Asymptotic Dipole Radiation Formulas

By W. HOWARD WISE

THE analysis of the radiation from dipoles as given by Sommerfeld and by von Hoerschelmann is deficient in one respect: it does not give the true [1] asymptotic expressions for the radiation leaving at a considerable angle from the horizontal. The correct asymptotic formulas have already been easily supplied by an appeal to the Reciprocal Theorem; lately M. J. O. Strutt [2] has got them directly from the boundary conditions and H. Weyl [3] has derived the correct asymptotic formula for a vertical dipole at the surface of the earth by a method quite different from Sommerfeld's. In the present paper it is shown how they can be got by merely improving the rigor of Sommerfeld's analysis.

The present analysis begins with the formulas of von Hoerschelmann for the wave potentials of vertical and horizontal dipoles at a finite distance above the surface of the earth and generally follows Sommerfeld. The derivation of an asymptotic approximation for the wave potential of a vertical dipole is considerably different from Sommerfeld's and results in the simpler and more precise formulas deduced from the reciprocal theorem.

Most of the analysis is somewhat simplified by taking the permeability of the earth to be unity.

The notation used is chiefly that of Bateman.[4]

τ = variable of integration, throughout the paper.

$$k^2 = \epsilon\mu\omega^2 + 4\pi\sigma\mu i\omega$$

$$l = \sqrt{\tau^2 - k_1^2}, \qquad m = \sqrt{\tau^2 - k_2^2}.$$

The subscripts 1 and 2 refer to air and ground respectively.

R_1, R_2, a, ρ, φ, w, x, y and z are adequately defined by Fig. 1.

$$\cos\theta_x = x/R, \qquad \cos\theta_y = y/R, \qquad \cos\theta_z = z/R, \qquad R^2 = x^2 + y^2 + z^2.$$

The wave potential of a horizontal dipole is [5, 4]

[1] See paragraph following equation (8).

[2] M. J. O. Strutt, *Ann. d. Phys.*, Bd. 1, p. 721, 1929.

[3] H. Weyl, *Ann. d. Phys.*, Bd. 60, p. 481, 1919.

[4] "Electrical and Optical Wave Motion," pp. 73–75.

[5] H. v. Hoerschelmann, *Jahrb. d. draht. Teleg.*, Bd. 5, 1912, pp. 14–188.

$$\Pi^h = i \times \left(- \frac{e^{ik_1R_1}}{R_1} + \frac{e^{ik_1R_2}}{R_2} - \int_0^\infty \frac{2}{l+m} J_0(\tau\rho) e^{-wl} \tau d\tau \right) + j \times 0$$
$$+ k \times 2 \cos \varphi \int_0^\infty \frac{k_2^2 - k_1^2}{(l+m)(k_2^2 l + k_1^2 m)} J_1(\tau\rho) e^{-wl} \tau^2 d\tau. \quad (1)$$

$- \dfrac{e^{ik_1R_1}}{R_1} + \dfrac{e^{ik_1R_1}}{R_2} = \Pi_x^\infty$ is the wave potential of a dipole placed parallel to a perfectly conducting plane, k_2 infinite. So, writing

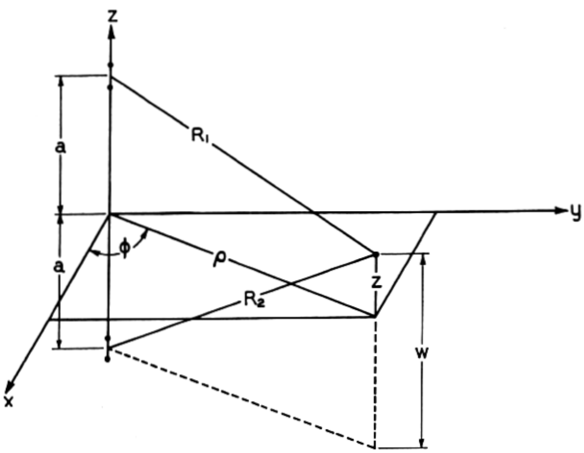

Fig. 1

$\Pi^h = \Pi_x^\infty + \Pi^\Delta$ and correspondingly $E^h = E^\infty + E^\Delta$

