A New Method for Obtaining Transient Solutions of Electrical Networks

By W. P. MASON

SYNOPSIS: A new method for obtaining transient solutions of electrical networks is developed in this paper which depends upon the fact that a distortionless line can be made to approach as a limit all three of the circuit elements, resistance, inductance and capacity. The process of solution consists in solving for the current in a distortionless line—which is ordinarily a simple process—and then proceeding to the limiting case of the distortionless line which approaches the element or elements of interest. Some examples are worked out and a derivation of the Laplacian integral solution is given. It is interesting to note that this method gives a formal solution of the Laplacian integral equation.

THE following paper sets forth a new method for obtaining the transient solutions of electrical networks, which it is believed has some advantages over other methods of solution, in that the operations required for solution are quite simple, and also because this method presents a more definite physical picture of the processes involved. By means of this method, the current at any time t can be obtained, due to an applied voltage which is zero when t is less than zero, and is $E_0 \cos(\omega t + \theta)$ when t is greater than zero. This type of voltage includes as a special case the applied voltage, which is zero when t is less than zero, and is unity when t is greater than zero, and hence the solutions obtained by this method reduce to the cases discussed by Heaviside,[1] when ω and θ are taken equal to zero.

This method gives directly the more compact Laplacian integral equation solution, first obtained by Carson, and in addition gives a method for solving this integral equation, if its solution is not already known.

I. METHOD OF SOLUTION

All practical schemes for solving the transient type of circuit problem, including the Laplacian integral equation, and the generalized Fourier integral solution, are made to depend on the known and easily determined steady state solution. This implies that all circuits which have the same steady state solution, have also the same transient or time solution. The method described in the present paper rests on the same basis.

The method of solution used here depends upon the fact that the

[1] Heaviside, "Electromagnetic Theory," Volume II.

distortionless line can be made to approach as a limit, all three of the electrical elements, resistance, inductance, and capacity, and that the complete solution for the current in a distortionless line can be obtained by adding the incident current and the sum of the reflected currents which can occur up to the time of interest. That is the distortionless line has a true velocity of propagation, and hence the current at any time will be the initial current and the sum of the reflections which can occur up to the time of interest. All of the three electrical elements, resistance, inductance, and capacity, can be considered as limiting cases of the distortionless line. Hence the process of solution consists in solving for the current in the distortionless line, and then proceeding to the limiting value of the line which coincides with the element of interest.

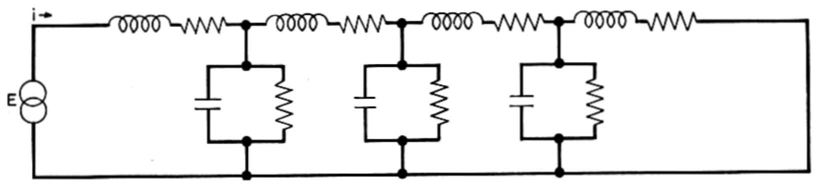


Fig. 1-A.

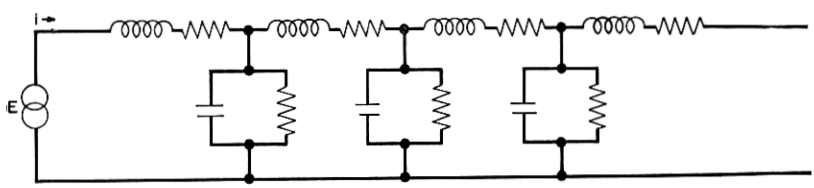


Fig. 1-B.
Diagrammatic representations of lines.

A. The Distortionless Line

Since the distortionless line is the tool by means of which problems are solved by this method, a brief discussion of lines [2] is given here. If a voltage is suddenly applied to a transmission line, the current at any point in the line is zero for a certain time and then begins to build up to its final or steady state value. If there is no distributed inductance in the line, the current begins to build up immediately.

