Mutual Impedance of Grounded Wires Lying on the Surface of the Earth *

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This paper presents a formula for the mutual impedance between two insulated wires of negligible diameter lying on the surface of the earth and grounded at their end-points. The formula holds for frequencies which are not too high to allow all displacement currents to be neglected. For any two elements dS, ds of the two wires the mutual impedance is obtained from their direct-current mutual impedance by introducing the complex factor $2(\gamma r)^{-2}[1 - (1 + \gamma r)e^{-\gamma r}]$ in the reactance term, γ being the propagation constant in the earth, and r the distance between the elements dS and ds.

THE mutual impedance of grounded circuits may be derived from certain results obtained by A. Sommerfeld,[1] who has developed formulæ for the electric and magnetic fields in the earth and in the air due to horizontal and vertical electric and magnetic antennæ situated at the surface of the earth. For our present problem we use his formulæ for the electric field in the earth due to a horizontal electric doublet, since this doublet may be regarded as a short element dS of a wire of negligible diameter carrying a finite current. At the end of this present paper we shall show how the same formula for the mutual impedance may be obtained directly from first principles.

Sommerfeld uses rectangular coordinates (x, y, z) and the corresponding cylindrical coordinates (r, ϕ, z), the surface of the earth, assumed flat, being the xy plane, and the z axis extending upward into the air. The doublet is at the origin, and its axis along the x axis. Then the components of the Hertzian vector [2] in the earth ($z < 0$) from which the electric field is determined are [3]

$$\Pi_x = C\frac{k_0^2}{k^2}\int_0^\infty \frac{J_0(\rho r)}{N'}\, e^{z\sqrt{\rho^2-k^2}}\rho d\rho, \tag{1}$$

$$\Pi_y = 0, \tag{2}$$

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[1] A. Sommerfeld, "Über die Ausbreitung der Wellen in der drahtlosen Telegraphie," *Annalen der Physik*, (4), **81**, 1135–1153 (December 1926). This paper is a summary and an extension of earlier work by Sommerfeld and von Hoerschelmann, references to which will be found in the paper.

[2] H. Abraham and A. Föppl, "Theorie der Elektrizität," 5th ed., Leipzig and Berlin, 1918; Vol. I, § 79, page 331.

[3] A. Sommerfeld, *loc. cit.*, pages 1145 and 1146, introducing the constant factor defined on page 1152.

$$\Pi_z = C(k^2 - k_0^2)\frac{k_0^2}{k^2}\cos\phi\int_0^\infty \frac{J_0'(\rho r)}{NN'}e^{z\sqrt{\rho^2-k^2}}\rho^2 d\rho, \tag{3}$$

where the time factor $e^{-i\omega t}$ is omitted throughout. J_0 is the Bessel function of order zero, and the constants k and k_0 are the propagation constants in the earth and in the air for plane waves varying with the time as $e^{-i\omega t}$. Their values in Heaviside units are given by Sommerfeld as

$$k^2 = \frac{1}{c^2}(\epsilon\omega^2 + i\sigma\omega), \qquad k_0^2 = \frac{1}{c^2}\epsilon_0\omega^2, \tag{4}$$

where ϵ and ϵ_0 are the dielectric constants of the earth and of the air, respectively, σ is the conductivity of the earth, assumed uniform, and c is the velocity of light. In both media the permeability is taken as unity. Also

$$N = k^2\sqrt{\rho^2 - k_0^2} + k_0^2\sqrt{\rho^2 - k^2}, \tag{5}$$

$$N' = \sqrt{\rho^2 - k_0^2} + \sqrt{\rho^2 - k^2}, \tag{6}$$

and C is a constant measuring the electric moment of the doublet.