$$E_z^\Delta = - i\omega \left[\Pi_z^\Delta + k_1^{-2} \frac{\partial}{\partial z} \left(\frac{\partial}{\partial x} \Pi_x^\Delta + \frac{\partial}{\partial y} \Pi_y^\Delta + \frac{\partial}{\partial z} \Pi_z^\Delta \right) \right]$$
$$= - i\omega 2 \cos \varphi \frac{1}{k_1^2} \int_0^\infty J_1(\tau\rho) \frac{e^{-wl}}{l+m} \left(\frac{(k_2^2 - k_1^2)\tau^2}{k_2^2 l + k_1^2 m} - l \right) \tau^2 d\tau$$
$$= - i\omega 2 \cos \varphi \int_0^\infty J_1(\tau\rho) e^{-wl} \frac{- m\tau^2}{k_2^2 l + k_1^2 m} d\tau,$$
$$= - i\omega \frac{2}{k_1^2} \frac{\partial}{\partial x} \int_0^\infty J_0(\tau\rho) e^{-wl} \left(\frac{- k_2^2 l}{k_2^2 l + k_1^2 m} + 1 \right) \tau d\tau$$
$$= - i\omega \frac{1}{k_1^2} \frac{\partial^2}{\partial x \partial w} \left(V - 2 \frac{e^{ik_1R_2}}{R_2} \right), \quad (2)$$

where

$$V = \int_0^\infty \frac{2k_2^2}{k_2^2 l + k_1^2 m} J_0(\tau\rho) e^{-wl} \tau d\tau. \quad (3)$$

When $k_2 = \infty$, $V = 2R_2^{-1} \exp ik_1R_2$ and E_z^Δ is zero. When $k_2 = k_1$, $V = 1R_2^{-1} \exp ik_1R_2$ and E_z^Δ just cancels the field of the image dipole in E^∞.

The wave potential of a vertical dipole is [5, 4]

$$\Pi_z^v = \frac{e^{ik_1R_1}}{R_1} - \frac{e^{ik_1R_2}}{R_2} + V. \tag{4}$$

V is the function Π_0 analysed by Sommerfeld in Riemann-Webers Differentialgleichungen der Physik. Sommerfeld transforms the integration from 0 to ∞ into an integration from $-\infty$ to $+\infty$ by replacing the Bessel function by its equivalent Hankel functions and then wraps the real axis path of integration around the zero of $k_2^2l + k_1^2m$ and the two branch cuts from k_1 and k_2 to $+i\infty$. He thus gets $V = P + Q_1 + Q_2$ where P is the integral around the zero of $k_2^2l + k_1^2m$, Q_1 is the integral around the branch cut from k_1, Q_2 is the integral around the branch cut from k_2 and the function integrated is

$$\frac{k_2^2}{k_2^2l + k_1^2m} H_0^1(\tau\rho)e^{-wl}\tau d\tau. \tag{5}$$

l and m are pure imaginaries on the branch cuts, which are rectangular hyperbolas, from k_1 and k_2 respectively. As one carries τ counterclockwise around the branch cut from k_1 to $+i\infty$, l travels up the right hand side of the imaginary axis from $-i\infty + \epsilon$ to $+i\infty + \epsilon$. A similar statement holds for m.

