A Generalization of Heaviside's Expansion Theorem

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The expansion theorem is one of the most frequently used methods of evaluating operational forms arising from the operational calculus developed by Heaviside. The original theorem, however, is applicable, in general, only to expressions containing integral powers of the operator d/dt. This paper describes an extension to, or a generalization of the original expansion theorem whereby, in general, operational forms with either fractional or integral powers of the operator can be evaluated. A number of operational equivalents are given to be used with the theorem, one of which is the equivalent used by Heaviside. Examples of the application of the theorem to electric circuit problems are shown.

THE well known expansion theorem given by Heaviside in Vol. II of his "Electromagnetic Theory" may be stated as follows:

An operational equation of the form $h = Y(p)/Z(p)$, may under certain well known restrictions on the functions Y and Z, have as its solution

$$h = \frac{Y(0)}{Z(0)} + \sum_n \frac{Y(p_n)}{p_n Z'(p_n)} e^{p_n t}, \qquad n = 1, 2, 3 \cdots. \tag{1}$$

p is the differential operator d/dt, and p_1, p_2 $\cdots$ are the roots of $Z(p) = 0$. $Z'(p_n)$ is the result of substituting p_n for p in $d(Z(p))/dp$. The theorem is true only when no root is zero and all roots are unequal. $Y(p)$ and $Z(p)$ must contain p to positive integral powers only. Various proofs of this theorem have been given and perhaps the simplest depends upon the expansion of $Y(p)/Z(p)$ by partial fractions.

The expansion theorem is valuable in the solution by operational methods, of problems in mathematical physics, and especially electric circuit theory problems.

Generalization of the Expansion Theorem

The generalization of this theorem may be stated as follows: Under certain circumstances it may be possible to write the operational equation

$$h = \frac{Y(p)}{Z(p)} \qquad \text{as} \qquad h = \frac{N(q)}{D(q)},$$

where q is a function of the operator p. With suitable restrictions on the functional forms of N and D the solution of the operational equation is given by

$$h = \frac{N(0)}{D(0)} + \sum_n \frac{N(q_n)}{q_n D'(q_n)} \psi(t, q_n), \tag{2}$$

where $\psi(t, q_n)$ is the equivalent of the operational expression $q/(q - q_n)$

and $q_1, q_2, \cdots$ represent the roots of $D(q) = 0$. If q is the differential operator, that is if

$$q = p = \frac{d}{dt},$$

then as is well known

$$\frac{q}{q - q_n} = \frac{p}{p - p_n} = e^{p_n t}$$

and (2) becomes the Heaviside expression (1).

A proof of the generalized theorem equation (2), is as follows:

By a theorem of partial fractions:

$$\frac{N(q)}{D(q)} = \frac{N(q_1)}{(q - q_1)D'(q_1)} + \frac{N(q_2)}{(q - q_2)D'(q_2)} + \cdots + \frac{N(q_n)}{(q - q_n)D'(q_n)} \tag{3}$$

where $q_1, q_2, \cdots q_n$ are the roots of $D(q) = 0$. The above theorem is true when $D(q)$ and $N(q)$ are rational polynomials and $N(q)$ is of a lower degree than $D(q)$. Further limitations are that no root can be zero and all roots must be unequal.

In writing the above identity in terms of operators it is tacitly assumed that the operators obey the three fundamental laws of algebra, the associative, commutative and distributive laws.

Now

$$\frac{1}{q - q_n} = -\frac{1}{q_n} + \frac{q}{q_n(q - q_n)} \cdot \tag{4}$$

Substituting (4) in (3)

$$\frac{N(q)}{D(q)} = \frac{N(q_1)}{(-q_1)D'(q_1)} + \frac{N(q_2)}{(-q_2)D'(q_2)} + \cdots + \frac{N(q_n)}{(-q_n)D'(q_n)} + \sum_n \frac{N(q_n)}{q_n D'(q_n)}\left(\frac{q}{q - q_n}\right) \tag{5}$$

$$= \frac{N(0)}{D(0)} + \sum_n \frac{N(q_n)}{q_n D'(q_n)} \psi(t, q_n), \tag{6}$$

where

$$\frac{q}{q - q_n} = \psi(t, q_n).$$

The expression fails where $N(0)/D(0)$ is infinite. When the operator

$$q = p = \frac{d}{dt}$$

then

$$\frac{p}{p - p_n} = e^{p_n t}$$

and (6) becomes the Heaviside Expansion theorem.