For a distortionless line, however, the current is zero for the time required to propagate the wave to the point of interest and then instantaneously reaches its steady state value. To show this let us consider the equations for a transmission line. A line has distributed

[2] For a more complete discussion of lines see "Transmission Circuits for Telephone Communication," K. S. Johnson, page 144.

series resistance R_u and inductance L_u, and distributed shunt capacity C_u and leakage conductance G_u, as shown on Fig. 1-A, where the letter u indicates the values per unit length. If i denotes the current and v the voltage at a distance x from one end of the line, the well known differential equations are

$$\left.\begin{aligned}\left(L_u \frac{d}{dt} + R_u\right) i &= -\frac{\partial}{\partial x} v,\\ \left(C_u \frac{d}{dt} + G_u\right) v &= -\frac{\partial}{\partial x} i.\end{aligned}\right\} \tag{1}$$

If we eliminate v from these two equations we have

$$L_u C_u \frac{d^2 i}{dt^2} + (R_u C_u + G_u L_u) \frac{di}{dt} + R_u G_u i = \frac{\partial^2 i}{\partial x^2}. \tag{2}$$

Similarly, if i is eliminated, the resulting equation is the same as (2) with i replaced by v. Since we are dealing with simple harmonic forces, the current i varies as $\cos(\omega t + \theta)$ where ω is 2π times the frequency of vibration and θ is an arbitrary phase angle. It is usually more convenient to consider i as varying according to the time factor

$$i = \dot{i} e^{j(\omega t + \theta)} = \dot{i}[\cos(\omega t + \theta) + j \sin(\omega t + \theta)],$$

where $\dot{i}$ is the maximum amplitude of i. The solution obtained on this assumption is called the symbolic solution, and the real solution is obtained from the symbolic solution by taking the real part. Substituting the symbolic form of i in equation (2), this equation reduces to

$$[(j\omega)^2 L_u C_u + j\omega[R_u C_u + G_u L_u] + R_u G_u] i = \frac{\partial^2 i}{dx^2}. \tag{3}$$

The solution for a line can be specified in terms of two parameters, the characteristic impedance and the propagation constant of the line. To show this we note that the solution of (3) is

$$i = Ae^{-\Gamma x} + Be^{\Gamma x}, \tag{4}$$

where $\Gamma^2 = [R_u + j\omega L_u][G_u + j\omega C_u]$ and A and B are constants. From the last of equations (1) we have

$$v = -\frac{\frac{\partial}{\partial x} i}{G_u + j\omega C_u} = \frac{\Gamma}{G_u + j\omega C_u}[Ae^{-\Gamma x} - Be^{\Gamma x}]. \tag{5}$$

When $x = 0$, $i = i_0$, and $v = v_0$. From (4) and (5) solving for A and B we have

$$A = \frac{i_0}{2} + \frac{v_0/2}{\sqrt{\frac{R_u + j\omega L_u}{G_u + j\omega C_u}}}; \qquad B = \frac{i_0}{2} - \frac{v_0/2}{\sqrt{\frac{R_u + j\omega L_u}{G_u + j\omega C_u}}}.$$

Substituting these values in (4) and (5), we have the equations

$$\begin{aligned} i &= i_0 \cosh \Gamma x - \frac{v_0 \sinh \Gamma x}{\sqrt{\frac{R_u + j\omega L_u}{G_u + j\omega C_u}}}, \\ v &= v_0 \cosh \Gamma x - i_0 \sqrt{\frac{R_u + j\omega L_u}{G_u + j\omega C_u}} \sinh \Gamma x. \end{aligned} \tag{6}$$

In this equation $\Gamma x = P$, the propagation constant of the line, and

$$\sqrt{\frac{R_u + j\omega L_u}{G_u + j\omega C_u}} = Z_0,$$

the characteristic impedance of the line. If we are interested in a given length of line l, the parameters are

$$P = \sqrt{(R + j\omega L)(G + j\omega C)}; \qquad Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}, \tag{7}$$

where R, L, G, and C are the total distributed constants for the length of line considered. The characteristic impedance is the impedance looking into a line of infinite length as can readily be seen from either of equations (6) by letting x, the length, approach infinity. In this case $\cosh \Gamma x = \cosh P$ approaches $\sinh P$ and both approach infinity. Then from (6)

$$\frac{v_0}{i_0} = \left(\frac{\cosh P - i/i_0}{\sinh P}\right) Z_0 \to Z_0 \quad \text{when} \quad x \to \infty,$$

since i/i_0 can never be larger than 1.