We now replace the doublet by a short element of wire dS carrying a current $Ie^{i\omega t}$, and at the same time we assume that ϵ and ϵ_0 are both negligible, so that all displacement currents are neglected. This is a simplification which is ordinarily made as a first approximation at power frequencies for the shorter transmission lines. Then, introducing c.g.s. electromagnetic units, in which the conductivity of the earth is λ, and noting that we have changed the sign of ω, formulæ (4)–(6) become

$$k^2 = -i4\pi\lambda\omega = -\gamma^2, \tag{7}$$

$$k_0^2 = 0, \tag{8}$$

$$N = -\gamma^2\rho, \tag{9}$$

$$N' = \rho + \sqrt{\rho^2 + \gamma^2}, \tag{10}$$

and the constant C is such that

$$\frac{Ck_0^2}{k^2} = \frac{1}{\sigma}\times \text{current} \times \text{effective length of doublet} \tag{11}$$

$$= \frac{IdS}{2\pi\lambda}.$$

Substituting from (7)–(11) in (1)–(3) we have, therefore,

$$\text{(12)}\qquad \Pi_x = \frac{IdS}{2\pi\lambda}\int_0^\infty \frac{J_0(r\rho)}{\rho + \sqrt{\rho^2+\gamma^2}}\epsilon^{z\sqrt{\rho^2+\gamma^2}}\rho d\rho = \frac{IdS}{2\pi\lambda\gamma^2}\left(\frac{\partial^2 P}{\partial z^2} + \frac{\partial^3 Q}{\partial x^2\partial z} + \frac{\partial^3 Q}{\partial y^2 \partial z}\right),$$

$$\text{(13)}\qquad \Pi_y = 0,$$

$$\text{(14)}\qquad \Pi_z = \frac{IdS}{2\pi\lambda}\frac{\partial}{\partial x}\int_0^\infty \frac{J_0(r\rho)}{\rho + \sqrt{\rho^2+\gamma^2}}\epsilon^{z\sqrt{\rho^2+\gamma^2}} d\rho = -\frac{IdS}{2\pi\lambda\gamma^2}\left(\frac{\partial^2 P}{\partial x \partial z} - \frac{\partial^3 Q}{\partial x \partial z^2}\right),$$

where

$$\text{(15)}\qquad P = \int_0^\infty J_0(r\rho)\epsilon^{z\sqrt{\rho^2+\gamma^2}}\frac{\rho d\rho}{\sqrt{\rho^2+\gamma^2}} = \frac{1}{R}e^{-\gamma R},$$

and

$$\text{(16)}\qquad Q = \int_0^\infty J_0(r\rho)\epsilon^{z\sqrt{\rho^2+\gamma^2}}\frac{d\rho}{\sqrt{\rho^2+\gamma^2}} = I_0[\tfrac{1}{2}\gamma(R+z)]K_0[\tfrac{1}{2}\gamma(R-z)],$$

with $R^2 = r^2 + z^2$.

The integral P is well known,[4] while Q is evaluated by a suitable transformation of a Fourier integral.[5] $I_0(z) = J_0(iz)$ and $K_0(z) = \frac{1}{2}\pi i H_0^{(1)}(iz)$ are the Bessel functions of the first and second kinds for imaginary arguments as defined by G. N. Watson.[6] In reducing Π_x to this form we use the differential equation [7] for J_0 to obtain the relation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)J_0(r\rho) + \rho^2 J_0(r\rho) = 0.$$

The components of the electric force in the earth are obtained from Π by the formula

$$\text{(17)}\qquad E = \text{grad div }\Pi - \gamma^2\Pi,$$

[4] See *e.g.* H. Bateman, "Electrical and Optical Wave-Motion," Cambridge, 1915, page 72; or G. N. Watson, "Theory of Bessel Functions," Cambridge, 1922, page 416, formula (2) of § 13.47, with $\mu = 0$ and $\nu = \frac{1}{2}$.

[5] G. A. Campbell, "The Practical Application of the Fourier Integral," *Bell System Technical Journal*, **7**, 639–707; using pair 936 of Table I, with $\alpha = \frac{1}{2}$, substituting x^2 for $(g^2 - 4)$ in the integral of G, and generalizing the resulting integral to include complex quantities.

[6] G. N. Watson, *op. cit.*, pages 77, 78.

