an analysis that perhaps will overcome the limitations which have thus far been levied in order that a working result might be obtained.

Appendix I [14]

Mathematical Theory of the Simple Graded Multiple

The mathematical analysis given in this appendix is based on the following assumptions:

1. Constant holding time per call.
2. "Lost calls held."
3. The load submitted by each subgroup of selectors varies about its average value "a" in accordance with the Poisson Law

[14] Prepared by E. C. Molina.

$$\frac{a^x e^{-a}}{\underline{|x}}$$

and is independent of the variations in any other subgroup.

4. " No-holes-in-the-multiple." [15]

The probability that a calling source fails to obtain an idle trunk immediately may be divided into two parts, corresponding to two essentially different sets of circumstances under which a call may be interfered with. Assume, to fix ideas, that the particular source under consideration belongs within subgroup No. 1 in Figure 7(*A*). Failure will occur if in addition to the call originating from this source,

1. At least $x + y$ calls originated by subgroup No. 1 occupy trunks (actually or potentially); or,
2. The number of calls placed on the trunks consists of $x + r$ originated from subgroup No. 1; at least $x + 1$ calls originated from each of s of the other subgroups; and, moreover, that said s other groups have, collectively, placed at least $sx + y - r$ calls. s is a number which may have any value from 1 to $(g - 1)$, inclusive.

This classification of the circumstances under which the call under consideration may be delayed gives, for the desired probability when " holes-in-the-multiple " are not possible, the equation

$$P = P_1 + P_2,$$

where

a. $P_1 = P(x + y, a)$

b. $P_2 = \sum_{r=0}^{y-1} \left(\frac{a^{x+r} e^{-a}}{\underline{|x + r}} \right) F(g - 1, x, y - r)$

c. $P(x + y, a)$ is the Poisson expansion $\sum_{t=x+y}^{\infty} \left(\frac{a^t e^{-a}}{\underline{|t}} \right)$

d. $F(g - 1, x, y - r)$

$$= \sum_{s=1}^{g-1} \binom{g-1}{s} [1 - P(x + 1, a)]^{g-1-s} \mathbf{S} \prod_{t=1}^{s} \left(\frac{a^{x+r_t} e^{-a}}{\underline{|x + r_t}} \right),$$

where **S** indicates that in the product

$$\prod_{t=1}^{s} \left(\frac{a^{x+r_t} e^{-a}}{\underline{|x + r_t}} \right), \; r_1, r_2, r_3 \cdots r_s$$

[15] Imagine that some method of transferring calls from common to individual trunks eliminates the possibility of "holes-in-the-multiple."

are to be given all values such that

$$r_t > 0, \qquad \sum_{t=1}^{s} r_t \nless (y - r).$$

For computing purposes, note that the function F satisfies the finite difference equation

$$\begin{aligned} F(g-1, x, y-r) = {} & P(x+y-r, a) \\ & + \sum_{R=0}^{y-r-1} \left(\frac{a^{x+} e^{-a}}{\lfloor x+R} \right) F(g-2, x, y-r-R) \\ & + [1 - P(x, a)] F(g-2, x, y-r) \end{aligned}$$

This difference equation becomes obvious if one considers the change which takes place in the value of P_2 when the number of subgroups in a graded multiple is increased from $(g - 1)$ to g.

Appendix II

Mathematical Theory of Graded Multiple with Unequal Subgroup Loads

If we deal with a single stage (or simple) graded multiple having g subgroups of " x " individual trunks each, and " y " common trunks; and if to each subgroup, " m " for instance, is submitted a particular load, " a_m," in average simultaneous calls which are originated at random according to the Poisson Distribution Law requirements; and if these calls are moreover of a constant holding time obeying the " lost calls held " assumption; and if they are at all times so arranged on the graded trunks that " no-holes-in-the-multiple " exist; then the proportion of calls not obtaining immediate service over the multiple taken as a whole may be approximated by:

$$P = \frac{a_1 P_1 + a_2 P_2 + \cdots + a_g P_g}{a_1 + a_2 + \cdots + a_g} = \frac{\sum_{m=1}^{g} a_m P_m}{\sum_{m=1}^{g} a_m}$$

where

$$\begin{aligned} P_m = P(x+y, a_m) + \sum_{r=0}^{y-1} \frac{a_m{}^{x+r} e^{-a_m}}{\lfloor x+r} F(g-1, a_1, a_2, \cdots, a_{m-1}, \\ a_{m+1}, \cdots, a_g, x, y-r), \end{aligned}$$

in which:

$$P(x+y, a_m) = \sum_{s=x+y}^{\infty} \frac{a_m{}^s e^{-a_m}}{\lfloor s}$$

and

$$F(g-1, a_1, a_2, \cdots, a_{m-1}, a_{m+1}, \cdots, a_g, x, y-r)$$
$$= P(x+y-r, a_1) + \sum_{R=0}^{y-r-1} \frac{a_1^{x+R}e^{-a_1}}{\underline{/x+R}} F(g-2, a_2, a_3, \cdots, a_{m-1}, a_{m+1}, \cdots, a_g, x, y-r-R)$$
$$+ [1 - P(x, a_1)]F(g-2, a_2, a_3, \cdots, a_{m-1}, a_{m+1}, \cdots, a_g, x, y-r).$$