The physical significance of the propagation constant is that e^{-P} represents the ratio of currents or voltages at the two ends of the line when the line is connected to an infinite line of the same characteristic impedance. To show this, suppose we terminate the section considered in an infinite line, which as we have seen above has an impedance Z_0. Then in equations (6), $v = v_1$ the output voltage and

$i = i_1$ the output current. We let $v_1 = i_1 Z_0$. Eliminating either v_0 or i_0, we have

$$v_1 = v_0 e^{-P} \qquad \text{or} \qquad i_1 = i_0 e^{-P}. \tag{8}$$

In the following work it is necessary to know the impedance of a short circuited line and that of an open circuited line. For the short circuited line, the voltage at the far end is zero, so putting $v = 0$ in the last of equation (6), we have

$$v_0/i_0 = Z_0 \tanh P. \tag{9}$$

For the open circuited line we put $i = 0$, obtaining

$$v_0/i_0 = Z_0 \coth P. \tag{10}$$

So far we have discussed the general transmission line. For the distortionless line there exists the relation

$$\frac{R}{L} = \frac{G}{C}. \tag{11}$$

Substituting this relation in equation (7), these parameters reduce to

$$Z_0 = R_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{R}{G}} \quad \text{and} \quad P = \sqrt{RG} + j\omega\sqrt{LC} = A + j\omega D. \tag{12}$$

This equation shows that the characteristic impedance becomes a resistance R_0, while the propagation constant equals a real constant A plus $j\omega$ times the constant D. To show how wave transmission takes place in an infinite line, these values are substituted in equation (8), giving

$$v_1 = v_0 e^{-(A+j\omega D)}; \qquad i_1 = i_0 e^{-(A+j\omega D)}.$$

To find the real solution, we take the real part of this symbolic solution, obtaining

$$v_1 = \bar{v}_0 e^{j(\omega t+\theta)} e^{-(A+j\omega D)} = \bar{v}_0 e^{-A} e^{j[\omega(t-D)+\theta]},$$

or, taking the real part,

$$\begin{aligned} v_1 &= \bar{v}_0 e^{-A} \cos\left[\omega(t - D) + \theta\right] \\ \text{and} \qquad i_1 &= \bar{i}_0 e^{-A} \cos\left[\omega(t - D) + \theta\right], \end{aligned} \tag{13}$$

where the dash over v_0 and i_0 indicate the maximum amplitude of these quantities. Since $v_0 = \bar{v}_0 \cos(\omega t + \theta)$, these equations show that either v_1 or i_1 has the same form as v_0 or i_0 respectively, attenuated by a factor e^{-A} and delayed in time by an amount D.

B. Condition for Obtaining Lumped Constants from a Distortionless Line

In a distortionless line there is only one necessary relation between the constants, equation (11). Hence, we are at liberty to vary the constants subject only to this relation. In the following work we wish to make the distortionless line degenerate into resistances, inductances, and capacities, or combinations of these.

For example, suppose that we wish to obtain a resistance from a distortionless line. To obtain this we take a short circuited line as shown on Fig. 1-A and let R be finite, $L \to 0$, $G \to 0$, and $C \to 0^2$ in order to satisfy equation (11). The shunt elements will all vanish and the series inductance disappears, leaving only the series resistance in the line. Since the line is short circuited, the line degenerates into a resistance. The line parameters for this case become

$$R_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}} \to \infty^{1/2}; \; P = \sqrt{RG} + j\omega\sqrt{LC} \to (0^{1/2} + j\omega 0^{3/2})$$

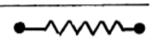
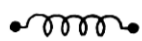
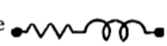
and

$$R_0 P = \sqrt{\frac{R}{G}}(\sqrt{RG}) + \sqrt{\frac{L}{C}}(j\omega\sqrt{LC}) \to R. \tag{14}$$

There are three combinations of lumped elements which can be obtained from the short circuited line and three combinations which can be obtained from the open circuited line. These are listed in the following table, together with the resulting line parameters.