[7] G. N. Watson, *op. cit.*, page 19, formula (1) of § 2.13.

and we thus obtain E_x, E_y, E_z in the compact form

$$(18)\qquad (E_x, E_y, E_z) = \frac{IdS}{2\pi\lambda}\left(-\frac{\partial^3 Q}{\partial y^2 \partial z} - \frac{\partial^2 P}{\partial z^2},\ \frac{\partial^3 Q}{\partial x \partial y \partial z},\ \frac{\partial^2 P}{\partial x \partial z} \right),$$

where P and Q are given by (15) and (16). In deriving this form we use the fact that Q satisfies the wave equation

$$\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2} - \gamma^2 Q = 0.$$

At the surface of the earth ($z = 0$) the electric force takes the simple form

$$(19)\qquad (E_x, E_y) = \frac{IdS}{2\pi\lambda}\left[-\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + \frac{1+\gamma r}{r^3} e^{-\gamma r},\ \frac{\partial^2}{\partial x \partial y}\left(\frac{1}{r}\right) \right],$$

where we have used the expressions for the derivatives [8] of the Bessel functions, $I_0'(z) = I_1(z)$, $K_0'(z) = -K_1(z)$, and also the identity [9] $I_0(z)K_1(z) + I_1(z)K_0(z) = 1/z$.

The mutual impedance dZ_{12} between two infinitesimal elements dS and ds is now written down as the ratio of the resulting electric force in one element to the current in the other, with sign reversed:

$$\begin{aligned}(20)\quad dZ_{12} &= \frac{dSds}{2\pi\lambda}\left[\cos\epsilon \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) - \cos\epsilon \frac{1+\gamma r}{r^3} e^{-\gamma r} - \sin\epsilon \frac{\partial^2}{\partial x \partial y}\left(\frac{1}{r}\right) \right] \\ &= \frac{dSds}{2\pi\lambda}\left[\frac{3\sin\Phi\sin\phi - \cos\epsilon}{r^3} - \frac{\cos\epsilon}{r^3}(1+\gamma r)e^{-\gamma r} \right] \\ &= \frac{dSds}{2\pi\lambda}\left\{ \frac{d^2}{dSds}\left(\frac{1}{r}\right) + \frac{\cos\epsilon}{r^3}[1 - (1+\gamma r)e^{-\gamma r}] \right\},\end{aligned}$$

where Φ and ϕ are the angles which the elements dS and ds make with r, and $\epsilon = \Phi - \phi$ is the angle they make with each other.

Integration over the two wires S and s gives a general formula for the mutual impedance of grounded wires lying on the surface of the earth:

$$\begin{aligned}(21)\quad Z_{12} &= \frac{1}{2\pi\lambda}\int\int \left\{ \frac{d^2}{dSds}\left(\frac{1}{r}\right) + \frac{\cos\epsilon}{r^3}[1 - (1+\gamma r)e^{-\gamma r}] \right\} dSds \\ &= \int\int \left[\frac{1}{2\pi\lambda} \cdot \frac{d^2}{dSds}\left(\frac{1}{r}\right) + i\omega \frac{\cos\epsilon}{r}\left\{ \frac{2}{(\gamma r)^2}[1 - (1+\gamma r)e^{-\gamma r}] \right\} \right] dSds.\end{aligned}$$

[8] G. N. Watson, *op. cit.*, page 79, formula (7) of § 3.71.
[9] G. N. Watson, *op. cit.*, page 80, formula (20) of § 3.71, with $\nu = 0$.

