Transients in Grounded Wires Lying on the Earth's Surface *

By JOHN RIORDAN

Voltages during transient conditions in a grounded wire lying on the earth's surface due to current in a second grounded wire also on the earth's surface are formulated for types of transient currents ordinarily obtained in a.-c. and d.-c. circuits. The fundamental formula is for voltage due to a unit step current, that is, a current zero for time less than zero, and unity for time greater than zero; curves are given for the function determining this voltage for a wide range of values of its two parameters. The formulas for other types of currents are not well adapted for numerical computation, which should be more conveniently carried out by numerical integration using the above curves.

I

A FORMULA for the mutual impedance of grounded wires lying on the earth's surface has recently been published by R. M. Foster.[1] The object of the present paper is to derive formulas for the voltages during transient conditions in one such grounded wire due to current in a second for types of transient currents ordinarily obtained in a.-c. and d.-c. circuits, and particularly for the voltage due to unit step current, zero for time less than zero, unity for time greater than zero.

The voltage due to unit step current is expressed in closed form for straight parallel wires; closed form expressions have not been obtained for straight parallel wires for the exponential forms of current for a.-c. and d.-c. transients. While the integrals might be evaluated numerically, or transformed to asymptotic expressions, it appears more desirable in practical calculation to use the curves given for the unit step voltage directly; a single integration is necessary to find the voltage for current of arbitrary wave form, from the unit step result.

The fundamental physical assumptions upon which the steady-state formula is based are as follows: The surface of the earth is assumed flat, the earth semi-infinite in extent, of uniform conductivity λ, unit

* A brief report of the results in this paper was given at the Summer Convention of the American Institute of Electrical Engineers, Toronto, Ontario, Canada, June 23–27, 1930, in Discussion of "Mutual Impedances of Ground Return Circuits—Some Experimental Studies," by A. E. Bowen and C. L. Gilkeson; *A. I. E. E. Trans.*, Oct. 1930.

[1] R. M. Foster: "Mutual Impedances of Grounded Circuits" (Abstract), *Bulletin of the American Mathematical Society*, May, 1930, pp. 367–368; "Mutual Impedance of Grounded Wires Lying on the Surface of the Earth," *Bell System Technical Journal*, July, 1931.

permeability and negligible dielectric constant. The air above the earth is of zero conductivity, unit permeability, and negligible dielectric constant. Because of the assumption of negligible dielectric constant, the formulas for voltages during transient conditions do not hold strictly for small values of the time, that is, during the initial stages of the transient. The wires are of negligible diameter, lying on the surface of the earth, and insulated from it except at the ends, where there is point contact.

In using the steady-state solution as the basis of transient solutions, the Heaviside operational calculus is employed after replacing $i\omega$, where $\omega = 2\pi f$ is the radian frequency and $i = \sqrt{-1}$, by $p = d/dt$, the time differentiator, since $(d^n/dt^n)(\exp i\omega t) = (i\omega)^n \exp i\omega t$, where n is integral.

II

The mutual impedance of grounded wires lying on the surface of the earth and insulated from it except at the ends is given by the following formula:[2]

$$Z_{12} = \frac{1}{2\pi\lambda}\int\int\left\{\frac{d^2}{dSds}\left(\frac{1}{r}\right) + \frac{\cos\epsilon}{r^3}[1 - (1 + \gamma r)e^{\gamma r}]\right\}dSds.$$

The integration is extended over the two wires S and s, having arbitrary paths, r and ϵ are the distance and angle, respectively, between differential elements dS and ds, and $\gamma = (4\pi\lambda i\omega)^{1/2}$; λ is the ground conductivity and $\omega = 2\pi f$ is the radian frequency.

Replacing $i\omega$ by $p = d/dt$ in γ, the resulting forms to be evaluated are $\exp(-\alpha\sqrt{p})$ and $\sqrt{p}\exp(-\alpha\sqrt{p})$ where $\alpha = r\sqrt{4\pi\lambda}$. The first of these is known and, following Heaviside,[3] may be developed as follows.