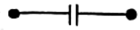
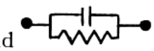
LIMITING CASES OF THE DISTORTIONLESS LINE

A. Limiting Values with Short Circuited Line

Equation	Assumed Line Constants				Resulting Line Parameters			Resulting Lumped Element
	R	L	G	C	R_0	P	R_0P	
(14)	Finite	0	0	0^2	$\infty^{1/2}$	$0^{1/2} + j\omega 0^{3/2}$	R	Resistance 
(15)	0	Finite	0^2	0	$\infty^{1/2}$	$0^{3/2} + j\omega 0^{1/2}$	$j\omega L$	Inductance 
(16)	Finite	Finite	0	0	$\infty^{1/2}$	$0^{1/2} + j\omega 0^{1/2}$	$R+j\omega L$	Resistance and Inductance 

The first three cases result from the suppression of the shunt elements in the short circuited line, and the line parameters are characterized by $R_0 \to \infty$; $P \to 0$. The last three cases result from the suppression of the series elements in the open circuited line, and are characterized by $R_0 \to 0$; $P \to 0$.

B. Limiting Values with Open Circuited Line

Equation	Assumed Line Constants				Resulting Line Parameters			Resulting Lumped Element
	R	L	G	C	R_0	P	R_0/P	
(17)	0	0^2	Finite	0	$0^{1/2}$	$0^{1/2} + j\omega 0^{3/2}$	$1/G$	Resistance 
(18)	0^2	0	0	Finite	$0^{1/2}$	$0^{3/2} + j\omega 0^{1/2}$	$1/j\omega C$	Capacity 
(19)	0	0	Finite	Finite	$0^{1/2}$	$0^{1/2} + j\omega 0^{1/2}$	$1/(G+j\omega C)$	Resistance and Capacity 

C. Solution for a Resistance and Inductance in Series

As a first example of a transient solution obtained by this method let us consider the case of an inductance and resistance in series with a source of alternating voltage. To solve this problem, consider the case of a voltage in series with a distortionless line, short circuited, as shown on Fig. 1-*A*. The current immediately flowing on application of the voltage will be i_0 where

$$i_0 = \frac{E}{R_0}. \tag{20}$$

This current is transmitted down the line and completely reflected at the far end, returning to the near end. The first reflected current entering the generator is then

$$i_1 = i_0 e^{-2P}, \tag{21}$$

where P is the propagation constant of the line. Upon reaching the near end, the current is completely reflected in the same phase and again enters the line. At the end of the first reflection, the current entering the line is

$$i = i_0(1 + 2e^{-2P}). \tag{22}$$

After $(n - 1)$ reflections and passages through the line, the current is

$$\begin{aligned} i &= i_0[1 + 2e^{-2P} + 2e^{-4P} + \cdots + 2e^{-2(n-1)P}] \\ &= i_0\left[2\left(\frac{1 - e^{-2nP}}{1 - e^{-2P}}\right) - 1\right]. \end{aligned} \tag{23}$$

Now the time at which the nth reflection occurs will be

$$t = n(2D),$$

where D is the time of delay in passing the network once. For a distortionless line

$$D = \sqrt{LC}. \tag{24}$$

Hence, we can replace n by

$$n = \frac{t}{2D} = \frac{t}{2\sqrt{LC}}. \tag{25}$$

Since $D \to 0$ and $n \to \infty$, the time scale becomes continuous. In equation (23) we insert the values given in (16) and (25), appropriate to the limiting case considered here, namely

$$P = \frac{R + j\omega L}{R_0}; \qquad R_0 \to \infty; \qquad n = \frac{t}{2\sqrt{LC}} = \frac{tR_0}{2L}$$

and note that $2P \to 0$ so that $e^{-2P} \to 1 - 2P$; then

$$i \to \frac{E}{R_0}\left[\frac{2(1 - e^{-tR_0/L[(R+j\omega L)/R_0]})}{1 - 1 + 2(R + j\omega L)/R_0} - 1\right]$$

$$= E\left[\frac{1 - e^{-t(R/L+j\omega)}}{R + j\omega L} - \frac{1}{R_0}\right].$$