Multiplying numerator and denominator of equation (5) by $k_2^2l - k_1^2m$ and then taking out a factor $(k_2^4 - k_1^4)^{-1}$ our integrand becomes

$$\frac{k_2^2}{k_2^4 - k_1^4} \frac{k_2^2\sqrt{\tau^2 - k_1^2} - k_1^2\sqrt{\tau^2 - k_2^2}}{(\tau - s)(\tau + s)} H_0^1(\tau\rho)e^{-w\sqrt{\tau^2-k_1^2}}\tau d\tau,$$

where $s = +k_1k_2 \div \sqrt{k_2^2 + k_1^2}$. Integrating around the pole at $\tau = s$, we get

$$\begin{aligned} P &= \frac{k_2^2}{k_2^4 - k_1^4} 2\pi i \frac{k_2^2\sqrt{s^2 - k_1^2} - k_1^2\sqrt{s^2 - k_2^2}}{2s} H_0^1(s\rho)e^{-w\sqrt{s^2-k_1^2}}s \\ &= 0 \quad \text{if} \quad k_2^2\sqrt{s^2 - k_1^2} - k_1^2\sqrt{s^2 - k_2^2} = 0 \\ &= -2\pi \frac{k_2^3k_1s}{k_2^4 - k_1^4} H_0^1(s\rho)e^{-iwsk_1/k_2}, \end{aligned} \tag{6}$$

if

$$k_2^2\sqrt{s^2 - k_1^2} - k_1^2\sqrt{s^2 - k_2^2} = 2k_2^2\sqrt{s^2 - k_1^2} = 2ik_1k_2s.$$

l and m must have their real parts positive and so in taking the square roots of $s^2 - k_1^2 = -k_1^4 \div (k_2^2 + k_1^2)$ and $s^2 - k_2^2 = -k_2^4 \div (k_2^2 + k_1^2)$ one halves the smallest angle with the positive real axis. If they both lie on the same side of the real axis P is zero. In order that they may lie on opposite sides of the real axis it is necessary that

$$\arg k_1^4 < \arg (k_1^2 + k_2^2) < \arg k_2^4.$$

Writing $k_1^2 = \alpha + i\beta$ and $k_2^2 = x + iy$ this means that

$$y > \frac{2\alpha\beta}{\alpha^2 - \beta^2} x + \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2} \beta.$$

The goal of the paper being asymptotic formulas for the sky waves of vertical and horizontal dipoles the ground wave, P, will hereafter be ignored. This is possible because at the high frequencies for which dipoles are useful the ground wave is very highly damped.

Sommerfeld gets an asymptotic expression for Q_1 by noting that if we are at a great distance from the source most of the value of the integral comes from that portion of the path of integration very close to k_1. The solution he arrives at is

$$-2(\Omega + \Omega^2 + \Omega^3 + + \cdots) \frac{e^{ik_1R}}{R} \quad \text{where} \quad \Omega = \frac{k_2^2}{k_1^2\sqrt{k_1^2 - k_2^2}} \frac{\partial}{\partial z}.^6 \qquad (7)$$

Neglecting higher powers of $1/R$ than the first, equation (7) sums up into

$$Q_1 \sim \frac{2k_2^2 \cos \theta_z}{k_2^2 \cos \theta_z + k_1\sqrt{k_2^2 - k_1^2}} \frac{e^{ik_1R}}{R}. \qquad (8)$$

But in getting equation (7) Sommerfeld has replaced $\sqrt{\tau^2 - k_2^2}$ by $\sqrt{k_1^2 - k_2^2}$. This is a needless approximation which ruins the symmetry, damages the utility and tends to hide the physical meaning of the final result. To get the true asymptotic formula for Q_1 it is necessary to confine the approximations to the purely operational variety, i.e. make no approximations of substitution before integrating but let the approximation reside wholly in the manner of integrating, as follows

$$Q_1 = \int \frac{k_2^2}{k_2^2 l + k_1^2 m} H_0^1(\tau\rho) e^{-wl} \tau d\tau$$

$$= -\frac{\partial}{\partial w} \int H_0^1(\tau\rho) \frac{e^{-wl}}{l} (C_0 + C_1 l + C_2 l^2 + \cdots) \tau d\tau$$

[6] *Annalen der Physik*, Band 28, 1909, page 705.

where

$$C_0 + C_1 l + C_2 l^2 + \cdots = k_2^2 \div (k_2^2 l + k_1^2 \sqrt{l^2 + k_1^2 - k_2^2}).$$