Expressing the exponential in series form:

$$\exp(-\alpha\sqrt{p}) = 1 - \alpha\sqrt{p} + \frac{\alpha^2 p}{2!} - \frac{\alpha^3 p\sqrt{p}}{3!} + \cdots.$$

Integral powers of p are neglected, since (omitting the discontinuity at $t = 0$) the operand is unity and the derivative of a constant is zero. Then:

$$\exp(-\alpha\sqrt{p}) = 1 - \alpha\sqrt{p}\left[1 + \frac{\alpha^2 p}{3!} + \frac{\alpha^4 p^2}{5!} + \cdots\right].$$

The bracketed terms may now be assumed to operate on $\sqrt{p} = (\pi t)^{-1/2}$

[2] Foster, loc. cit.
[3] Heaviside: "Electromagnetic Theory," Vol. II, pp. 49–51, equations (4) and (12).

and, if p^n is replaced by d^n/dt^n,

$$\begin{aligned}\exp(-\alpha\sqrt{p}) &= 1 - \left[1 + \frac{\alpha^2}{3!}\frac{d}{dt} + \frac{\alpha^4}{5!}\frac{d^2}{dt^2} + \cdots\right]\frac{\alpha}{\sqrt{\pi t}} \\ &= 1 - \left[1 - \frac{1}{3x1!}\left(\frac{\alpha^2}{4t}\right) + \frac{1}{5x2!}\left(\frac{\alpha^2}{4t}\right)^2 - \cdots\right]\frac{\alpha}{\sqrt{\pi t}} \\ &= 1 - \operatorname{erf}\frac{\alpha}{2\sqrt{t}},\end{aligned}$$

since the term in brackets with its accompanying multiplier is the absolutely convergent expansion of the error function (erf);

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-z^2)dz.$$

The result may also be established either by use of an integral equation [4] or the Fourier integral; it is given as pair 803, Table I, in tables published by G. A. Campbell.[5] In the present use of the tables, for unit step current, the mate of $F(p)/p$, where $F(p)$ is a function of p to be evaluated, is taken since the unit step function is expressed by p^{-1} (pair 415).

The second operational form required may be derived from the first by differentiating with respect to α, since $(d/d\alpha)F(p) = (d/d\alpha)f(t)$ where $F(p)$ and $f(t)$ are corresponding functions of p and t. Thus,

$$\alpha\sqrt{p}\exp(-\alpha\sqrt{p}) = \frac{\alpha}{\sqrt{\pi t}}\exp\left(-\frac{\alpha^2}{4t}\right),$$

since

$$\frac{d}{dt}\operatorname{erf}[\psi(t)] = \frac{2}{\sqrt{\pi}}\psi'(t)\exp\left\{-[\psi(t)]^2\right\}.$$

The unit step voltage may now be expressed, by substitution of these results, by the following formula:

$$V_{12}(t) = \frac{1}{2\pi\lambda}\int\int\left\{\frac{d^2}{dSds}\left(\frac{1}{r}\right) + \frac{\cos\epsilon}{r^3}\left[\operatorname{erf}\left(r\sqrt{\frac{\pi\lambda}{t}}\right) - 2r\sqrt{\frac{\lambda}{t}}\exp\left(-\frac{\pi\lambda r^2}{t}\right)\right]\right\}dSds. \quad (1)$$

In equation (1), as in the steady-state formula from which it is derived, the wires are unrestricted in path or length on the surface of

[4] J. R. Carson: "Electric Circuit Theory and The Operational Calculus," McGraw Hill Co., 1926, p. 19, eq. 29.

[5] "The Practical Application of the Fourier Integral," *Bell System Technical Journal*, October, 1928.

the earth. The formula for straight parallel wires, wire S extending along the z axis from $-a$ to $+a$, and wire s from z_1 to z_2 at distance x from it, is obtained by double integration between these limits with $r^2 = x^2 + (S - s)^2$, $\cos \epsilon = 1$.

The result of integrating once, with respect to S, is:

$$V_{12}(t) = \frac{1}{2\pi\lambda}\int \left\{ \frac{d}{ds}\left[\frac{1}{\sqrt{x^2 + (a - s)^2}} - \frac{1}{\sqrt{x^2 + (\alpha + s)^2}} \right] + \phi(s + a) - \phi(s - a) \right\} ds, \quad (2)$$

where

$$\phi(u) = \frac{u}{x^2\sqrt{(x^2 + u^2)}} \operatorname{erf}\left(\sqrt{x^2 + u^2}\sqrt{\frac{\pi\lambda}{t}} \right) - \frac{1}{x^2}\exp\left(-\frac{\pi\lambda x^2}{t} \right) \operatorname{erf}\left(u\sqrt{\frac{\pi\lambda}{t}} \right),$$

where u is to be replaced by $s + a$ and $s - a$ in equation (2).