But $R_0 \to \infty$ and hence the solution is

$$i = E\left[\frac{1 - e^{-t(R/L+j\omega)}}{R + j\omega L}\right]. \tag{26}$$

This is the symbolic or complex algebra solution of the equation

$$L\frac{di}{dt} + Ri = E. \tag{27}$$

In general it is desirable to obtain the current due to an applied voltage of the form

$$E = E_0 \cos(\omega t + \theta).$$

This solution can be obtained directly from the symbolic solution given in (26) by taking the real part. The result is

$$i = E_0\left[\frac{\cos(\omega t + \theta - \varphi) - \cos(\theta - \varphi)e^{-tR/L}}{\sqrt{R^2 + \omega^2 L^2}}\right]. \tag{28}$$

It will be found that (28) is a solution of (27) for an applied voltage $E_0 \cos(\omega t + \theta)$.

D. General Method for Determining Reflections

The method for obtaining the successive current reflections of the line given in the preceding section, is laborious to carry out in complicated cases and hence it is desirable to obtain a simple method for determining the reflections. Such a method is the expansion of the expression for the impedance by the binomial theorem in order to get the successive reflections. In the above example the current i is

$$i = E/(R + j\omega L).$$

We note that the expression for the impedance is approached by that of a short circuited line when the R and L of the line are finite and the capacity and leakance approach zero. Hence

$$i = \frac{E}{R + j\omega L} \to \frac{E}{R_0 \tanh P} = \frac{E(1 + e^{-2P})}{R_0(1 - e^{-2P})}.$$

Now the expansion of

$$\frac{1}{1 - e^{-2P}} = 1 + e^{-2P} + e^{-4P} + \cdots.$$

Hence

$$i \to \frac{E}{R_0}(1 + e^{-2P})(1 + e^{-2P} + e^{-4P} + \cdots)$$

$$= \frac{E}{R_0}[1 + 2e^{-2P} + 2e^{-4P} + \cdots + 2e^{-2(n-1)P} + \cdots],$$

which is the expression for the reflections given by equation (23).

In all the following problems it will be found that a similar process for obtaining the reflections can be followed. It is evident that any method which gives an expression for the current in the form

$$i = E[a_0 + a_1 e^{-2j\omega D} + a_2 e^{-4j\omega D} + \cdots + a_n e^{-2nj\omega D} + \cdots] \quad (29)$$

will give the reflections, for if we take the real part of this expression we have

$$i = E_0[a_0 \cos(\omega t + \theta) + a_1 \cos[\omega(t - 2D) + \theta] + \cdots + a_n \cos[\omega(t - 2nD) + \theta] + \cdots].$$

Each term represents a current which adds to the original current after a time of delay $2D$, $4D$, $\cdots$ $2nD$, and hence the nth terms represents the nth reflection. Therefore any method, such as the above, which gives the current in the form of equation (29), will give the reflections.

E. Simpler Form for Replacing an Impedance

In the preceding section, the transient solution of an inductance and resistance in series was obtained by replacing the impedance $R + j\omega L$ by the expression

$$R_0 \tanh P, \text{ where } R_0 P = R + j\omega L \text{ and } R_0 \to \infty ; \; P \to 0.$$

Tanh P has a numerator and a denominator both of which must be expanded in order to obtain the reflections. If a single term can be used, the expansion becomes simpler. In order to effect such a simplification, it is necessary to find a physical structure, which has only one term in its impedance expression and which approaches a resistance and inductance as a limit.

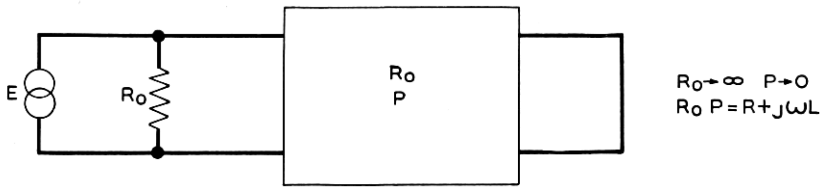


Fig. 2—Short circuited line and shunt resistance.

