Effect of Ground Permeability on Ground Return Circuits

By W. HOWARD WISE

The formulas for the self and mutual impedances of ground return circuits are derived without restricting the ground permeability. Curves are given to show the effect of a ground permeability 1.7 on the mutual impedance between two parallel ground return circuits with the wires lying on the ground.

ON account of the irregular and heterogeneous character of the major portion of the earth's surface and the consequent difficulty in choosing a conductivity to be used in a computation of ground return circuit impedance it has heretofore been considered useless to take into consideration the possibility of an earth permeability greater than unity. However, since the permeability may sometimes be known to be appreciably different from unity and it is always desirable to reduce the probable error in a computation and since the inclusion of the permeability in the formulas may sometimes lead to a better agreement between the theory and experiments it seems worth while to provide formulas which include the permeability.

The self impedance of a ground return circuit is

$$Z = z + i2\omega \log \rho''/a + 4\omega(P + iQ),$$

where $z + i2\omega \log \rho''/a$ is the self impedance with a perfectly conducting ground and $4\omega(P + iQ)$ contains the effect of the finite conductivity and permeability of the ground. Carson [1] has derived an infinite integral and series expansions for $P + iQ$ on the basis of unit permeability. The infinite integral derived here is arrived at merely by going through Carson's paper and writing in the permeability wherever Carson has replaced it by unity. The reader will be expected to have a copy of Carson's paper at hand as not all of the steps in his paper will be here reproduced.

Equations (23) and (24) respectively are the new infinite integral formulas for self and mutual impedance. Equations (A) and (C) respectively are the new asymptotic and convergent series formulas for P and Q. The functions m and l occurring in equations (C) are functions of the permeability. Since some of them are defined by series and their computation is consequently rather laborious, enough

[1] John R. Carson, *Bell System Technical Journal*, Oct., 1926.

of them are tabulated for values of μ from 1 to 1.7 to provide for the computation of P and Q for values of r_1 up to 2.

Equation (1)[1] is unchanged but there is a new definition for α

$$\alpha = 4\pi\lambda\mu\omega.$$

Since curl $E = -(\partial/\partial t)\mu H$ equations (2) and (3) have the factor μ added to their left hand sides.

The next change is in the application of the boundary conditions. At the surface of the ground H_x and μH_y must be continuous. The equations to be solved for $F(\tau)$ and $\phi(\tau)$ now become

$$\frac{1}{\mu i\omega}\sqrt{\tau^2 + i\alpha}\, F(\tau) = 2Ie^{-h\tau} + \phi(\tau),$$

$$\frac{1}{i\omega}\tau F(\tau) = 2Ie^{-h\tau} - \phi(\tau),$$

whence

$$F(\tau) = \frac{\mu i\omega e^{-h\tau}}{\sqrt{\tau^2 + i\alpha} + \mu\tau} 4I, \tag{11}$$

$$\phi(\tau) = \frac{\sqrt{\tau^2 + i\alpha} - \mu\tau}{\sqrt{\tau^2 + i\alpha} + \mu\tau} e^{-h\tau} 2I. \tag{12}$$

The new equations (13), (14), (18), (19), (20), (23) and (24) are

$$E_z = -i4\omega I\mu \int_0^\infty \frac{\cos x\tau}{\sqrt{\tau^2 + i\alpha} + \mu\tau} e^{-\tau h + y\sqrt{\tau^2 + i\alpha}} d\tau, \tag{13}$$

$$E_z = -i4\omega I\mu \int_0^\infty \frac{\cos x'\tau}{\sqrt{\tau^2 + i} + \mu\tau} e^{-\tau h' + y'\sqrt{\tau^2 + i}} d\tau, \tag{14}$$

$$E_z = -i4\omega I\mu \int_0^\infty \frac{e^{-(h'+y')\tau}}{\sqrt{\tau^2 + i} + \mu\tau} \cos x'\tau d\tau - i2\omega I \log\frac{\rho''}{\rho'} - \frac{\partial}{\partial z} V, \tag{18}$$

$$zI = -i4\omega I\mu \int_0^\infty \frac{e^{-(h'+y')\tau}}{\sqrt{\tau^2 + i} + \mu\tau} d\tau - i2\omega I \log\frac{\rho''}{a} + \Gamma V, \tag{19}$$

$$\Gamma^2 = (G + i\omega C)\left[z + i2\omega \log\frac{\rho''}{a} + i4\omega\mu \int_0^\infty \frac{e^{-2h'\tau}}{\sqrt{\tau^2 + i} + \mu\tau} d\tau \right], \tag{20}$$

$$R + iX = Z = z + i2\omega \log \frac{\rho''}{a} + i4\omega\mu \int_0^\infty \frac{e^{-2h'\tau}}{\sqrt{\tau^2 + i} + \mu\tau} d\tau \tag{23}$$
$$= z + i2\omega \log \frac{\rho''}{a} + 4\omega(P + iQ),$$

$$Z_{12} = i2\omega \log \frac{\rho''}{\rho'} + i4\omega\mu \int_0^\infty \frac{e^{-(h_1'+h_2')\tau}}{\sqrt{\tau^2 + i} + \mu\tau} \cos x'\tau d\tau \tag{24}$$
$$= i2\omega \log \frac{\rho''}{\rho'} + 4\omega(P + iQ).$$

The principal steps in the derivation of equation (18) are given in Appendix I.