Equation (2) is checked as follows. In the first term substitute limits after removing differentiation and integration with respect to S, which cancel each other. In the second term integrate by parts:

$$\int \frac{1}{[x^2 + (S - s)^2]^{3/2}} \operatorname{erf}\sqrt{\frac{\pi\lambda}{t}[x^2 + (S - s)^2]}\, dS$$
$$= \frac{S - s}{x^2\sqrt{x^2 + (S - s)^2}} \operatorname{erf}\sqrt{\frac{\pi\lambda}{t}[x^2 + (S - s)^2]} - 2\sqrt{\frac{\lambda}{t}}\int \frac{(S - s)^2}{x^2[x^2 + (S - s)^2]} \exp\left\{ -\frac{\pi\lambda}{t}[x^2 + (S - s)^2] \right\} dS.$$

The integral coming from this operation combines with the remaining term to give:

$$-2\sqrt{\frac{\lambda}{t}}\int \frac{1}{x^2}\exp\left\{ -\frac{\pi\lambda}{t}[x^2 + (S - s)^2] \right\} dS,$$

which can be simplified in terms of the error function to the form in equation (2).

Integration from z_1 to z_2 gives the result:

$$V_{12}(t) = \frac{1}{2\pi\lambda x}[\psi(z_2 + a) - \psi(z_2 - a) - \psi(z_1 + a) + \psi(z_1 - a)], \quad (3)$$

where

$$\psi(u) = -\frac{x}{\sqrt{x^2 + u^2}} + \frac{\sqrt{x^2 + u^2}}{x} \operatorname{erf}\left(\sqrt{x^2 + u^2}\sqrt{\frac{\pi\lambda}{t}} \right) - \frac{u}{x}\exp\left(-\frac{\pi\lambda x^2}{t} \right) \operatorname{erf}\left(u\sqrt{\frac{\pi\lambda}{t}} \right).$$

As before, u is to be replaced in the equation by the functional arguments, which are the four sums of the z-coordinates of position. The factor x in $\psi(u)$ is introduced to make it a function of two parameters, ux^{-1} and $\pi\lambda x^2 t^{-1}$; the result of integration is $x^{-1}\psi(u)$. The result has the dimensions of abohms when all quantities are in electromagnetic c.g.s. units.

To check equation (3) notice that the integration of the first term of equation (2) is effected by removal of differentiation and integration signs, and substitution of limits; its contribution is identical with the d.-c. mutual resistance.[6] The integration of $\phi(u)$ may be effected by integrating the first term by parts and employing the indefinite integral:

$$\int \operatorname{erf}(ax)dx = x \operatorname{erf}(ax) + \frac{1}{a\sqrt{\pi}} \exp(-a^2x^2) + \text{const.}$$

The result is checked by differentiating, that is, by the relation:

$$\frac{d}{du}\left[x^{-1}\psi(u) + \frac{1}{\sqrt{x^2+u^2}}\right] = \phi(u).$$

For large values of u,

$$\psi(u) \sim \frac{|u|}{x}\left[1 - \exp\left(-\frac{\pi\lambda x^2}{t}\right)\right],$$

since

$$\operatorname{erf}(\pm\infty) = \pm 1,$$

so that for $a = \infty$ the unit step voltage approaches the limit:

$$V_{12}(t) = \frac{z_2 - z_1}{\pi\lambda x^2}\left[1 - \exp\left(-\frac{\pi\lambda x^2}{t}\right)\right]$$
$$= \frac{l}{\pi\lambda x^2}\left[1 - \exp\left(-\frac{\pi\lambda x^2}{t}\right)\right],$$

where $l = z_2 - z_1$ is the length of the second wire.

This result is in agreement with a result published by F. Ollendorff, *Elektrische Nachrichten—Technik*, October, 1930, eq. (26), and by L. C. Peterson, *Bell System Technical Journal*, October, 1930, equation (5).