To the extent that most of the value of the integral comes from that portion of the path of integration very close to $\tau^2 = k_1^2$ the expansion in powers of l is valid. Replacing each l by $-(\partial/\partial w)$ we have then

$$Q_1 \sim -2 \frac{\partial}{\partial w}\left(C_0 - C_1 \frac{\partial}{\partial w} + C_2 \frac{\partial^2}{\partial w^2} - + \cdots \right) \frac{e^{ik_1 R_2}}{R_2}. \tag{9}$$

Now

$$-\frac{\partial}{\partial w}\frac{e^{ik_1R_2}}{R_2} = (-ik_1 \cos\theta_z)\frac{e^{ik_1R_2}}{R_2} + \frac{\cos\theta_z}{R_2}\frac{e^{ik_1R_2}}{R_2}$$

$$= \left[\gamma + \frac{\cos\theta_z}{R_2}\right]\frac{e^{ik_1R_2}}{R_2} \quad \text{where} \quad \gamma = -ik_1\cos\theta_z,$$

$$\frac{\partial^2}{\partial w^2}\frac{e^{ik_1R_2}}{R_2} = \left(\gamma^2 + 2\gamma^1\frac{\cos\theta_z}{R_2} + 1\cdot 2\gamma^0\frac{ik_1\sin^2\theta_z}{2R_2} + \cdots\right)\frac{e^{ik_1R_2}}{R_2},$$

$$-\frac{\partial^3}{\partial w^3}\frac{e^{ik_1R_2}}{R_2} = \left(\gamma^3 + 3\gamma^2\frac{\cos\theta_z}{R_2} + 2\cdot 3\gamma^1\frac{ik_1\sin^2\theta_z}{2R_2} + \cdots\right)\frac{e^{ik_1R_2}}{R_2},$$

. . . .

etc., and so

$$Q_1 \sim 2(\gamma^1 C_0 + \gamma^2 C_1 + \gamma^3 C_2 + + \cdots)\frac{e^{ik_1R_2}}{R_2}$$

$$+ 2\cos\theta_z(1\gamma^0 C_0 + 2\gamma^1 C_1 + 3\gamma^2 C_2 + + \cdots)\frac{e^{ik_2R_2}}{R_2^2}$$

$$+ ik_1\sin^2\theta_z(1\cdot 2\gamma^0 C_1 + 2\cdot 3\gamma^1 C_2 + 3\cdot 4\gamma^2 C_3 + + \cdots)\frac{e^{ik_1R_2}}{R_2^2}$$

$$+ \text{higher order terms}$$

$$= \frac{2k_2^2\gamma}{k_2^2\gamma + k_1^2\sqrt{\gamma^2 + k_1^2 - k_2^2}}\frac{e^{ik_1R_2}}{R_2}$$

$$+ \left(\cos\theta_z\frac{\partial}{\partial\gamma} + \frac{ik_1\sin^2\theta_z}{2}\frac{\partial^2}{\partial\gamma^2}\right)\frac{2k_2^2\gamma}{k_2^2\gamma + k_1^2\sqrt{\gamma^2 + k_1^2 - k_2^2}}\frac{e^{ik_1R_2}}{R_2^2}$$

$$+ \text{higher order terms.} \tag{10}$$

The second order term can be neglected if $k_1R_2 >> 1$, say if $R > 20\lambda$. It will hereafter be supposed that this is the case.

Multiplying numerator and denominator of the first order term by i and canceling a k_1 we have finally

$$Q_1 \sim \frac{2k_2^2 \cos \theta_z}{k_2^2 \cos \theta_z + k_1\sqrt{k_2^2 - k_1^2 \sin^2 \theta_z}} \frac{e^{ik_1R_2}}{R_2}. \qquad (11)$$

This Q_1 behaves as one would rightfully expect a true asymptotic formula to behave. It is $2(\exp ik_1R_2)/R_2$ at $k_2 = \infty$ and $1(\exp ik_1R_2)/R_2$ at $k_2 = k_1$.