The new definition of $P + iQ$ is

$$P + iQ = i\mu \int_0^\infty \frac{e^{-(h'+y')\tau}}{\sqrt{\tau^2 + i} + \mu\tau} \cos x'\tau d\tau.$$

Replacing i by v^2 and assuming that v is a real quantity this is

$$P + iQ = \mu v^2 R \int_0^\infty \frac{e^{-v(h'+y'+ix')\tau}}{\sqrt{\tau^2 + 1} + \mu\tau} d\tau,$$

where R is used to indicate that the real part is to be taken.

The asymptotic expansion is easiest derived by expanding $1/(\sqrt{\tau^2 + 1} + \mu\tau)$ into an ascending power series in τ and integrating termwise.

$$1/(\sqrt{\tau^2 + 1} + \mu\tau) = 1 - \mu\tau + (\mu^2 - \tfrac{1}{2})\tau^2 - (\mu^3 - \mu)\tau^3$$
$$+ (\mu^4 - \tfrac{3}{2}\mu^2 + \tfrac{3}{8})\tau^4 - (\mu^5 - 2\mu^3 + \mu)\tau^5 + - \cdots,$$

whence, writing $h' + y' + ix' = re^{i\theta}$,

$$P + iQ = \mu v^2 \Big[\frac{\cos\theta}{vr} - \mu \frac{\cos 2\theta}{v^2r^2} + (\mu^2 - \tfrac{1}{2}) \frac{\cos 3\theta}{v^3r^3} 2!$$
$$- (\mu^3 - \mu) \frac{\cos 4\theta}{v^4r^4} 3! + (\mu^4 - \tfrac{3}{2}\mu^2 + \tfrac{3}{8}) \frac{\cos 5\theta}{v^5r^5} 4!$$
$$- \mu(\mu^2 - 1)^2 \frac{\cos 6\theta}{v^6r^6} 5! + \cdots \Big],$$

whence, separating the real and imaginary parts,

$$P = \frac{\mu}{\sqrt{2}} \Big[\frac{\cos\theta}{r} + (2\mu^2 - 1) \frac{\cos 3\theta}{r^3}$$
$$+ 3(12\mu^2 - 8\mu^4 - 3) \frac{\cos 5\theta}{r^5} + \cdots \Big]$$

$$-\frac{\mu^2}{\mu^2-1}\left[\left(\frac{\mu^2-1}{r^2}\right)1!\cos 2\theta-\left(\frac{\mu^2-1}{r^2}\right)^3 5!\cos 6\theta\right.$$
$$\left.+\left(\frac{\mu^2-1}{r^2}\right)^5 9!\cos 10\theta-+\cdots\right], \qquad (A)$$

$$Q=\frac{\mu}{\sqrt{2}}\left[\frac{\cos\theta}{r}-(2\mu^2-1)\frac{\cos 3\theta}{r^3}\right.$$
$$\left.+3(12\mu^2-8\mu^4-3)\frac{\cos 5\theta}{r^5}-\cdots\right]$$
$$+\frac{\mu^2}{\mu^2-1}\left[\left(\frac{\mu^2-1}{r^2}\right)^2 3!\cos 4\theta-\left(\frac{\mu^2-1}{r^2}\right)^4 7!\cos 8\theta\right.$$
$$\left.+\left(\frac{\mu^2-1}{r^2}\right)^6 11!\cos 12\theta-+\cdots\right].$$

It is worth noticing that when r is so large that only the leading terms in P are of importance

$$P=[\mu+(h_1+h_2)\sqrt{2\pi\lambda\omega\mu}]/4\pi\lambda\omega[x^2+(h_1+h_2)^2].$$

At power frequencies $(h_1+h_2)\sqrt{2\pi\lambda\omega\mu}$ is small in comparison with μ.

When $\mu-1$ is small a series in powers of $\mu-1$ is a convenient form of solution. This is readily arrived at by writing

$$\frac{1}{\sqrt{\tau^2+1}+\mu\tau}=\frac{1}{\sqrt{\tau^2+1}+\tau}\left[1-\left(\frac{\epsilon\tau}{\sqrt{\tau^2+1}+\tau}\right)\right.$$
$$\left.+\left(\frac{\epsilon\tau}{\sqrt{\tau^2+1}+\tau}\right)^2-+\cdots\right].$$

The expansion is absolutely convergent for all values of τ if $\epsilon=\mu-1<2$.

$$\begin{aligned}1/(\sqrt{\tau^2+1}+\mu\tau)&=(\sqrt{\tau^2+1}-\tau)-\epsilon\tau(\sqrt{\tau^2+1}-\tau)^2\\&\quad+\epsilon^2\tau^2(\sqrt{\tau^2+1}-\tau)^3-+\cdots\\&=\sqrt{\tau^2+1}[1+\epsilon 2\tau^2+\epsilon^2(4\tau^4+\tau^2)+\epsilon^3(8\tau^6+4\tau^4)\\&\quad+\epsilon^4(16\tau^8+12\tau^6+\tau^4)+\epsilon^5(32\tau^{10}+32\tau^8+6\tau^6)\\&\quad+\epsilon^6(64\tau^{12}+80\tau^{10}+24\tau^8+\tau^6)\\&\quad+\epsilon^7(128\tau^{14}+192\tau^{12}+80\tau^{10}+8\tau^8)+\cdots]\\&\quad-[\tau+\epsilon(2\tau^3+\tau)+\epsilon^2(4\tau^5+3\tau^3)\\&\quad+\epsilon^3(8\tau^7+8\tau^5+\tau^3)+\epsilon^4(16\tau^9+20\tau^7+5\tau^5)\\&\quad+\epsilon^5(32\tau^{11}+48\tau^9+18\tau^7+\tau^5)+\cdots].\end{aligned}$$