The case of collinear straight wires is obtained by taking the limit $x = 0$, which gives

$$\lim_{x=0} x^{-1}\psi(u) = \frac{1}{u}\left[-1 + \left(\frac{1}{2} + \frac{\pi\lambda u^2}{t}\right)\operatorname{erf}\left(u\sqrt{\frac{\pi\lambda}{t}}\right) + u\sqrt{\frac{\pi\lambda}{t}}\exp\left(-\frac{\pi\lambda u^2}{t}\right)\right]$$
$$= u^{-1}\varsigma(u).$$

This result involves the evaluation of an indeterminate form.

[6] G. A. Campbell: "Mutual Impedances of Grounded Circuits," *Bell System Technical Journal*, October, 1923, eq. (3), p. 5.

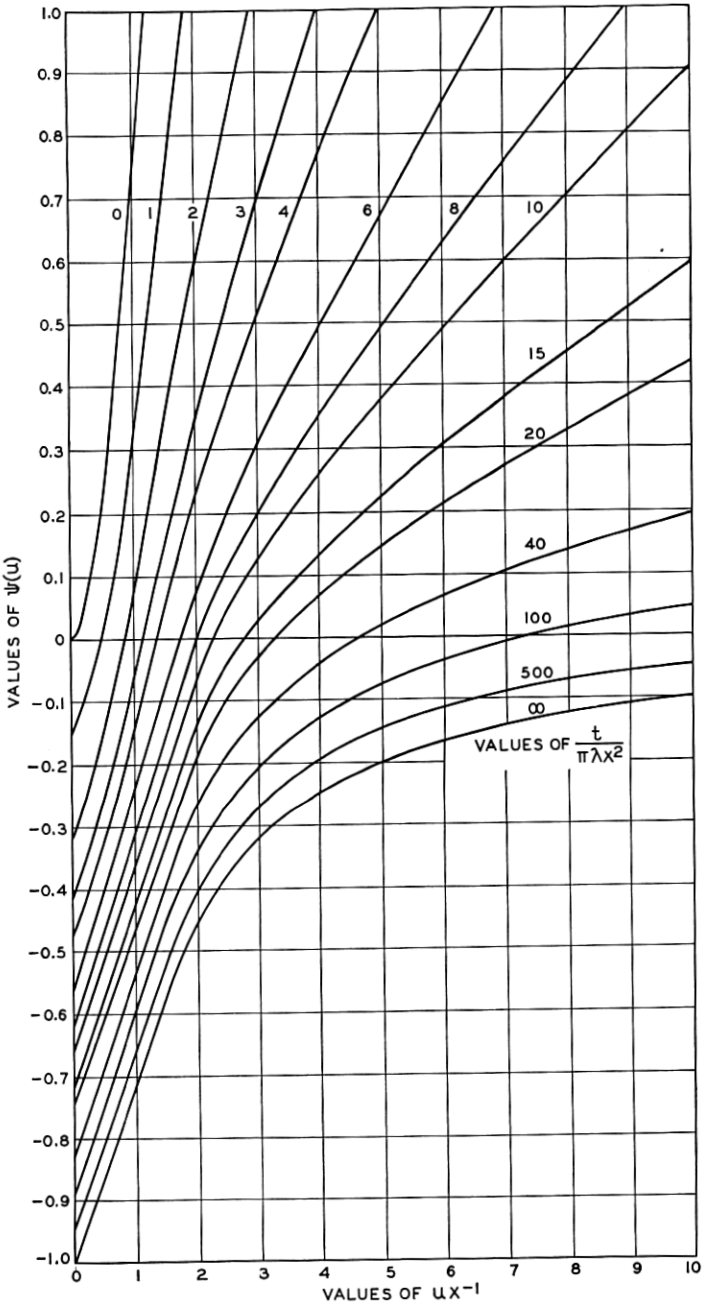

Fig. 1—$\psi(u)$ for the range in which $\psi(u) \leq 1$, $0 \leq ux^{-1} \leq 10$.