Such a structure is shown on Fig. 2. It consists of a short circuited line shunted by a resistance R_0. The current into the combination is

$$i = \frac{E}{\dfrac{R_0 \times R_0 \tanh P}{R_0 + R_0 \tanh P}} = \frac{E}{\dfrac{R_0(1 - e^{-2P})}{2}}. \tag{30}$$

If now in the short circuited line, we let R and L be finite and $G \to 0$, $C \to 0$, the combination obviously reduces to a resistance and inductance in series, since the infinite shunt will not affect the result. Hence the replacement of a resistance and inductance in the equation

$$i = \frac{E}{R + j\omega L}$$

by the expression in (30), is justified.

The solution of (30) is gotten by expanding the expression and is

$$i = \frac{2E}{R_0}\left[1 + e^{-2P} + e^{-4P} + \cdots + e^{-2(n-1)P} + \cdots\right] = \frac{2E[1 - e^{-2nP}]}{R_0[1 - e^{-2P}]}.$$

Upon substituting in the values $R_0 P = R + j\omega L$, $n = t/2D$, and letting $R_0 \to \infty$; $P \to 0$, we have

$$i = E\left[\frac{1 - e^{-t[R/L + j\omega]}}{R + j\omega L}\right]$$

in agreement with equation (26).

Similarly, when we have the expression

$$\frac{1}{G + j\omega C}$$

we can replace it by

$$\frac{2R_0}{1 - e^{-2P}} \quad \text{where} \quad \frac{R_0}{P} = \frac{1}{G + j\omega C} \quad \text{and} \quad R_0 \to 0,\ P \to 0. \tag{31}$$

The structure which gives the impedance in (31) is an open circuited line in series with a resistance R_0. The impedance of the combination is

$$R_0 + R_0 \coth P = R_0 \left[1 + \frac{1 + e^{-2P}}{1 - e^{-2P}} \right] = \frac{2R_0}{1 - e^{-2P}} \cdot$$

If then the series impedances of the line approach zero, $R_0 \to 0$ and the impedance of the combination approaches

$$\frac{1}{G + j\omega C} \cdot$$

F. Solution for a Resistance and Capacity in Series

As a second example let us obtain the solution of a resistance and capacity in series. To obtain the solution we solve the case of a

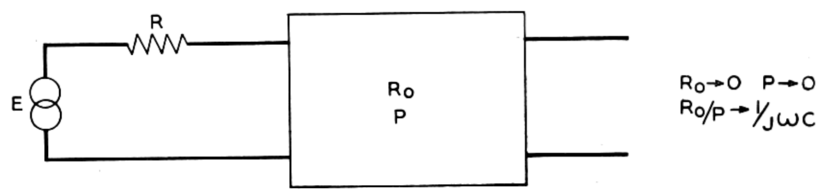


Fig. 3—Open circuited line and series resistance.

resistance in series with an open circuited line as shown on Fig. 3. The steady state solution for the current in this circuit is

$$i = \frac{E}{R + \dfrac{1}{j\omega C}} \cdot$$

Replacing $\frac{1}{j\omega C}$ by $\frac{2R_0}{1 - e^{-2P}}$, and substituting in the above equation there results

$$i = \frac{E}{R + \dfrac{2R_0}{1 - e^{-2P}}} \quad \text{when} \quad R_0 \to 0 \quad \text{and} \quad P \to 0, \quad \text{and} \quad \frac{R_0}{P} \to \frac{1}{j\omega C}$$

in accordance with equation (18).