Although the above proof of (6) is for cases where $D(q)$ is a polynomial, if $D(q)$ is a transcendental function which can be expanded by the process shown, the equation will still hold. It is shown in treatises on trigonometry that tan at, cot at, 1/sin at, 1/cos at, 1/sinh at, 1/cosh at, tanh at, and coth at, all can be expanded in an infinite series of partial fractions which are identical[1] with the expansions obtained by applying the process of equation (3).

Equivalents to be Used in Generalized Theorem

In applying this theorem the following operational equivalents are useful:

Equivalent No. 1:

Let

$$q = p = \frac{d}{dt} \cdot$$

Then

$$\frac{q}{q-a} = \frac{p}{p-a} = e^{at}. \tag{7}$$

This is the equivalent used in the expansion theorem by Heaviside.

Equivalent No. 2:

Let

$$q = p^{1/2} = \left(\frac{d}{dt}\right)^{1/2} \cdot$$

Then

$$\frac{q}{q-a} = \frac{p^{1/2}}{p^{1/2}-a} = e^{a^2t}[1 + \text{erf}\,(at^{1/2})] \tag{8}$$

where

$$\text{erf}\,(at^{1/2}) = \frac{2}{\sqrt{\pi}} \int_0^{at^{1/2}} e^{-\lambda^2} d\lambda.$$

Equivalent No. 3:

Let

$$q = p^{1/s} = \left(\frac{d}{dt}\right)^{1/s} \qquad s = \text{a positive integer.}$$

Then

$$\frac{q}{q-a} = \frac{p^{1/s}}{p^{1/s}-a} = e^{a^st}[1 + \psi_1(t, a) + \psi_2(t, a) + \cdots + \psi_{s-1}(t, a)], \tag{9}$$

where

$$\psi_1(t, a) = \frac{1}{\Gamma(1/s + 1)} \int_0^{at^{1/s}} \epsilon^{-\lambda^s} d\lambda,$$

[1] Except in some cases for the first term $Y(0)/Z(0)$.

$$\psi_2(t, a) = \frac{1}{\Gamma(2/s + 1)} \int_0^{a^2 t^{2/s}} e^{-\lambda^{s/2}} d\lambda,$$

. . . .

$$\psi_{s-1}(t, a) = \frac{1}{\Gamma\left(\frac{s-1}{s} + 1\right)} \int_0^{a^{s-1} t^{(s-1)/s}} \epsilon^{-\lambda^{s/(s-1)}} d\lambda.$$

Equivalent No. 4:

Let

$$q = p^2 = \left(\frac{d}{dt}\right)^2.$$

Then

$$\frac{q}{q-a} = \frac{p^2}{p^2 - a} = \cosh a^{1/2} t. \tag{10}$$

Equivalent No. 5:

Let

$$q = p^3 = \left(\frac{d}{dt}\right)^3.$$

Then

$$\frac{q}{q-a} = \frac{p^3}{p^3 - a} = (1/3)e^{a^{1/3}t} + (2/3)e^{-a^{1/3}t/2} \cos\left(a^{1/3} t \frac{\sqrt{3}}{2}\right). \tag{11}$$

Equivalent No. 6:

Let

$$q = p^4 = \left(\frac{d}{dt}\right)^4.$$

Then

$$\frac{q}{q-a} = \frac{p^4}{p^4 - a} = \tfrac{1}{2} \cosh (a^{1/4} t) + \tfrac{1}{2} \cos (a^{1/4} t). \tag{12}$$

Equivalent No. 7:

Let

$$q = (p + b)^{1/2} = \left(\frac{d}{dt} + b\right)^{1/2}.$$

Then

$$\frac{q}{q-a} = \frac{b}{b - a^2} - \frac{a^2}{b - a^2} e^{(a^2 - b)t} + \frac{a\sqrt{b}}{b - a^2} \operatorname{erf}(\sqrt{bt}) - \frac{a^2}{b - a^2} e^{(a^2-b)t} \operatorname{erf}(a\sqrt{t}).$$

where $b \neq a^2$

Equivalent No. 8:

Let

$$q = \left(\frac{p}{p + b}\right)^{1/2}.$$

Then

$$\frac{q}{q-a} = \frac{1}{1-a^2}\left[e^{a^2bt/(1-a^2)} + ae^{-(bt/2)}I_0\left(\frac{bt}{2}\right) + \frac{ab}{(1-a^2)^2} e^{a^2bt/(1-a^2)} \int_0^t e^{(a^2+1)bt/2(a^2-1)} I_0\left(\frac{bt}{2}\right) dt \right]$$

where $a^2 \neq 1$

and $I_0\left(\frac{bt}{2}\right) = J_0\left(\frac{ibt}{2}\right) =$ Bessel Function of the first kind.

The above equivalents can be obtained by known operational methods and their derivation will not be given here.

In the application of the generalized theorem to electrical problems, equivalents No. 1, No. 2 and No. 3, especially No. 1 and No. 2, are the ones which will be most frequently used. Equivalents No. 4, No. 5, and No. 6, since they involve only integral powers of p are of use in reducing the labor of applying the original expansion theorem to expressions containing only these powers of the operator p or multiples of these powers. Their use in such cases is illustrated by example No. 3 below.

Equivalent No. 7 enables expressions like the following to be evaluated in closed form.

$$\frac{1}{p + c(p+b)^{1/2} + d}, \qquad \frac{1}{(p+b)^{3/2} + cp + d},$$

$$\frac{(p+b)^{1/2}}{(p+b)^{3/2} + cp + d}, \qquad \frac{1}{\cosh (p+b)^{1/2}}, \quad \text{etc.}$$

In applying equivalents No. 2 and No. 7 some of the following properties of the error function are often conveniently used.

$$\operatorname{erf}(-t) = -\operatorname{erf}(t) \qquad \text{and} \qquad \operatorname{erf}(it) = \frac{2i}{\sqrt{\pi}}\int_0^t e^{\lambda^2} d\lambda;$$

also

$$\frac{d}{dt}\operatorname{erf}[\psi(t)] = \frac{2}{\sqrt{\pi}} e^{-[\psi(t)]^2}\psi'(t)$$

and

$$\frac{d}{dt}\operatorname{erf}(at^{1/2}) = \frac{e^{-a^2t}}{\sqrt{\pi}} at^{-1/2}.$$

The value of erf (t) for different values of t may be obtained from tables of the probability integral as for example Pierce Table of Integrals.

The values of erf (it) for values of t from .01 to 2 are given in a table in London Mathematical Society Vol. 29, 1897–98, page 519. The values of erf $(te^{i\pi/4})$ and erf $(te^{-i\pi/4})$ are given by the following formulæ:

$$\text{erf}\,(te^{i\pi/4}) = \sqrt{2i}[C(t\sqrt{2/\pi}) - iS(t\sqrt{2/\pi})], \tag{14}$$

$$\text{erf}\,(te^{-i\pi/4}) = \sqrt{2i}[-iC(t\sqrt{2/\pi}) + S(t\sqrt{2/\pi})], \tag{15}$$

where

$$C(t\sqrt{2/\pi}) = \int_0^{t\sqrt{2/\pi}} \cos\left(\frac{\pi t^2}{2}\right) dt,$$

and

$$S(t\sqrt{2/\pi}) = \int_0^{t\sqrt{2/\pi}} \sin\left(\frac{\pi t^2}{2}\right) dt,$$

and

$$C(-it\sqrt{2/\pi}) = -iC(t\sqrt{2/\pi})$$

and

$$S(-it\sqrt{2/\pi}) = iS(t\sqrt{2/\pi}).$$

Tables of the values of these two integrals known as the Fresnel Integrals are given in various handbooks such as Jahnke and Emde.