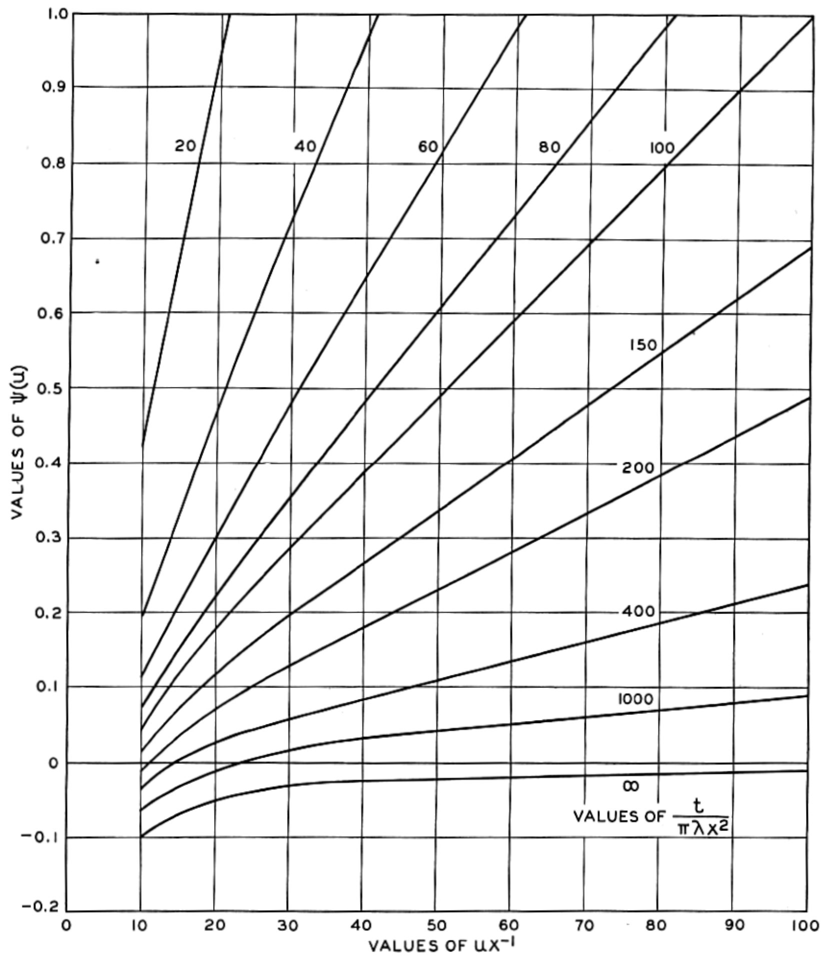

Fig. 2—$\psi(u)$ for the range in which $\psi(u) \leq 1$, $10 \leq ux^{-1} \leq 100$.

Curves for $\psi(u)$ as a function of ux^{-1} with $t/(\pi\lambda x^2)$ as parameter of the curve families are shown on Figures 1, 2, and 3. The range $\psi(u) \leq 1$, is shown on Figures 1 and 2 for $ux^{-1} \leq 10$ and 100, respectively; both figures cover the entire range of $t/(\pi\lambda x^2)$ in the intervals. The remaining range $\psi(u) > 1$ is shown on Figure 3. For the greater part of the range on Figure 3 the function is determined by its limiting form for ux^{-1} large, that is, by the equation