After some rearrangements this can be put into the form

$$i = \frac{E}{R + R_0}\left[1 - \frac{R_0}{R}\frac{e^{-2F}(1 + e^{-2P})}{1 - e^{-2(F+P)}}\right], \tag{32}$$

where

$$e^{-2F} = \frac{R}{R + 2R_0}.$$

Expanding equation (32) in the form of a series, there results

$$i = \frac{E}{R+R_0}\left[\frac{R+3R_0}{R+2R_0} - \frac{R_0}{R}\left(\frac{2R+2R_0}{R+2R_0}\right)(1+e^{-2(F+P)}+e^{-4(F+P)} + \cdots)\right].$$

Summing up n terms of this series, we have

$$i = \frac{E}{R + R_0}\left[\frac{R + 3R_0}{R + 2R_0} - \frac{R_0}{R}\left(\frac{2R + 2R_0}{R + 2R_0}\right)\left(\frac{1 - e^{-2n(F+P)}}{1 - e^{-2(F+P)}}\right)\right]. \tag{33}$$

Since in the above expression $R_0 \to 0$, we can obtain the value of F by writing the first terms of the expansion for the exponential

$$e^{-2F} = 1 - 2F + \frac{(2F)^2}{2!} + \cdots = \frac{R}{R + 2R_0} = 1 - \frac{2R_0}{R} + \cdots.$$

Hence

$$F \to \frac{R_0}{R}.$$

If now in equation (33) we proceed to the limit, letting

$$R_0 \to 0; \quad P \to 0; \quad \frac{R_0}{P} = \frac{1}{j\omega C}; \quad n = \frac{t}{2D}$$

there results the equation

$$i = \frac{E}{R}\left[1 - \frac{1 - e^{-t(1/RC+j\omega)}}{1 + j\omega CR}\right]. \tag{34}$$

This equation is the symbolic solution of the integral equation

$$Ri + \frac{1}{C}\int idt = E_0 e^{j(\omega t+\theta)}.$$

If we wish the solution corresponding to the impressed voltage, $E_0 \cos(\omega t + \theta)$, we take the real part of (34), obtaining

$$i = E_0 \frac{\cos(\omega t + \theta + \delta) - [\sin(\theta - \delta)\tan\delta]e^{-t/RC}}{\sqrt{R^2 + 1/\omega^2 C^2}},$$

where

$$\tan\delta = \frac{1}{\omega RC}.$$

II. Solution for M Sections of All-Pass Lattice Network

The process for solving any type of problem is to replace any resistance and inductance in series by $R_{0_1}(1 - e^{-2P_1})/2$, and any capacity and leakance in parallel by $(1 - e^{-2P_2})/2R_{0_2}$, where $R_{0_1} \to \infty$; $R_{0_2} \to 0$; $R_{0_1}P_1 = R + j\omega L$; $R_{0_2}/P_2 = 1/(G + j\omega C)$.

The next problem considered here is the solution for any number of sections of all-pass lattice type network [3] as shown on Figure 4.

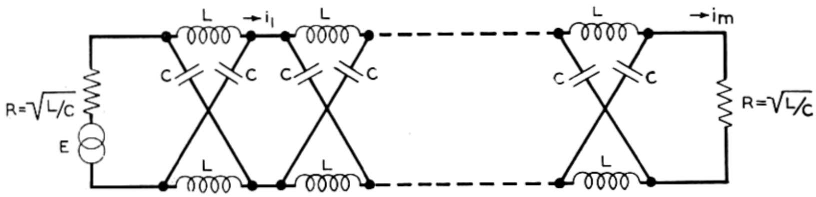


Fig. 4—Sections of all pass lattice network.

These networks have the property of passing all frequencies without attenuation, and they are much used as phase equalizers.

The steady state equation for the current at the end of the first section, when this section is terminated at each end by a resistance $R = \sqrt{Z_1 Z_2}$, is

$$i_1 = \frac{E}{2R}\left[\frac{\sqrt{Z_2} - \sqrt{Z_1}}{\sqrt{Z_2} + \sqrt{Z_1}}\right].$$

The current flowing out of the mth section of such a structure takes the form

$$i_m = \frac{E}{2R}\left[\frac{\sqrt{Z_2} - \sqrt{Z_1}}{\sqrt{Z_2} + \sqrt{Z_1}}\right]^m. \qquad (35)$$

In the structure considered $Z_1 = j\omega L$; $Z_2 = 1/j\omega C$, and $\sqrt{L/C} = R$. In accordance with section I-B, we replace the inductance by a short circuited line, and the capacity by an open circuited line. For the first line, in the limiting case, we have by equation (15),

$$R_0 \to \infty; \quad R_0 P = j\omega L.$$

For the second line, by equation (18), we have

$$\bar{R}_0 \to 0; \quad \frac{\bar{R}_0}{\bar{P}} = \frac{1}{j\omega C}.$$

There is no loss of generality if we take the propagation constants for

[3] See for example *B. S. T. J.*, July 1928, page 510.