Writing $c = v(h' + y' + ix') = vre^{i\theta}$ we have then

$$P + iQ = \mu v^2 R \int_0^\infty \frac{e^{-c\tau}}{\sqrt{\tau^2+1} + \mu\tau} d\tau$$

$$= \mu v^2 R \left[1 + \epsilon 2 \frac{d^2}{dc^2} + \epsilon^2 \left(4 \frac{d^4}{dc^4} + \frac{d^2}{dc^2} \right) + \cdots \right] f(c) \qquad (B)$$

$$- \mu v^2 R \left[\frac{1}{c^2} + \epsilon \left(2 \frac{3!}{c^4} + \frac{1}{c^2} \right) + \epsilon^2 \left(4 \frac{5!}{c^6} + 3 \frac{3!}{c^4} \right) + \cdots \right],$$

where $f(c) = \int_0^\infty \sqrt{\tau^2+1}\, e^{-c\tau} d\tau = \int_0^\infty \cosh^2\phi\, e^{-c \sinh \phi} d\phi$

$$= \frac{K_1(c)}{c} + \frac{c}{3} - \frac{c^3}{3^2 5} + \frac{c^5}{3^2 5^2 7} - + \cdots .^2$$

$\mu v^2 R[f(c) - (1/c^2)] = \mu$ times Carson's $P + iQ$ with $\alpha = \mu 4\pi\lambda\omega$.

The problem is now reduced to the tedious procedure of differentiating $f(c)$ and separating real and imaginary parts twice for each power of ϵ. The chief steps are given in Appendix II. The result is best written in the form

$$P = \frac{\pi}{8} \left[m_0 - \frac{m_4}{2!3!} \left(\frac{r_1}{2} \right)^4 \cos 4\theta + \frac{m_8}{4!5!} \left(\frac{r_1}{2} \right)^8 \cos 8\theta - \frac{m_{12}}{6!7!} \left(\frac{r_1}{2} \right)^{12} \cos 12\theta + - \cdots \right]$$

$$+ \frac{\theta}{2} \left[\frac{m_2}{1!2!} \left(\frac{r_1}{2} \right)^2 \sin 2\theta - \frac{m_6}{3!4!} \left(\frac{r_1}{2} \right)^6 \sin 6\theta + \frac{m_{10}}{5!6!} \left(\frac{r_1}{2} \right)^{10} \sin 10\theta - + \cdots \right]$$

$$- \frac{1}{\sqrt{2}} \left[m_1 \frac{r_1 \cos \theta}{3} - m_3 \frac{r_1^3 \cos 3\theta}{3^2 5} - m_5 \frac{r_1^5 \cos 5\theta}{3^2 5^2 7} + m_7 \frac{r_1^7 \cos 7\theta}{3^2 5^2 7^2 9} + - \cdots \right]$$

$$+ \frac{r_1^2 \cos 2\theta}{2^3 1!2!} \left(l_2 + m_2 \log \frac{2}{r_1} \right) - \frac{r_1^6 \cos 6\theta}{2^7 3!4!} \left(l_6 + m_6 \log \frac{2}{r_1} \right) + \frac{r_1^{10} \cos 10\theta}{2^{11} 5!6!} \left(l_{10} + m_{10} \log \frac{2}{r_1} \right) - + \cdots, \qquad (C)$$

$$Q = -\frac{\pi}{8} \left[\frac{m_2}{1!2!} \left(\frac{r_1}{2} \right)^2 \cos 2\theta - \frac{m_6}{3!4!} \left(\frac{r_1}{2} \right)^6 \cos 6\theta + \frac{m_{10}}{5!6!} \left(\frac{r_1}{2} \right)^{10} \cos 10\theta - + \cdots \right]$$

[2] See Jahnke & Emde, "Funktionentafeln," pages 171 and 93.

$$-\frac{\theta}{2}\left[\frac{m_4}{2!3!}\left(\frac{r_1}{2}\right)^4\sin 4\theta - \frac{m_8}{4!5!}\left(\frac{r_1}{2}\right)^8\sin 8\theta + \frac{m_{12}}{6!7!}\left(\frac{r_1}{2}\right)^{12}\sin 12\theta - + \cdots\right]$$

$$+\frac{1}{\sqrt{2}}\left[m_1\frac{r_1\cos\theta}{3} + m_3\frac{r_1^3\cos 3\theta}{3^2 5} - m_5\frac{r_1^5\cos 5\theta}{3^2 5^2 7} - m_7\frac{r_1^7\cos 7\theta}{3^2 5^2 7^2 9} + + \cdots\right]$$

$$+\frac{1}{2\cdot 0!1!}\left(l_0 + m_0\log\frac{2}{r_1}\right) - \frac{r_1^4\cos 4\theta}{2^5 2!3!}\left(l_4 + m_4\log\frac{2}{r_1}\right) + \frac{r_1^8\cos 8\theta}{2^9 4!5!}\left(l_8 + m_8\log\frac{2}{r_1}\right) - + \cdots,$$

where $r_1 = r/\sqrt{\mu} = \sqrt{4\pi\lambda\omega[x^2 + (h+y)^2]}$ = Carson's r and the permeability is contained in the functions m_x and l_x.