$$\psi(u) = ux^{-1}\left[1 - \exp\left(-\frac{\pi\lambda x^2}{t}\right)\right]$$

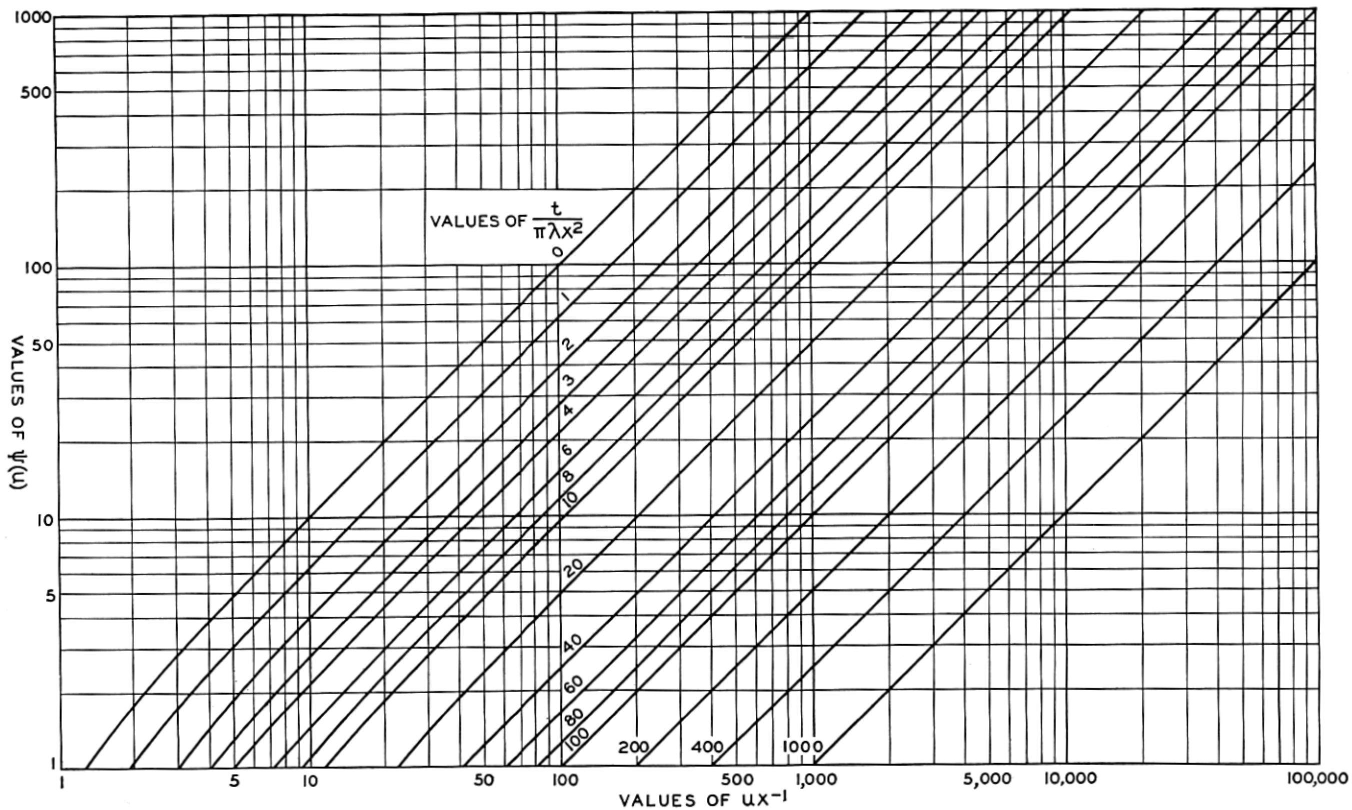

Fig. 3—$\psi(u)$ for a range in which $\psi(u) \geqq 1$, $1 \leqq ux^{-1} \leqq 100{,}000$. The straight line portion of the curve plots the equation

$$\psi(u) = ux^{-1}\left[1 - \exp\left(-\frac{\pi\lambda x^2}{t}\right)\right]$$

which is the limiting form for large values of u.

or

$$\log \psi(u) = \log ux^{-1} + \log \left[1 - \exp\left(- \frac{\pi\lambda x^2}{t} \right) \right].$$

Thus Figure 3 may be used to indicate the range of applicability of the limiting form, which is quite large; in this range the unit step voltage is simplified as shown above.

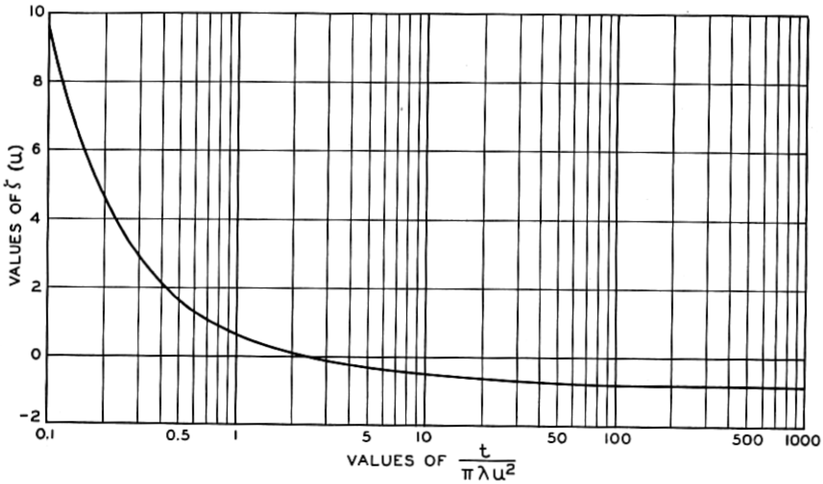

Fig. 4—The function $\zeta(u)$, for collinear straight wires; for values below the range shown $\zeta(u) \sim -\frac{1}{2} + \frac{\pi\lambda u^2}{t}$.