Examples of Application of Theorem

A few applications of the theorem will be given.

Example 1:

The operational solution for the current entering an infinitely long ideal cable with a given impressed voltage of the form Ee^{-at} is

$$K\frac{p^{3/2}}{p+a},$$

K being a constant and

$$p = \frac{d}{dt}.$$

To evaluate $p^{3/2}/(p+a)$ in closed form call $p^{1/2} = q$ then

$$\frac{p^{3/2}}{p+a} = \frac{q^3}{q^2+a}.$$

Since the theorem applies in general only when the degree of the numerator is less than that of the denominator we will write

$$\frac{q^3}{q^2+a} = q - \frac{aq}{q^2+a}$$

and

$$\frac{q}{q^2 + a} = \frac{Y(q)}{Z(q)} = \frac{Y(0)}{Z(0)} + \sum_n \frac{Y(q_n)}{q_n Z'(q_n)} \psi(q_n, t),$$

$$\frac{Y(0)}{Z(0)} = 0. \qquad \frac{Y(q_n)}{q_n Z'(q_n)} = \frac{q_n}{2q_n^2} = \frac{1}{2q_n},$$

$$Z(q) = q^2 + a \text{ and the roots of } Z(q) = 0 \text{ are } q_n = \pm ia^{1/2},$$

$$\psi(q_n, t) = e^{q_n^2 t}[1 + \operatorname{erf} q_n t^{1/2}] \qquad \text{(see equivalent No. 2).}$$

So

$$\frac{q}{q^2 + a} = \frac{1}{2ia^{1/2}} e^{-at}[1 + \operatorname{erf}(ia^{1/2}t^{1/2})] - \frac{1}{2ia^{1/2}} e^{-at}[1 + \operatorname{erf}(-ia^{1/2}t^{1/2})]$$

$$= \frac{1}{ia^{1/2}} e^{-at} \operatorname{erf}(ia^{1/2}t^{1/2}).$$

Hence

$$\frac{q^3}{q^2 + a} = q - \frac{aq}{q^2 + a} = \frac{t^{-1/2}}{\Gamma(\tfrac{1}{2})} - \frac{a^{1/2}}{i} e^{-at} \operatorname{erf}(ia^{1/2}t^{1/2}),$$

since

$$q = p^{1/2} = \left(\frac{d}{dt}\right)^{1/2} = \frac{t^{-1/2}}{\Gamma(\tfrac{1}{2})}.$$

Example 2:

The operational expression for the current entering at time t in a cable of distributed resistance R and capacity C with an electromotive force $\sin \omega t$ impressed is given by

$$I = \sqrt{C/R}\, \frac{\omega p^{3/2}}{p^2 + \omega^2}$$

where

$$p = \frac{d}{dt}.$$

Put $q = p^{1/2}$. Then

$$\frac{p^{3/2}}{p^2 + \omega^2} = \frac{q^3}{q^4 + \omega^2}.$$

Here

$$\frac{Y(0)}{Z(0)} = 0, \qquad \frac{Y(q_n)}{q_n Z'(q_n)} = \frac{q_n^3}{4q_n^4} = \frac{1}{4q_n}.$$

If

$$q^4 + \omega^2 = 0, \quad q_n = \omega^{1/2}e^{i(\pi/4)}, \quad \omega^{1/2}e^{-i(\pi/4)}, \quad -\omega^{1/2}e^{i(\pi/4)}, \quad -\omega^{1/2}e^{-i(\pi/4)}.$$

So

$$\sqrt{C/R}\,\frac{\omega p^{3/2}}{p^2+\omega^2} = \omega\sqrt{C/R}\sum_n \frac{1}{4q_n} e^{q_n^2 t}[1+\operatorname{erf}(q_n t^{1/2})]$$

$$= \omega\sqrt{C/R}\Bigg[\frac{1}{4\omega^{1/2}e^{i(\pi/4)}} e^{i\omega t}\{1+\operatorname{erf}(\omega^{1/2}e^{i(\pi/4)}t^{1/2})\}$$