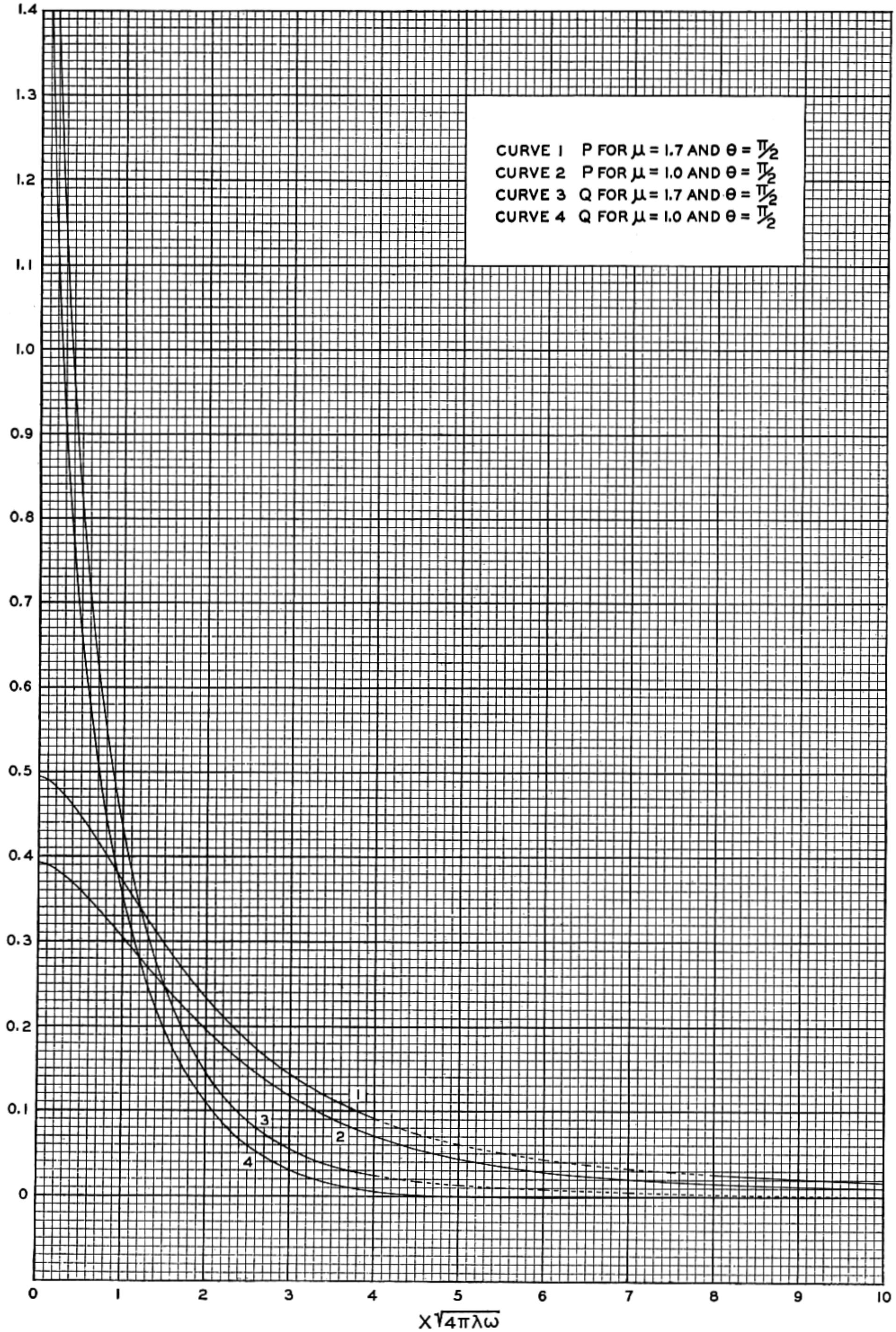
The definitions of the m_x and l_x will be found in Appendix II. The table of numerical values should suffice for most needs.