The function $\zeta(u)$, for the case of collinear straight wires, is shown on Fig. 4 for values of the argument $t/(\pi\lambda u^2)$ from 0.1 to 1000; for small values of the argument, the function is approximately

$$\zeta(u) \sim -\frac{1}{2} + \frac{\pi\lambda u^2}{t} \qquad \left(\frac{t}{\pi\lambda u^2} < 0.4 \right).$$

These curves may be employed to obtain voltages due to other forms of disturbing currents by numerical or mechanical integration of the following integral:[7]

$$E_{12}(t) = \frac{d}{dt}\int_0^t I(\tau) V_{12}(t - \tau) d\tau$$
$$= \frac{d}{dt}\int_0^t I(t - \tau) V_{12}(\tau) d\tau,$$

where $I(t)$ is the disturbing current as a function of time.

[7] J. R. Carson: loc. cit., p. 16, eq. (20) and (20a).

III

The equation above may be used to obtain a formula for voltage due to suddenly applied current $\exp i\omega t$; or the operational product, of which it is an expression in terms of t, may be carried out directly in terms of p. The current is expressed in terms of p by:

$$\exp i\omega t = \frac{p}{p - i\omega}.$$

The second term in $\psi(u)$ is transformed by the operational equivalent already developed:

$$\operatorname{erf} \frac{\alpha}{2\sqrt{t}} = 1 - \exp(-\alpha\sqrt{p}).$$

The last term in $\psi(u)$ is not known in closed form in p.

The operational product of $\exp i\omega t$ and the second term is evaluated by

$$\frac{p[1 - \exp(-\alpha\sqrt{p})]}{p - i\omega} = \frac{p}{p - i\omega} - \frac{p \exp(-\alpha\sqrt{p})}{p - i\omega}$$
$$= \exp i\omega t - \frac{1}{2}\left[\exp(i\omega t - \alpha\sqrt{i\omega}) \operatorname{erfc}\left(\frac{\alpha}{2\sqrt{t}} - \sqrt{i\omega t}\right)\right.$$
$$\left. + \exp(i\omega t + \alpha\sqrt{i\omega}) \operatorname{erfc}\left(\frac{\alpha}{2\sqrt{t}} + \sqrt{i\omega t}\right)\right],$$

the last term of which is given by pair 819 (with $\beta = 0$) in the tables referred to. Erfc is the error function complement;

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z).$$

The operational product of $\exp i\omega t$ and the last term in $\psi(u)$ may be expressed in integral form by the formula:

$$\frac{p}{p - i\omega} f(t) = \left[1 + \frac{i\omega}{p - i\omega}\right] f(t)$$
$$= f(t) + i\omega \exp i\omega t \int_0^t \exp(-i\omega t) f(t) dt.$$

The complete expression for the voltage due to cisoidal current is as follows:

$$E_{12}(t) = \frac{1}{2\pi\lambda x}[\Phi(z_2 + a) - \Phi(z_2 - a) - \Phi(z_1 + a) + \Phi(z_1 - a)], \quad (4)$$

where

$$\Phi(u) = \frac{u^2}{x\sqrt{x^2+u^2}} - \frac{u}{x}\exp\left(-\frac{\pi\lambda x^2}{t}\right)\operatorname{erf}\left(u\sqrt{\frac{\pi\lambda}{t}}\right)$$
$$-\frac{\sqrt{x^2+u^2}}{2x}\left[\exp\left(i\omega t-\gamma\sqrt{x^2+u^2}\right)\operatorname{erfc}\left(\sqrt{\frac{\pi\lambda}{t}(x^2+u^2)}-\sqrt{i\omega t}\right)\right.$$
$$\left.+\exp\left(i\omega t+\gamma\sqrt{x^2+u^2}\right)\operatorname{erfc}\left(\sqrt{\frac{\pi\lambda}{t}(x^2+u^2)}+\sqrt{i\omega t}\right)\right]$$
$$-\frac{u}{x}i\omega\exp i\omega t\int_0^t\exp\left(-i\omega t-\frac{\pi\lambda x^2}{t}\right)\operatorname{erf}\left(u\sqrt{\frac{\pi\lambda}{t}}\right)dt.$$

The integral appearing in $\Phi(u)$ apparently cannot be expressed in closed form in terms of known functions; for numerical results series or asymptotic expressions may be derived but it appears more desirable to employ numerical or mechanical integration using the unit step voltage since tables or charts of the error function of complex variable which also appears in $\Phi(u)$ are not available.