$$+\frac{1}{4\omega^{1/2}e^{-i(\pi/4)}} e^{-i\omega t}\{1+\operatorname{erf}(\omega^{1/2}e^{-i(\pi/4)}t^{1/2})\}$$

$$-\frac{1}{4\omega^{1/2}e^{i(\pi/4)}} e^{i\omega t}\{1+\operatorname{erf}(-\omega^{1/2}e^{i(\pi/4)}t^{1/2})\}$$

$$-\frac{1}{4\omega^{1/2}e^{-i(\pi/4)}} e^{-i\omega t}\{1+\operatorname{erf}(-\omega^{1/2}e^{-i(\pi/4)}t^{1/2})\}\Bigg]$$

$$= \frac{\omega^{1/2}}{2}\sqrt{C/R}[e^{i(\omega t-\pi/4)}\operatorname{erf}(\omega^{1/2}e^{i(\pi/4)}t^{1/2})$$

$$+ e^{-i(\omega t-\pi/4)}\operatorname{erf}(\omega^{1/2}e^{-i(\pi/4)}t^{1/2})]$$

$$= \sqrt{\frac{2C\omega}{R}}[\sin(\omega t)S(\omega^{1/2}t^{1/2}\sqrt{2/\pi})$$

$$+\cos(\omega t)C(\omega^{1/2}t^{1/2}\sqrt{2/\pi})].$$

The last transformation is obtained by means of formulæ 14 and 15.

Example 3:

Evaluate

$$y = \frac{1}{p^4-3p^2+2}.$$

This can be solved by the expansion theorem in the usual way. A somewhat shorter method is to use the generalized theorem with the operator $q = p^2$. Then

$$y = \frac{1}{q^2-3q+2} = \tfrac{1}{2} + \sum_n \frac{1}{2q_n^2-3q_n}\psi(q_n, t)$$

$$q_n = 1, 2;\ \psi(q_n, t) = \cosh p_n^{1/2}t.$$

See equivalent No. 4. So

$$y = \tfrac{1}{2} + \tfrac{1}{2}\cosh t\sqrt{2} - \cosh t.$$

Example 4:

Evaluate

$$\frac{\sinh bp^{1/2}}{\sinh ap^{1/2}} = \frac{\sinh bq}{\sinh aq} = \frac{b}{a} + \sum \frac{\sinh bq_n}{aq_n\cosh aq_n} e^{q_n^2 t}[1+\operatorname{erf}(q_n t^{1/2})],$$

$$-\sinh aq = i\sin iaq.$$

The roots[1] of sin $iaq = 0$ are

$$q_n = \frac{n\pi}{ia}, \qquad n = \pm 1, \pm 2, \qquad \text{etc.}$$

Substituting these values of q_n in above we get

$$\frac{\sinh bq}{\sinh aq} = \frac{b}{a} + \sum_{\substack{n=\pm 1\\ n=\pm 2\\ \text{etc.,}}} \frac{\sinh \frac{n\pi b}{ia}}{\frac{n\pi}{i}\cosh\frac{n\pi}{i}} e^{-(n^2\pi^2/a^2)t}\left[1 + \text{erf}\left(\frac{n\pi}{ia}t^{1/2}\right)\right]$$

$$= \frac{b}{a} + \frac{2}{\pi}\sum_{n=1,2,3,\cdots} \frac{(-1)^n \sin\frac{n\pi b}{a}}{n} e^{-(n^2\pi^2/a^2)t}.$$

If $(\sinh bp^{1/2})/(\sinh ap^{1/2})$ is solved by the expansion theorem and the summation is extended over both positive and negative roots, the result is

$$\frac{b}{a} + \frac{4}{\pi}\sum_{n=1,2,3,\cdots} (-1)^n \frac{\sin\frac{n\pi b}{a}}{n} e^{-(n^2\pi^2/a^2)t}.$$

In other words the summation quantity is just double what it should be. In order to correct this in practice, those who have used the theorem for such cases have extended the summation only over the positive roots, notwithstanding the fact that in similar cases with integral exponents such as, for example, $1/\cosh ap$ the summation is extended over all the roots. The truth is the original expansion theorem is not applicable if either numerator or denominator contains p to a fractional form. In the above case were the problems to evaluate $(\sinh bp^{2/3}/\sinh ap^{2/3})$ the expansion theorem gives an entirely incorrect answer, while the correct answer is obtained from the extension to the theorem.