TABLE 1

μ	$-\frac{1}{2}l_0$	l_2	l_4	l_6	l_8	l_{10}
1	0.03861	0.67278	1.08945	1.38112	1.60612	1.78945
1.1	0.04619	0.70382	1.23834	1.71141	2.1758	2.6549
1.2	0.05264	0.72954	1.38429	2.07062	2.8568	3.7890
1.3	0.05808	0.75059	1.52655	2.45663	3.6558	5.2371
1.4	0.06261	0.76756	1.66456	2.86745	4.5785	7.0466
1.5	0.06631	0.78095	1.79799	3.30122	5.6305	9.2669
1.6	0.06923	0.79121	1.92660	3.75623	6.8167	11.9492
1.7	0.07159	0.79871	2.05026	4.23089	8.1417	15.1467

μ	m_0	m_1	m_2	m_3	m_4	m_5
1	1	1	1	1	1	1
1.1	1.04762	1.06700	1.09751	1.13529	1.17851	1.22625
1.2	1.09091	1.12837	1.19008	1.26928	1.36318	1.47078
1.3	1.13043	1.18469	1.27788	1.40147	1.55291	1.73236
1.4	1.16667	1.23643	1.36111	1.53153	1.74676	2.00996
1.5	1.20000	1.28403	1.44000	1.65922	1.94400	2.30254
1.6	1.23077	1.32793	1.51479	1.78442	2.14402	2.60929
1.7	1.25926	1.36845	1.58573	1.90706	2.34630	2.92942

μ	m_6	m_7	m_8	m_9	m_{10}	m_{11}
1	1	1	1	1	1	1
1.1	1.27799	1.3334	1.3926	1.4554	1.5217	1.5918
1.2	1.59192	1.7270	1.8767	2.0420	2.2241	2.4244
1.3	1.94162	2.1834	2.4613	2.7798	3.1441	3.5602
1.4	2.32690	2.7055	3.1556	3.6895	4.3217	5.0696
1.5	2.74752	3.2957	3.9685	4.7925	5.8003	7.0324
1.6	3.20326	3.9566	4.9089	6.1108	7.6264	9.5370
1.7	3.69388	4.6903	5.9856	7.6674	9.8497	12.6816


CURVE 1 P FOR μ = 1.7 AND θ = π/2
CURVE 2 P FOR μ = 1.0 AND θ = π/2
CURVE 3 Q FOR μ = 1.7 AND θ = π/2
CURVE 4 Q FOR μ = 1.0 AND θ = π/2
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
0
1
2
3
4
5
6
7
8
9
10
X√4πλω

The curves show the effect of $\mu = 1.7$ on the mutual impedance between two parrallel ground return circuits with the wires lying on the ground. The dashed portions of the curves were not computed.

Appendix I

Equations (4), (7) and (17) substituted into (16) give

$$E_z(x, y) = E_z(x, 0) - i\omega \int_0^y \left[\frac{2I(h - y)}{x^2 + (h - y)^2} + \int_0^\infty \phi(\tau) \cos x\tau \cdot e^{-y\tau} d\tau \right] dy - \frac{\partial V}{\partial z}$$

$$= E_z(x, 0) + i\omega \int_0^\infty \phi(\tau) \cos x\tau (e^{-y\tau} - 1) \frac{d\tau}{\tau} + i\omega I \log \frac{x^2 + (h - y)^2}{x^2 + h^2} - \frac{\partial V}{\partial z}$$

$$= - i4\omega I \int_0^\infty \frac{\mu e^{-h\tau} \cos x\tau}{\sqrt{\tau^2 + i\alpha} + \mu\tau} d\tau + i2\omega I \int_0^\infty \frac{\sqrt{\tau^2 + i\alpha} - \mu\tau}{\sqrt{\tau^2 + i\alpha} + \mu\tau} (e^{-(h+y)\tau} - e^{-h\tau}) \frac{\cos x\tau}{\tau} d\tau + i\omega I \log \frac{x^2 + (h - y)^2}{x^2 + h^2} - \frac{\partial V}{\partial z}$$

$$= - i4\omega I \int_0^\infty \frac{\mu e^{-(h+y)\tau}}{\sqrt{\tau^2 + i\alpha} + \mu\tau} \cos x\tau d\tau + i2\omega I \int_0^\infty (e^{-(h+y)\tau} - e^{-h\tau}) \frac{\cos x\tau}{\tau} d\tau + i\omega I \log \frac{x^2 + (h - y)^2}{x^2 + h^2} - \frac{\partial V}{\partial z}$$

$$= - i4\omega I \int_0^\infty \frac{\mu e^{-(h+y)\tau}}{\sqrt{\tau^2 + i\alpha} + \mu\tau} \cos x\tau d\tau + i\omega I \log \frac{x^2 + (h - y)^2}{x^2 + (h + y)^2} - \frac{\partial V}{\partial z}.$$

Appendix II

The succeeding analysis has been considerably shortened by writing

$$\zeta_{nm} = \omega_n - \omega_{2n} + \frac{1}{2n - 1} + \omega_{2(n-1-m)},$$

where

$$\omega_n = \sum_1^n \frac{1}{s}, \qquad \omega_0 = 0.$$

$$f(c) = \frac{1}{c^2} + \left[\frac{c}{3} - \frac{c^3}{3^2 5} + \frac{c^5}{3^2 5^2 7} - \frac{c^7}{3^2 5^2 7^2 9} + - \cdots \right]$$

$$+ \frac{1}{2} \left[\zeta_{10} - \zeta_{20} \frac{1}{1!2!} \left(\frac{c}{2} \right)^2 + \zeta_{30} \frac{1}{2!3!} \left(\frac{c}{2} \right)^4 - \zeta_{40} \frac{1}{3!4!} \left(\frac{c}{2} \right)^6 + - \cdots \right]$$

$$+ \frac{1}{2} \left[1 - \frac{1}{1!2!} \left(\frac{c}{2} \right)^2 + \frac{1}{2!3!} \left(\frac{c}{2} \right)^4 - \frac{1}{3!4!} \left(\frac{c}{2} \right)^6 + - \cdots \right] \log \frac{2}{\gamma c}$$