The factor

$$\frac{2}{(\gamma r)^2}[1 - (1 + \gamma r)e^{-\gamma r}] \tag{22}$$

approaches unity as ω approaches zero, and Z_{12} then agrees with the

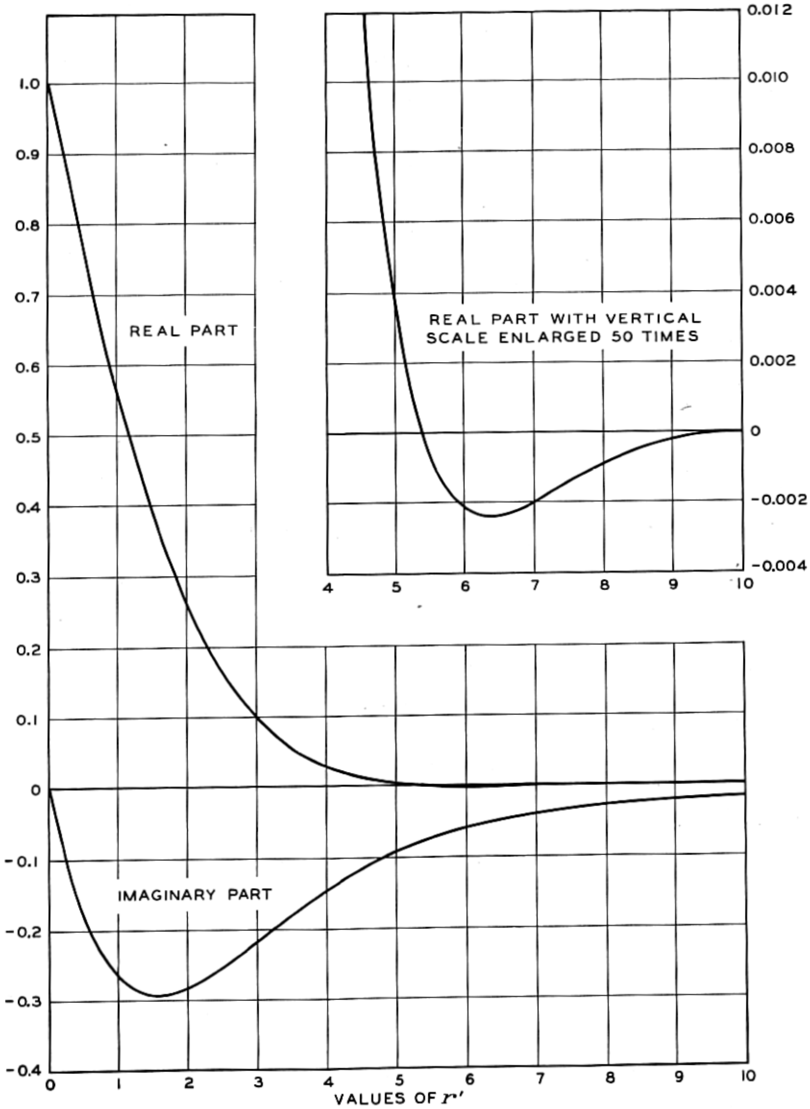

Fig. 1—Real and imaginary parts of the complex factor,

$$\frac{2}{(\gamma r)^2}[1 - (1 + \gamma r)e^{-\gamma r}],$$

plotted as functions of $r' = |\gamma r| = (4\pi\lambda\omega)^{1/2}r$.

direct-current mutual impedance as given by G. A. Campbell.[10] Introducing this factor, which is a function of γr only, into the reactance term for the direct-current mutual impedance between two elements dS and ds gives the general expression for their mutual impedance corresponding to the propagation constant γ. It is interesting also to determine, for any given value of γ, the variation of the factor (22) for increasing values of r. This is shown very clearly in Fig. 1, where the real and imaginary parts of (22) are plotted for increasing values of $r' = |\gamma r| = (4\pi\lambda\omega)^{1/2}r$. The real part, we note, decreases rapidly from the initial value unity as r' increases, while the imaginary part is always negative, decreasing from zero to a minimum value (approximately -0.3 for $r' = 1.5$) and then increasing towards zero, although it does not approach zero so rapidly as the real part does.

The first three terms in the expansion of Z_{12} for low frequencies are given by

$$(23)\quad Z_{12} = \frac{1}{2\pi\lambda}\left(\frac{1}{Aa} - \frac{1}{Ab} - \frac{1}{Ba} + \frac{1}{Bb}\right) + i\omega N_{Ss} + (1-i)\tfrac{1}{3}(8\pi\lambda\omega^3)^{1/2}AB\,ab\cos\theta + \cdots,$$

where N_{Ss} is the mutual Neumann integral between the two wires S and s of arbitrary form but with end-points A, B and a, b respectively; θ is the angle between the straight lines AB and ab. The first two terms in this expansion are precisely the direct-current mutual impedance as given by G. A. Campbell.