A useful check on the above formula is obtained by taking the limit for $t = \infty$, which gives the steady-state mutual impedance between straight parallel wires; the result is as follows:

$$Z_{12} = E_{12}(t)\exp(-i\omega t)$$
$$= \frac{1}{2\pi\lambda x}\left[\Psi(z_2+a)-\Psi(z_2-a)-\Psi(z_1+a)+\Psi(z_1-a)\right], \quad (5)$$

where

$$\Psi(u) = \frac{u^2}{x\sqrt{x^2+u^2}} - \frac{\sqrt{x^2+u^2}}{x}\exp\left(-\gamma\sqrt{x^2+u^2}\right)$$
$$-\frac{u}{x}\int_0^\infty\exp\left(-w-\frac{\gamma^2x^2}{4w}\right)\operatorname{erf}\frac{\gamma u}{2\sqrt{w}}\,dw$$
$$= -\frac{x}{\sqrt{x^2+u^2}}+\frac{\sqrt{x^2+u^2}}{x}\left[1-\exp\left(-\gamma\sqrt{x^2+u^2}\right)\right]$$
$$-\frac{\gamma u}{x}\int_0^u\exp\left(-\gamma\sqrt{x^2+w^2}\right)dw,$$

where as before $\gamma^2 = 4\pi\lambda i\omega$.

The third term in $\Phi(u)$ approaches the limit given because $\operatorname{erfc}(-\sqrt{i\,\infty}) = 2$, $\operatorname{erfc}(\sqrt{i\,\infty}) = 0$; the integral term as given in the first form of $\Psi(u)$ has been transformed by the substitution $w = i\omega t$.

The first form of $\Psi(u)$ may be checked directly from equation (3) by introducing $i\omega = p$ in the operationally equivalent function of p; the third term of (3) being expressed by the infinite integral:

$$F(p) = p \int_0^\infty e^{-pt} f(t) dt.$$

The second form of $\Psi(u)$ is obtained by separating the d.-c. mutual resistance term, and transforming the infinite integral as follows: express the error function in integral form, put $y = \gamma v/(2\sqrt{w})$ where y is the variable of integration for the error function, and invert the order of integration; thus

$$\begin{aligned}\int_0^\infty \exp\left(-w - \frac{\gamma^2 x^2}{4w}\right) \operatorname{erf} \frac{\gamma u}{2\sqrt{w}} dw &= \frac{\gamma}{\sqrt{\pi}} \int_0^u dv \int_0^\infty \exp\left(-w - \frac{\gamma^2(x^2+v^2)}{4w}\right) \frac{dw}{\sqrt{w}} \\ &= \frac{2\gamma}{\sqrt{\pi}} \int_0^u dv \int_0^\infty \exp\left(-z^2 - \frac{\gamma^2(x^2+v^2)}{4z^2}\right) dz \qquad (z = \sqrt{w}) \\ &= \gamma \int_0^u \exp(-\gamma\sqrt{x^2+v^2}) dv.\end{aligned}$$

The infinite integral evaluated in the third line is No. 495 in Peirce's "Short Table of Integrals," third edition.

The second form of $\Psi(u)$ may be verified by direct double integration of the mutual impedance; it agrees with the known result in the limit for one wire infinite, and, when expanded in powers of γ, with the terms given in the second form for the mutual impedance by R. M. Foster, loc. cit.

Expressions for voltages due to suddenly applied currents $\exp(-kt) \sin \omega t$ or $1 - \exp(-kt)$, which are important forms for a.-c. and d.-c. networks, may be readily obtained from equation (4), the first by use of the expression:

$$\exp(-kt) \sin \omega t = \frac{1}{2i}[\exp(-kt + i\omega t) + \exp(-kt - i\omega t)]$$

and the second by the substitution $-k = i\omega$ and subtraction from the unit step voltage.

The results attained in this paper depend in appreciable measure on advice and suggestions received from Mr. R. M. Foster of the American Telephone and Telegraph Company; I am also appreciative of the interest and advice of Messrs. K. L. Maurer and H. M. Trueblood of this company.