In so far as k_2 is considerably larger than k_1 and the expansion in powers of l is valid Q_2 is negligible in comparison with Q_1.[7] Perhaps the easiest way of seeing this is to note that equation (10) might just as well have been obtained directly from V instead of from Q_1.

Substituting equation (11) in equation (4) we get

$$\Pi_z^v \sim \left(1 + \frac{k_2^2 \cos \theta_z - k_1\sqrt{k_2^2 - k_1^2 \sin^2 \theta_z}}{k_2^2 \cos \theta_z + k_1\sqrt{k_2^2 - k_1^2 \sin^2 \theta_z}} e^{ik_1(R_2 - R_1)}\right) \frac{e^{ik_1R_1}}{R_1}, \qquad (12)$$

whence

$$E_z^v = -i\omega\left(\Pi_z^v + k_1^{-2}\frac{\partial^2}{\partial z^2}\Pi_z^v\right)$$
$$\sim -i\omega \sin^2 \theta_z(1 + R_1 e^{ik_1 2a \cos \theta_z}) \frac{e^{ik_1R}}{R}, \qquad (13)$$

where

$$R_1 = \frac{k_2^2 \cos \theta_z - k_1\sqrt{k_2^2 - k_1^2 \sin^2 \theta_z}}{k_2^2 \cos \theta_z + k_1\sqrt{k_2^2 - k_1^2 \sin^2 \theta_z}}.$$

Substituting equation (11) in equation (2) and adding E_z^∞ we get

$$E_z^h \sim -i\omega \cos \theta_x \cos \theta_z(1 - R_1 e^{ik_1 2a \cos \theta_z}) \frac{e^{ik_1R}}{R}. \qquad (14)$$

R_1 is the coefficient of reflection for a plane wave polarized in the plane of incidence.

The horizontal fields of a horizontal dipole are

$$E_y^h = -i\omega\left[\Pi_y^h + k_1^{-2}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\Pi_x^h + \frac{\partial}{\partial y}\Pi_y^h + \frac{\partial}{\partial z}\Pi_z^h\right)\right]$$
$$= -i\omega k_1^{-2}\left[\frac{\partial^2}{\partial y \partial x}\Pi_x^\infty + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\Pi_x^\Delta + \frac{\partial}{\partial z}\Pi_z^\Delta\right)\right] \qquad (15)$$

and

$$E_x^h = -i\omega\left[\Pi_x^h + k_1^{-2}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\Pi_x^h + \frac{\partial}{\partial y}\Pi_y^h + \frac{\partial}{\partial z}\Pi_z^h\right)\right]$$
$$= -i\omega\left[\Pi_x^\infty + k_1^{-2}\frac{\partial^2}{\partial x^2}\Pi_x^\infty\right]$$

[7] Riemann-Weber's "Differentialgleichungen der Physik," p. 556.

$$- i\omega \left[\Pi_x{}^\Delta + k_1{}^{-2} \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \Pi_x{}^\Delta + \frac{\partial}{\partial z} \Pi_z{}^\Delta \right) \right]$$

$$\sim - i\omega(\Pi_x{}^\infty + \Pi_x{}^\Delta) + \frac{\cos \theta_x}{\cos \theta_y} E_y{}^h$$

$$= - i\omega(\Pi_x{}^\infty + \Pi_x{}^\Delta) + \cot \varphi E_y{}^h; \quad (16)$$

$$\therefore \quad E_\varphi{}^h = E_x{}^h \sin \varphi - E_y{}^h \cos \varphi \sim - i\omega \sin \varphi(\Pi_x{}^\infty + \Pi_x{}^\Delta). \quad (17)$$