$$f^{(n)}(c) = \frac{(n+1)!}{c^{n+2}} + \frac{1}{2} \frac{(n-1)!}{c^n} - \frac{1}{2 \cdot 4} \frac{(n-3)!}{c^{n-2}} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} \frac{(n-5)!}{c^{n-4}} - + \cdots \frac{1 \cdot 3 \cdot 5 \cdots (n-3)}{2 \cdot 4 \cdot 6 \cdots n} \frac{1!}{c^2}$$

$$+ (-2)^{n/2} \left[\frac{(n/2)!\, c}{3 \cdot 5 \cdot 7 \cdots (n+3)} - \frac{(1 + n/2)!\, c^3}{1! 3^2 5 \cdot 7 \cdots (n+5)} + \frac{(2 + n/2)!\, c^5}{2! 3^2 5^2 7 \cdot 9 \cdots (n+7)} - + \cdots \right]$$

$$- \left(-\frac{1}{2} \right)^{(n/2)+1} \left[\zeta_{[(n/2)+1]n/2} \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{0! \left(\frac{n}{2} + 1 \right)!} - \zeta_{[(n/2)+2]n/2} \frac{3 \cdot 5 \cdot 7 \cdots (n+1)}{1! \left(\frac{n}{2} + 2 \right)!} \left(\frac{c}{2} \right)^2 + - \cdots \right]$$

$$- \left(-\frac{1}{2} \right)^{(n/2)+1} \left[\frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{0! \left(\frac{n}{2} + 1 \right)!} - \frac{3 \cdot 5 \cdot 7 \cdots (n+1)}{1! \left(\frac{n}{2} + 2 \right)!} \times \left(\frac{c}{2} \right)^2 + \frac{5 \cdot 7 \cdot 9 \cdots (n+3)}{2! \left(\frac{n}{2} + 3 \right)!} \left(\frac{c}{2} \right)^4 - + \cdots \right] \log \frac{2}{\gamma c},$$

the n being an even integer.

The inverse powers of c all cancel out, in equation (B), and there remains

$$P + iQ = \mu v^2 R \left\{ \frac{c}{3} q_1 - \frac{c^3}{3^2 5} q_3 + \frac{c^5}{3^2 5^2 7} q_5 - \frac{c^7}{3^2 5^2 7^2 9} q_7 + - \cdots \right.$$

$$+ \frac{1}{2} \left[\zeta_{10} \frac{1}{0!1!} p_0 - \zeta_{20} \frac{(c/2)^2}{1!2!} p_2 + \zeta_{30} \frac{(c/2)^4}{2!3!} p_4 \right.$$

$$\left. - \zeta_{40} \frac{(c/2)^6}{3!4!} p_6 + - \cdots \right] \quad (D)$$

$$+ \frac{1}{2} \log \frac{2}{\gamma c} \left[q_0 - \frac{(c/2)^2}{1!2!} q_2 + \frac{(c/2)^4}{2!3!} q_4 \right.$$

$$\left. \left. - \frac{(c/2)^6}{3!4!} q_6 + - \cdots \right] \right\},$$

where

$$\zeta_{10} p_0 = \zeta_{10} - \epsilon 2 \zeta_{21} \frac{1}{2 \cdot 2!} + \epsilon^2 \left(4 \zeta_{32} \frac{1 \cdot 3}{2^2 \cdot 3!} - \zeta_{21} \frac{1}{2 \cdot 2!} \right)$$

$$- \epsilon^3 \left(8 \zeta_{43} \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 4!} - 4 \zeta_{32} \frac{1 \cdot 3}{2^2 \cdot 3!} \right)$$

$$+ \epsilon^4 \left(16 \zeta_{54} \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4 \cdot 5!} - 12 \zeta_{43} \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 4!} + \zeta_{32} \frac{1 \cdot 3}{2^2 \cdot 3!} \right) - + \cdots,$$

$$\zeta_{20} \frac{1}{2!} p_2 = \zeta_{20} \frac{1}{2!} - \epsilon 2 \zeta_{31} \frac{3}{2 \cdot 3!} + \epsilon^2 \left(4 \zeta_{42} \frac{3 \cdot 5}{2^2 \cdot 4!} - \zeta_{31} \frac{3}{2 \cdot 3!} \right)$$

$$- \epsilon^3 \left(8 \zeta_{53} \frac{3 \cdot 5 \cdot 7}{2^3 5!} - 4 \zeta_{42} \frac{3 \cdot 5}{2^2 \cdot 4!} \right) + \cdots,$$

$$\zeta_{30} \frac{1}{3!} p_4 = \zeta_{30} \frac{1}{3!} - \epsilon 2 \zeta_{41} \frac{5}{2 \cdot 4!} + \epsilon^2 \left(4 \zeta_{52} \frac{5 \cdot 7}{2^2 \cdot 5!} - \zeta_{41} \frac{5}{2 \cdot 4!} \right)$$

$$- \epsilon^3 \left(8 \zeta_{63} \frac{5 \cdot 7 \cdot 9}{2^3 \cdot 6!} - 4 \zeta_{52} \frac{5 \cdot 7}{2^2 \cdot 5!} \right) + \cdots,$$