The first term in the expansion of Z_{12} for a long straight wire S and any wire s located near the midpoint of S is

$$(24)\qquad \int\left[\frac{1}{\pi\lambda x^2} - \frac{\gamma}{\pi\lambda x}K_1(\gamma x)\right]\cos\epsilon\,ds,$$

x being the positive distance from ds to S, and ϵ the angle between ds and S. $K_1(z) = -\tfrac{1}{2}\pi H_1^{(1)}(iz)$ is the Bessel function of the second kind for imaginary argument as defined by G. N. Watson.[11] In obtaining (24) from (21) we use the derivative with respect to x of the integral

$$\int_0^\infty \frac{e^{-\gamma r}}{r}\,dz = K_0(\gamma x),$$

which is a special case of the integral used above in evaluating Q, with x assumed positive.

[10] G. A. Campbell, "Mutual Impedances of Grounded Circuits," *Bell System Technical Journal*, **2**, 1–30 (October 1923).
[11] G. N. Watson, *op. cit.*, page 78.

The expression in square brackets in (24) is the mutual impedance gradient parallel to an infinite wire at a positive distance x from the wire. It agrees with the results published independently by F. Pollaczek,[12] J. R. Carson,[13] and G. Haberland,[14] and has been employed by us to obtain numerical results since 1917. Pollaczek has also investigated the case of two gounded circuits of finite length.[15]

The mutual impedance dZ_{12} between a short grounded circuit dS and a counterclockwise small loop of area da, on the surface of the earth, is given by the formula

$$dZ_{12} = \frac{dSda}{2\pi\lambda} \cdot \frac{\sin\phi}{r^4}[3 - (3 + 3\gamma r + \gamma^2 r^2)e^{-\gamma r}], \tag{25}$$

where ϕ is the angle which dS makes with r, the line from da to dS. This may be obtained from Sommerfeld's formulæ for the horizontal electric force due to a vertical magnetic antenna, or it may be obtained by an application of Stokes's theorem to formula (20) above.

By a further application of Stokes's theorem we may obtain the mutual impedance between two counterclockwise small loops dA and da, namely,

$$dZ_{12} = \frac{dAda}{2\pi\lambda} \cdot \frac{1}{r^5}[(9 + 9\gamma r + 4\gamma^2 r^2 + \gamma^3 r^3)e^{-\gamma r} - 9]. \tag{26}$$

This result might also be derived from Sommerfeld's formula for the vertical magnetic force due to a vertical magnetic antenna.

We shall now indicate briefly how the same value of E as given in (18) above may be obtained directly, though more laboriously, from first principles. In this method we start from the fundamental solution [16]

$$u = e^{lx+my+nz}e^{i\omega t} \tag{27}$$

of the wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \gamma^2 u = 0, \tag{28}$$

[12] F. Pollaczek, "Über das Feld einer unendlich langen wechselstromdurchflossenen Einfachleitung," *Elektrische Nachrichten-technik*, **3**, 339–359 (September 1926).

[13] J. R. Carson, "Wave Propagation in Overhead Wires with Ground Return," *Bell System Technical Journal*, **5**, 539–554 (October 1926).

[14] G. Haberland, "Theorie der Leitung von Wechselstrom durch die Erde," *Zeitschrift für angewandte Mathematik und Mechanik*, **6**, 366–379 (October 1926).

[15] F. Pollaczek, "Gegenseitige Induktion zwischen Wechselstromfreileitungen von endlicher Länge," *Annalen der Physik*, (4), **87**, 965–999 (December 1928). His assumptions regarding conditions at the ground connections seem to depart considerably from the conditions assumed in the present paper, and moreover his results are not expressed in convenient form for direct comparison with the formula given above for Z_{12}.