Evidently the procedure which yielded the true asymptotic expres-

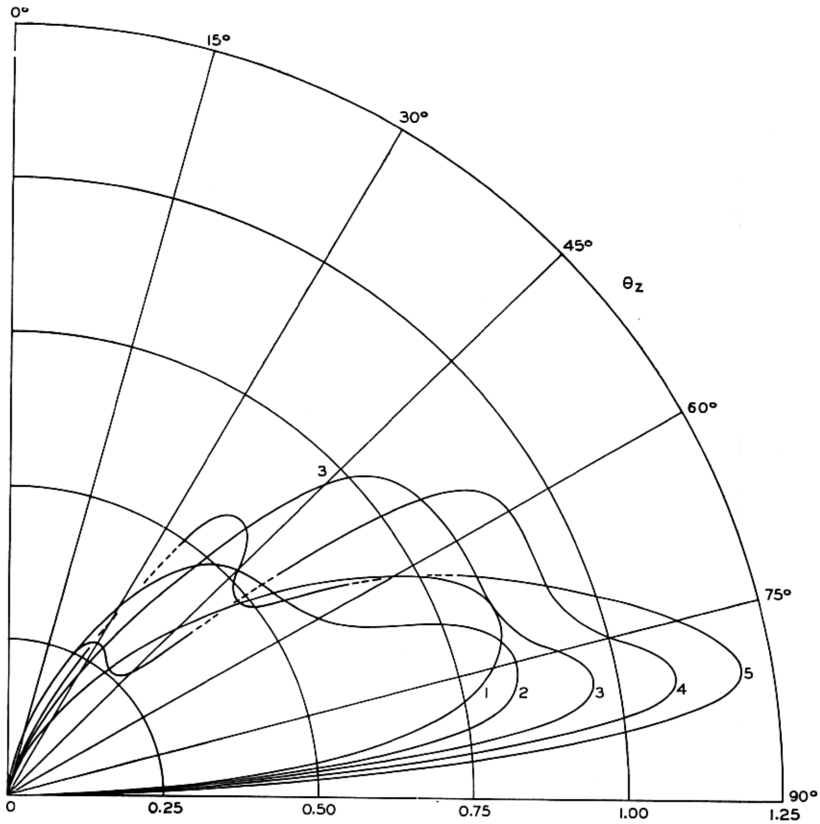

Fig. 2—Vertical dipole polar diagrams computed for $\lambda = 6$ meters, $\epsilon = 9$, $\mu = 1$, and $\sigma = 10^{-13}$

sion for V will do the same for $\Pi_x{}^\Delta$. The details are not interesting. The result is

$$\Pi_x{}^\Delta \sim - \frac{2k_1 \cos \theta_z}{k_1 \cos \theta_z + \sqrt{k_2{}^2 - k_1{}^2 \sin^2 \theta_z}} \frac{e^{ik_1R_2}}{R_2}. \quad (18)$$

Substituting equation (18) in equation (17) we get

$$E_\varphi^h \sim + i\omega \sin \varphi(1 - R_2 e^{ik_1 2a \cos \theta_z}) \frac{e^{ik_1 R}}{R}, \tag{19}$$

where

$$R_2 = \frac{\sqrt{k_2^2 - k_1^2 \sin^2 \theta_z} - k_1 \cos \theta_z}{\sqrt{k_2^2 - k_1^2 \sin^2 \theta_z} + k_1 \cos \theta_z}.$$

R_2 is the coefficient of reflection for a plane wave polarized perpendicular to the plane of incidence.