. . . .

$$q_0 = 1 - \epsilon 2 \frac{1}{2 \cdot 2!} + \epsilon^2 \left(4 \frac{1 \cdot 3}{2^2 \cdot 3!} - \frac{1}{2 \cdot 2!} \right)$$

$$- \epsilon^3 \left(8 \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 4!} - 4 \frac{1 \cdot 3}{2^2 \cdot 3!} \right) + - \cdots,$$

$$\frac{1}{2!} q_2 = \frac{1}{2!} - \epsilon 2 \frac{3}{2 \cdot 3!} + \epsilon^2 \left(4 \frac{3 \cdot 5}{2^2 \cdot 4!} - \frac{3}{2 \cdot 3!} \right)$$

$$- \epsilon^3 \left(8 \frac{3 \cdot 5 \cdot 7}{2^3 \cdot 5!} - 4 \frac{3 \cdot 5}{2^2 \cdot 4!} \right) + - \cdots,$$

$$\frac{1}{3!}q_4 = \frac{1}{3!} - \epsilon 2\frac{5}{2\cdot 4!} + \epsilon^2\left(4\frac{5\cdot 7}{2^2\cdot 5!} - \frac{5}{2\cdot 4!}\right) - \epsilon^3\left(8\frac{5\cdot 7\cdot 9}{2^3\cdot 6!} - 4\frac{5\cdot 7}{2^2\cdot 5!}\right) + - \cdots,$$

. . . .

$$q_1 = 1 - \epsilon 2\frac{2^1\cdot 1!}{5} + \epsilon^2\left(4\frac{2^2\cdot 2!}{5\cdot 7} - \frac{2^1\cdot 1!}{5}\right) - \epsilon^3\left(8\frac{2^3\cdot 3!}{5\cdot 7\cdot 9} - 4\frac{2^2\cdot 2!}{5\cdot 7}\right) + - \cdots,$$

$$1!q_3 = 1! - \epsilon 2\frac{2^1\cdot 2!}{7} + \epsilon^2\left(4\frac{2^2\cdot 3!}{7\cdot 9} - \frac{2^1\cdot 2!}{7}\right) - \epsilon^3\left(8\frac{2^3\cdot 4!}{7\cdot 9\cdot 11} - 4\frac{2^2\cdot 3!}{7\cdot 9}\right) + - \cdots,$$

$$2!q_5 = 2! - \epsilon 2\frac{2^1\cdot 3!}{9} + \epsilon^2\left(4\frac{2^2\cdot 4!}{9\cdot 11} - \frac{2^1\cdot 3!}{9}\right) - \epsilon^3\left(8\frac{2^3\cdot 5!}{9\cdot 11\cdot 13} - 4\frac{2^2\cdot 4!}{9\cdot 11}\right) + - \cdots.$$

. . . .

The way in which the succeeding terms of each series are to be formed will be made clear by comparing the numbers just preceding the ζs in the p series with the numbers in the expansion of $1/(\sqrt{\tau^2 + 1} + \mu\tau)$ into a power series in ϵ.

The series converge if $\epsilon = \mu - 1 < 2$.

The q series are all represented by the single formula

$$\begin{aligned} q_x &= \left(\frac{2}{1+\mu}\right)^{1+(x/2)} F\left(\frac{x}{2}, 1 - \frac{x}{2}, 2 + \frac{x}{2}, -\frac{\epsilon}{2}\right) \\ &= \left(\frac{2}{1+\mu}\right)^{1+(x/2)} \left[1 + \frac{x}{x+4}\left(\frac{\epsilon}{2}\right)\frac{x-2}{2} \right. \\ &\quad + \frac{x(x+2)}{(x+4)(x+6)}\left(\frac{\epsilon}{2}\right)^2 \frac{(x-2)(x-4)}{2\cdot 4} \\ &\quad + \frac{x(x+2)}{(x+6)(x+8)}\left(\frac{\epsilon}{2}\right)^3 \frac{(x-2)(x-4)(x-6)}{2\cdot 4\cdot 6} \\ &\quad \left. + \frac{x(x+2)}{(x+8)(x+10)}\left(\frac{\epsilon}{2}\right)^4 \frac{(x-2)(x-4)(x-6)(x-8)}{2\cdot 4\cdot 6\cdot 8} + \cdots \right], \end{aligned}$$

good for all values of ϵ if x is even; good for $\epsilon \leq 2$ if x is odd. Other series are available for odd x and $2 < \epsilon$ but there is little likelihood of their being needed.