[16] H. Bateman, *op. cit.*, § 4, pages 6, 7; § 11, page 26.

which is satisfied by the electric force in the earth; $\gamma = (i4\pi\lambda\omega)^{1/2}$ is the propagation constant for plane waves which vary with the time as $e^{i\omega t}$. The parameters l, m, n satisfy the relation

$$l^2 + m^2 + n^2 - \gamma^2 = 0. \tag{29}$$

In the air, the same equations hold, but with the propagation constant γ equal to zero, and we note that the solution in the air must be chosen to vanish at an infinite height, while in the earth the solution must vanish at an infinite depth.

For convenience in this method we start with a short straight wire of length $2a$ lying along the x axis, later allowing a to approach zero. Thus we suppose that the current $Ie^{i\omega t}$ enters the earth at the point $(a, 0, 0)$ and leaves it at the point $(-a, 0, 0)$. The factor $e^{i\omega t}$ will be omitted, however, throughout the following work. The current flow in this system is symmetrical with respect to the vertical plane through the wire, the xz plane, and is also symmetrical, but with sign reversed, with respect to the vertical plane normal to the wire at its midpoint, the yz plane. Then if we replace the three parameters l, m, n of (27) by two independent parameters μ, ν, such that

$$l = \pm i\mu, \qquad m = \pm i\nu, \qquad n = \pm\sqrt{\mu^2 + \nu^2 + \gamma^2}, \tag{30}$$

formula (29) is identically satisfied, and we can then replace the four solutions $e^{\pm i\mu x \pm i\nu y}$ by their corresponding expressions in terms of sines and cosines, namely,

$$\sin x\mu \sin y\nu, \qquad \sin x\mu \cos y\nu, \qquad \cos x\mu \sin y\nu, \qquad \cos x\mu \cos y\nu.$$

The above considerations of symmetry will eliminate, for each component of the electric force, all but one of these forms. With the remaining solution as a basis we build up, by means of the Fourier integral, a general expression for any possible steady harmonic oscillation. Hence we may write down the general solutions for the total electric force in the earth ($z < 0$), as follows.

$$E_x = \int_0^\infty\int_0^\infty F_x(\mu, \nu)e^{z\sqrt{\mu^2+\nu^2+\gamma^2}}\cos x\mu \cos y\nu \, d\mu d\nu, \tag{31}$$

$$E_y = \int_0^\infty\int_0^\infty F_y(\mu, \nu)e^{z\sqrt{\mu^2+\nu^2+\gamma^2}}\sin x\mu \sin y\nu \, d\mu d\nu, \tag{32}$$

$$E_z = \int_0^\infty\int_0^\infty F_z(\mu, \nu)e^{z\sqrt{\mu^2+\nu^2+\gamma^2}}\sin x\mu \cos y\nu \, d\mu d\nu, \tag{33}$$

where the positive sign is chosen in the exponential term containing z since the solution must vanish at an infinite depth, z being negative in the earth; and that value of the radical is taken which has a positive real part. F_x, F_y, F_z are arbitrary functions of their arguments, to be determined by the physical conditions of the problem.

In the air $(0 < z)$ we may formulate the corresponding solutions for the total electric force as

$$E_x = \int_0^\infty \int_0^\infty P_x(\mu, \nu) e^{-z\sqrt{\mu^2+\nu^2}} \cos x\mu \cos y\nu \, d\mu d\nu, \tag{34}$$

$$E_y = \int_0^\infty \int_0^\infty P_y(\mu, \nu) e^{-z\sqrt{\mu^2+\nu^2}} \sin x\mu \sin y\nu \, d\mu d\nu, \tag{35}$$

$$E_z = \int_0^\infty \int_0^\infty P_z(\mu, \nu) e^{-z\sqrt{\mu^2+\nu^2}} \sin x\mu \cos y\nu \, d\mu d\nu, \tag{36}$$

where the propagation constant is zero in the air; the negative sign is chosen in the exponential term containing z since the solution must vanish at an infinite height, z being positive in the air; and P_x, P_y, P_z are arbitrary functions of their arguments.