Formulas (13), (14) and (19) are just what one would get by applying

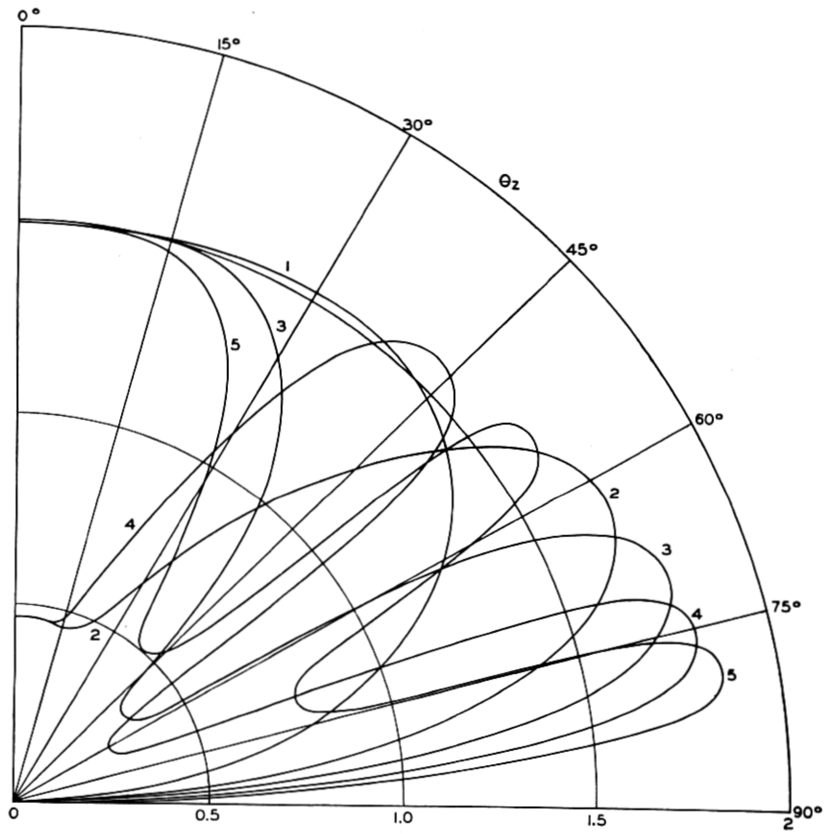

Fig. 3—Polar diagrams of the horizontal fields of horizontal dipoles computed for $\lambda = 6$ meters, $\epsilon = 9$, $\mu = 1$ and $\sigma = 10^{-13}$

the coefficient of reflection from an imperfect reflector to the reflected waves of the corresponding electrostatic formulas.

Returning to equation (15)

$$E_y{}^h = -i\omega k_1{}^{-2}\frac{\partial^2}{\partial y \partial x}\left[\Pi_x{}^\infty - \int_0^\infty \frac{2}{m+l}\left(1-\frac{(k_2{}^2-k_1{}^2)l}{k_2{}^2 l + k_1{}^2 m}\right)J_0(\tau\rho)e^{-wl}\tau d\tau\right]$$

$$= -i\omega\frac{\partial^2}{\partial y \partial x}(k_1{}^{-2}\Pi_x{}^\infty - k_2{}^{-2}V)$$

$$\sim -i\omega \cos\theta_x \cos\theta_y\left(-1+\left[1-\frac{k_1{}^2}{k_2{}^2}(1+R_1)\right]e^{ik_1 2a\cos\theta_z}\right)\frac{e^{ik_1 r}}{R}. \quad (20)$$

Usually one cares only for $E_\varphi{}^h$ at $\varphi = \pi/2$ and $E_x{}^h$ and $E_y{}^h$ are of no particular interest. Figs. 2, 3 and 4 are polar diagrams in the

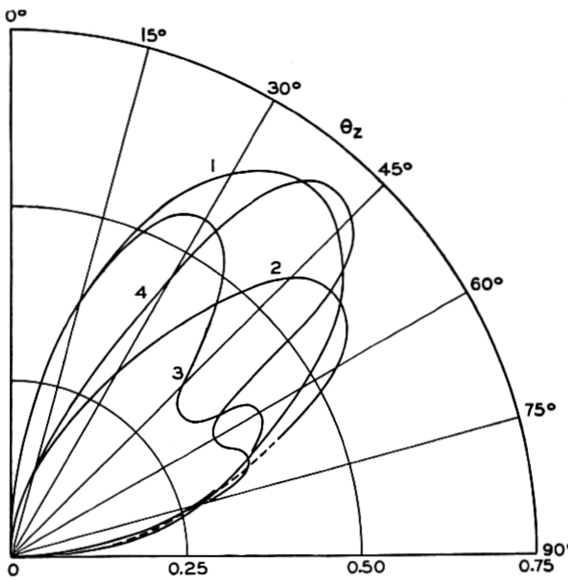

Fig. 4—Polar diagrams of the vertical fields of horizontal dipoles computed for $\lambda = 6$ meters, $\epsilon = 9$, $\mu = 1$ and $\sigma = 10^{-13}$