The p series are all comprised in the single formula

$$\zeta_{(1+x/2)0}p_x = \left(1+\frac{x}{2}\right)!\left\{\zeta_{(1+x/2)0}\frac{1}{\left(1+\frac{x}{2}\right)!} - \epsilon 2\zeta_{(2+x/2)1}\frac{x+1}{2\left(2+\frac{x}{2}\right)!} + \epsilon^2\left(4\zeta_{(3+x/2)2}\frac{(x+1)(x+3)}{2^2\left(3+\frac{x}{2}\right)!} - \zeta_{(2+x/2)1}\frac{x+1}{2\left(2+\frac{x}{2}\right)!}\right) - \cdots\right\}.$$

Since $\zeta_{nm} = \zeta_{n(n-1)} + \omega_{2(n-1-m)}$ we can write this

$$\zeta_{(1+x/2)0}p_x = \omega_x q_x + \delta_x,$$

where

$$\delta_x = \left(1+\frac{x}{2}\right)!\left\{\zeta_{(1+x/2)x/2}\frac{1}{\left(1+\frac{x}{2}\right)!} - \epsilon 2\zeta_{(2+x/2)(1+x/2)}\frac{x+1}{2\left(2+\frac{x}{2}\right)!} + \epsilon^2\left(4\zeta_{(3+x/2)(2+x/2)}\frac{(x+1)(x+3)}{2^2\left(3+\frac{x}{2}\right)!} - \zeta_{(2+x/2)(1+x/2)}\frac{x+1}{2\left(2+\frac{x}{2}\right)!}\right) - \cdots\right\}.$$

$$\delta_0 = \frac{2\mu}{\mu^2-1}\log\frac{1+\mu}{2},$$

$$\delta_2 = \left(\frac{2}{1+\mu}\right)^2\left(\frac{1}{2}+\frac{1}{\mu-1}-\frac{2\mu}{(\mu-1)^2}\log\frac{1+\mu}{2}\right),$$

$$\delta_x = \frac{1}{\mu^2-1}\cdot\frac{x+2}{x-1}\left(\zeta_{x/2(x/2-1)} - \delta_{x-2}\right) \text{ for } 4 \leq x.$$

By separating the real and imaginary parts one gets from equation (D)

$$\begin{aligned}P = {} & \frac{\pi}{8}\mu\left[q_0 - \frac{1}{2!3!}\left(\frac{r}{2}\right)^4\cos 4\theta\cdot q_4 + \frac{1}{4!5!}\left(\frac{r}{2}\right)^8\cos 8\theta\cdot q_8 - + \cdots\right] \\ & + \frac{\theta}{2}\mu\left[\frac{1}{1!2!}\left(\frac{r}{2}\right)^2\sin 2\theta\cdot q_2 - \frac{1}{3!4!}\left(\frac{r}{2}\right)^6\sin 6\theta\cdot q_6 + - \cdots\right] \\ & - \frac{\mu}{\sqrt{2}}\left[\frac{r\cos\theta}{3}q_1 - \frac{r^3\cos 3\theta}{3^25}q_3 - \frac{r^5\cos 5\theta}{3^25^27}q_5 + + - - \cdots\right] \\ & + \frac{1}{2}\frac{\mu}{1!2!}\left(\frac{r}{2}\right)^2\cos 2\theta\left(\zeta_{20}p_2 + q_2\log\frac{2}{\gamma r}\right) \\ & - \frac{1}{2}\frac{\mu}{3!4!}\left(\frac{r}{2}\right)^6\cos 6\theta\left(\zeta_{40}p_6 + q_6\log\frac{2}{\gamma r}\right) \\ & + \frac{1}{2}\frac{\mu}{5!6!}\left(\frac{r}{2}\right)^{10}\cos 10\theta\left(\zeta_{60}p_{10} + q_{10}\log\frac{2}{\gamma r}\right) - + \cdots,\end{aligned}$$

$$
\begin{aligned}
Q = & -\frac{\pi}{8}\mu\left[\frac{1}{1!2!}\left(\frac{r}{2}\right)^2 \cos 2\theta\cdot q_2 - \frac{1}{3!4!}\left(\frac{r}{2}\right)^6 \cos 6\theta\cdot q_6 + - \cdots\right] \\
& - \frac{\theta}{2}\mu\left[\frac{1}{2!3!}\left(\frac{r}{2}\right)^4 \sin 4\theta\cdot q_4 - \frac{1}{4!5!}\left(\frac{r}{2}\right)^8 \sin 8\theta\cdot q_8 + - \cdots\right] \\
& + \frac{\mu}{\sqrt{2}}\left[\frac{r\cos\theta}{3} q_1 + \frac{r^3 \cos 3\theta}{3^2 5} q_3 - \frac{r^5\cos 5\theta}{3^2 5^2 7} q_5 - + + \cdots\right] \\
& + \frac{1}{2}\frac{\mu}{0!1!}\left(\zeta_{10} p_0 + q_0 \log\frac{2}{\gamma r}\right) \\
& - \frac{1}{2}\frac{\mu}{2!3!}\left(\frac{r}{2}\right)^4 \cos 4\theta\left(\zeta_{30} p_4 + q_4 \log\frac{2}{\gamma r}\right) \\
& + \frac{1}{2}\frac{\mu}{4!5!}\left(\frac{r}{2}\right)^8 \cos 8\theta\left(\zeta_{50} p_8 + q_8 \log\frac{2}{\gamma r}\right) - + \cdots .
\end{aligned}
$$

Equations (C) are now got by writing $r = r_1\sqrt{\mu}$, $m_x = \mu^{1+(x/2)} q_x$ and

$$
\begin{aligned}
l_x &= \mu^{1+(x/2)}(\zeta_{[1+(x/2)]0} p_x - q_x \log \gamma\sqrt{\mu}). \\
l_x &= m_x(\omega_x - \log_e \gamma\sqrt{\mu}) + \mu^{1+(x/2)}\delta_x
\end{aligned}
$$

$$\log_e \gamma = 0.5772157.$$