Example 5.

$$\frac{p^{1/3}}{p^{2/3} - 1} = \frac{q}{q^2 - 1} \qquad \sum \frac{q_n}{2q_n^2}\psi(t, q_n).$$

Here

$$\psi(t, q_n) = e^{q_n^3 t}\left[1 + \frac{1}{\Gamma(4/3)}\int_0^{q_n t^{1/3}} e^{-\lambda^3}d\lambda + \frac{1}{\Gamma(5/3)}\int_0^{q_n^2 t^{2/3}} e^{-\lambda^{3/2}}d\lambda\right],$$

$$q_n = \pm 1,$$

[1] Excluding the root $q_n = 0$ which is not used in this case.

$$\frac{q}{q^2-1} = \tfrac{1}{2}e^t\left[1+\frac{1}{\Gamma(4/3)}\int_0^{t^{1/3}} e^{-\lambda^3}d\lambda + \frac{1}{\Gamma(5/3)}\int_0^{t^{2/3}} e^{-\lambda^{3/2}}d\lambda\right]$$

$$-\tfrac{1}{2}e^{-t}\left[1+\frac{1}{\Gamma(4/3)}\int_0^{-t^{1/3}} e^{-\lambda^3}d\lambda + \frac{1}{\Gamma(5/3)}\int_0^{t^{2/3}} e^{-\lambda^{3/2}}d\lambda\right]$$

$$= \sinh t + \frac{\sinh t}{\Gamma(5/3)}\int_0^{t^{2/3}} e^{-\lambda^{3/2}}d\lambda + \tfrac{1}{2}\frac{e^t}{\Gamma(4/3)}\int_0^{t^{1/3}} e^{-\lambda^3}d\lambda$$

$$-\tfrac{1}{2}\frac{e^{-t}}{\Gamma(4/3)}\int_0^{-t^{1/3}} e^{-\lambda^3}d\lambda.$$

Example 6:

If the problem is to evaluate

$$\frac{\sinh b(p+c)^{1/2}}{\sinh a\,(p+c)^{1/2}}$$

Equivalent No. 7 is used. The details will not be worked out since they are quite similar to Example No. 4. The answer is

$$\frac{\sinh b(p+c)^{1/2}}{\sinh a(p+c)^{1/2}} = \frac{b}{a}$$

$$+\frac{2}{\pi}\sum_{n=1,2,3\ldots}(-1)^n\frac{\sin\left(\frac{n\pi b}{a}\right)}{n}\left[\frac{c}{c+\frac{n^2\pi^2}{a^2}} + \frac{n^2\pi^2}{a^2c+n^2\pi^2}e^{-[(n^2\pi^2/a^2)+c]t}\right].$$

If $c = 0$ the above equivalent reduces to the answer of Example No. 4.

Final Remarks

Operational methods were used by Euler and other mathematicians prior to Heaviside. Their use, however, depended in general upon a formal definition of the operator. Heaviside, on the other hand, adopted a different procedure. In the differential equation of the problem he replaced the operator d/dt by p and obtained the solution of the resulting algebraic equation. He then determined the significance of the operator by the condition that it should give the complete solution of the original differential equation subject to equilibrium boundary condition.

While Heaviside developed the operational calculus in a fairly workable and complete form he failed to correlate it or reconcile it with conventional mathematics or to put its theorems on a rigorous basis. The development since Heaviside's day has been due to a considerable extent to the engineer and mathematical physicist rather than to the pure mathematician.

There are now available a number of methods of evaluating operational forms, among which may be mentioned

The original Heaviside expansion theorem,
Operational Division which gives a series solution,
Contour Integration of the Bromwich-Fourier Integral,
Carson's Integral Equation.

It is thought that this extension to the expansion theorem will be of value as another way of evaluating in closed form certain operational expressions, especially those involving fractional exponents.

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