vertical plane of equations (13), (14) and (19). Assuming the conductivity of the air to be zero and the dielectric constant to be unity

$$k_2{}^2 = k_1{}^2(\epsilon + i2c\lambda\sigma), \qquad k_1 = 2\pi/\lambda,$$

where ϵ is the dielectric constant of the ground referred to air as unity and σ is the conductivity of the ground in electromagnetic units. The values of ϵ and σ used in computing the polar diagrams are generally supposed to be somewhere in the neighborhood of their average values for earth but they vary so much with the locality that the diagrams can scarcely be regarded as giving more than a general idea as to what may be expected of the formulas.

The number attached to each curve is the height of its dipole in quarter wave-lengths.

Formulas (13), (14) and (19) are just what one would get by applying the Reciprocal Theorem to two dipoles, one near the earth and the other far away. The electric field acting on the one near the earth is composed of a direct field and a reflected field which is R_1 or R_2, as the case may be, times the direct field.

When the depth to groundwater, bedrock, an orebody or any marked discontinuity in the electrical properties of the ground is known and is not too great the effect of this discontinuity on the polar diagram ought not to be ignored. The asymptotic formulas for any stratified ground are got by putting the coefficients of reflection for a plane wave reflected from the surface of that ground in place of the corresponding coefficients in equations (13), (14) and (19). For a number of rather obvious reasons it would usually be out of the question to deal with more than one plane of discontinuity; one is bad enough. The coefficients for a single plane of discontinuity at a depth Δ are

$$R_1{}^1 = \frac{k_2{}^2 \cos \theta_z - \eta_1 k_1 \sqrt{k_2{}^2 - k_1{}^2 \sin^2 \theta_z}}{k_2{}^2 \cos \theta_z + \eta_1 k_1 \sqrt{k_2{}^2 - k_1{}^2 \sin^2 \theta_z}}$$

and

$$R_2{}^1 = \frac{\sqrt{k_2{}^2 - k_1{}^2 \sin^2 \theta_z} - \eta_2 k_1 \cos \theta_z}{\sqrt{k_2{}^2 - k_1{}^2 \sin^2 \theta_z} - \eta_2 k_1 \cos \theta_z},$$

where

$$\eta_1 = \frac{\mu_2}{\mu_1} \frac{1 + \delta_1}{1 - \delta_1}, \qquad \eta_2 = \frac{\mu_2}{\mu_1} \frac{1 + \delta_2}{1 - \delta_2},$$

$$\delta_1 = \frac{k_2{}^2 \mu_3 \sqrt{k_3{}^2 - k_1{}^2 \sin^2 \theta_z} - k_3{}^2 \mu_2 \sqrt{k_2{}^2 - k_1{}^2 \sin^2 \theta_z}}{k_2{}^2 \mu_3 \sqrt{k_3{}^2 - k_1{}^2 \sin^2 \theta_z} - k_3{}^2 \mu_2 \sqrt{k_2{}^2 - k_1{}^2 \sin^2 \theta_z}} e^{i 2 \Delta \sqrt{k_2{}^2 - k_1{}^2 \sin^2 \theta_z}},$$

and

$$\delta_2 = \frac{\mu_3 \sqrt{k_2{}^2 - k_1{}^2 \sin^2 \theta_z} - \mu_2 \sqrt{k_3{}^2 - k_1{}^2 \sin^2 \theta_z}}{\mu_3 \sqrt{k_2{}^2 - k_1{}^2 \sin^2 \theta_z} + \mu_2 \sqrt{k_3{}^2 - k_1{}^2 \sin^2 \theta_z}} e^{i 2 \Delta \sqrt{k_2{}^2 - k_1{}^2 \sin^2 \theta_z}}.$$

If Δ is not large and k_3 is considerably different from k_2 then η_1 and η_2 will differ considerably from unity.