To determine these six arbitrary functions we need six independent relations among them. Two of these relations are obtained by utilizing the fact that the divergence of the electric force either in the earth or in the air is equal to zero, that is,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0.$$

By means of this we obtain from (31)–(33),

$$- \mu F_x + \nu F_y + \sqrt{\mu^2 + \nu^2 + \gamma^2} F_z = 0, \tag{37}$$

and from (34)–(36),

$$- \mu P_x + \nu P_y - \sqrt{\mu^2 + \nu^2} P_z = 0. \tag{38}$$

Since the horizontal components of the electric force are continuous at the surface of the earth $(z = 0)$ we see that we must also have, from (31) and (34),

$$F_x = P_x, \tag{39}$$

and from (32) and (35),

$$F_y = P_y. \tag{40}$$

We may obtain a fifth relation from the fact that the current I flows through the earth from one grounding point to the other. To utilize this fact let us compute the total current flowing out through five faces of a rectangular prism in the earth, the sixth face being a rectangle in the surface of the earth surrounding the grounding point $(a, 0, 0)$, the prism extending from $x = a - \xi$ to $x = a + \xi$, from $y = -\eta$ to $y = \eta$, and from $z = -\zeta$ to $z = 0$. The components of the electric force being given by (31)-(33), and λ being the conductivity of the earth, we obtain for this current the expression

$$-4\lambda \int_0^\infty \int_0^\infty F_z \frac{\sin a\mu \sin \xi\mu \sin \eta\nu}{\mu\nu} d\mu d\nu, \tag{41}$$

after simplifying by means of the divergence condition (37). This current flowing out through the prism is I if the face in the surface of the earth includes only the one grounding point $(a, 0, 0)$, but is zero if it includes both grounding points; that is, the above integral (41) equals I if $\xi < 2a$, but equals zero if $2a < \xi$, for any positive value of η. It is readily verified that the Fourier integral

$$\frac{8I}{\pi^2} \int_0^\infty \int_0^\infty \frac{\sin^2 a\mu \sin \xi\mu \sin \eta\nu}{\mu\nu} d\mu d\nu \tag{42}$$

has the desired properties. Accordingly, we must have

$$F_z = -\frac{2I}{\pi^2\lambda} \sin a\mu. \tag{43}$$

To obtain the one additional relation which is needed, we make use of the fact that the current I flows through the straight wire from one grounding point to the other. Let us integrate the magnetic force around a rectangle in a plane perpendicular to the wire, that is, perpendicular to the x axis, the rectangle extending from $y = -\eta$ to $y = \eta$ and from $z = -\zeta$ to $z = \zeta$, the path of integration being taken in the clockwise direction looking along the positive direction of the x axis, and then equate this integral to 4π times the total current threading the rectangle. The components of the magnetic force which we need, H_y and H_z, are found from the fact that curl $E = -i\omega H$, that is,

$$i\omega H_y = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}, \tag{44}$$

$$i\omega H_z = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}, \tag{45}$$

where the E's are given by (31)–(33) for $z < 0$ and by (34)–(36) for $0 < z$. We now subtract from this integral 4π times the current *in the earth* which threads the rectangle, this quantity being found by the appropriate integration of E_x, as given by (31), over that portion of the area of the rectangle which lies below the surface of the earth. As a final result we obtain the expression

$$\frac{2}{i\omega}\int_0^\infty\int_0^\infty \frac{1}{\nu}\left(-\sqrt{\mu^2+\nu^2+\gamma^2}F_x+\mu F_z-\sqrt{\mu^2+\nu^2}P_x-\mu P_z\right) \times \cos x\mu \sin \eta\nu \, d\mu d\nu, \tag{46}$$

after simplifying by means of the divergence conditions (37) and (38). The net current threading the rectangle, after subtracting the current in the earth, is I if the rectangle is situated between the two grounding points, but is zero if it is outside them; that is, the above integral (46) equals $4\pi I$ if $|x| < a$, but equals zero if $a < |x|$, for any positive value of η. It is readily verified that the Fourier integral

$$\frac{16I}{\pi}\int_0^\infty\int_0^\infty \frac{\sin a\mu \cos x\mu \sin \eta\nu}{\mu\nu}\, d\mu d\nu$$

has the desired properties. Accordingly we must have

$$-\sqrt{\mu^2+\nu^2+\gamma^2}F_x+\mu F_z-\sqrt{\mu^2+\nu^2}P_x-\mu P_z = \frac{8i\omega I}{\pi}\cdot\frac{\sin a\mu}{\mu}. \tag{47}$$

We can now solve equations (37)–(40), (43), and (47) for the six arbitrary functions, obtaining

$$F_x = P_x = \frac{2I}{\pi^2\lambda}\left[\frac{\nu^2}{\mu\sqrt{\mu^2+\nu^2}} - \frac{\sqrt{\mu^2+\nu^2+\gamma^2}}{\mu}\right]\sin a\mu, \tag{48}$$

$$F_y = P_y = \frac{2I}{\pi^2\lambda}\cdot\frac{\nu}{\sqrt{\mu^2+\nu^2}}\sin a\mu, \tag{49}$$

$$F_z = -\frac{2I}{\pi^2\lambda}\sin a\mu, \tag{43}$$

$$P_z = \frac{2I}{\pi^2\lambda}\cdot\frac{\sqrt{\mu^2+\nu^2+\gamma^2}}{\sqrt{\mu^2+\nu^2}}\sin a\mu. \tag{50}$$

Substituting these values in equations (31)–(33) and letting a approach zero such that $2a = dS$, we find, for the electric force in the earth,

$$(51)\quad E_x = \frac{IdS}{\pi^2\lambda}\int_0^\infty\int_0^\infty\left[\frac{\nu^2}{\sqrt{\mu^2+\nu^2}} - \sqrt{\mu^2+\nu^2+\gamma^2}\right]e^{z\sqrt{\mu^2+\nu^2+\gamma^2}} \times \cos x\mu \cos y\nu \, d\mu d\nu,$$

$$(52)\quad E_y = \frac{IdS}{\pi^2\lambda}\int_0^\infty\int_0^\infty \frac{\mu\nu}{\sqrt{\mu^2+\nu^2}} e^{z\sqrt{\mu^2+\nu^2+\gamma^2}} \sin x\mu \sin y\nu \, d\mu d\nu,$$

$$(53)\quad E_z = -\frac{IdS}{\pi^2\lambda}\int_0^\infty\int_0^\infty \mu e^{z\sqrt{\mu^2+\nu^2+\gamma^2}} \sin x\mu \cos y\nu \, d\mu d\nu.$$

These are precisely the values found by the former method, for the integrals P and Q may be expressed as double integrals by substituting for $J_0(r\rho)$ the integral expression given by the formula [17]

$$(54)\qquad J_0(r\sqrt{\mu^2+\nu^2}) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\cos(r\mu\cos\theta)\cos(r\nu\sin\theta)d\theta,$$

and introducing rectangular coordinates in place of r, θ. These integrals may, therefore, be written in the equivalent forms,

$$(55)\quad P = \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{e^{z\sqrt{\mu^2+\nu^2+\gamma^2}}}{\sqrt{\mu^2+\nu^2+\gamma^2}}\cos x\mu\cos y\nu \, d\mu d\nu,$$

$$(56)\quad Q = \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{e^{z\sqrt{\mu^2+\nu^2+\gamma^2}}}{\sqrt{\mu^2+\nu^2}\sqrt{\mu^2+\nu^2+\gamma^2}}\cos x\mu\cos y\nu \, d\mu d\nu,$$

and comparison with (51)–(53) again leads to the values

$$(18)\quad (E_x, E_y, E_z) = \frac{IdS}{2\pi\lambda}\left(-\frac{\partial^3 Q}{\partial y^2\partial z} - \frac{\partial^2 P}{\partial z^2}, \ \frac{\partial^3 Q}{\partial x\partial y\partial z}, \ \frac{\partial^2 P}{\partial x\partial z}\right),$$

where P and Q are evaluated in (15) and (16). Thus the mutual impedance formula presented in this paper may be derived directly from first principles, without reference to the work of Sommerfeld.

I am greatly indebted to my colleague, Dr. Marion C. Gray, for putting into its present form the derivation of my formula from Sommerfeld's results.

[17] G. N. Watson, *op. cit.*, page 21, formula (1) of § 2.21.