9

the two lines equal so that

$$\frac{\overline{R}_0}{\overline{P}} \times R_0 P = j\omega L \times \frac{1}{j\omega C} = \frac{L}{C} = R^2.$$

Hence $\overline{R}_0 = R^2/R_0$. Substituting these values in equation (35), we have

$$i_m = \frac{E}{2R}\left[\frac{1-[R_0/2R](1-e^{-2P})}{1+[R_0/2R](1-e^{-2P})}\right]^m. \tag{36}$$

After some simple rearrangements, equation (36) takes the form

$$i_m = \frac{E}{2R}\left[\frac{4R}{R_0}\left(\frac{1}{1-e^{-2[(R/R_0)+P]}}\right)-1\right]^m. \tag{37}$$

If m equals 1, the solution can be obtained exactly as discussed in the first example in section (1), and it is

$$i_1 = \frac{E}{2R}\left[\frac{1-j\omega\sqrt{LC}-2e^{-t[(1/\sqrt{LC})+j\omega]}}{1+j\omega\sqrt{LC}}\right].$$

The solution for m sections of lattice network is discussed in the Appendix, and it is there shown that the solution can be written in the form

$$\begin{aligned} i_m = \frac{E}{2R}\Bigg[&\left(\frac{1-j\omega\sqrt{LC}}{1+j\omega\sqrt{LC}}\right)^m - e^{-t[(1/\sqrt{LC})+j\omega]}\Bigg[\frac{\left(\frac{t}{\sqrt{LC}}\right)^{m-1}}{(m-1)!}\times\frac{2^m}{1+j\omega\sqrt{LC}} \\ &+\frac{\left(\frac{t}{\sqrt{LC}}\right)^{m-2}}{(m-2)!}\left(\frac{2^m}{(1+j\omega\sqrt{LC})^2}-\frac{m2^{m-1}}{1+j\omega\sqrt{LC}}\right) \\ &+\cdots+\frac{t}{\sqrt{LC}}\left[\frac{(1-j\omega\sqrt{LC})^m}{(1+j\omega\sqrt{LC})^{m-1}}-(-1)^m\left(m+1-\frac{m}{1+j\omega\sqrt{LC}}\right)\right] \\ &+\left(\frac{1-j\omega\sqrt{LC}}{1+j\omega\sqrt{LC}}\right)^m-(-1)^m\Bigg]\Bigg]. \end{aligned} \tag{38}$$

Equation (38) represents the symbolic or complex algebra solution for the current at the end of the mth section of a lattice network as shown on Fig. 4. It is usually desirable to obtain the current due to an applied voltage $E_0 \cos(\omega t + \theta)$. This can be obtained from equation (38) by taking the real part of the equation and rejecting the imaginary part. The process of doing this is simple and the result obtained is

$$i_m = \frac{E}{2R}\left[\cos(\omega t + \theta - 2m\varphi) - 2e^{-(t/\sqrt{LC})}\left[\frac{\left(\frac{2t}{\sqrt{LC}}\right)^{m-1}}{(m-1)!}\cos(\theta-\varphi)\cos\varphi\right.\right.$$

$$+ \frac{\left(\frac{2t}{\sqrt{LC}}\right)^{m-2}}{(m-2)!}[2\cos^2\varphi\cos(\theta - 2\varphi) - m\cos\varphi\cos(\theta-\varphi)]$$

$$+ \frac{\left(\frac{2t}{\sqrt{LC}}\right)^{m-3}}{(m-3)!}\left[4\cos^3\varphi\cos(\theta - 3\varphi) - 2m\cos^2\varphi\cos(\theta - 2\varphi)\right.$$

$$\left.\left.\left. + \frac{m(m-1)}{2!}\cos\varphi\cos(\theta-\varphi)\right] + \cdots\right]\right], \tag{39}$$

where $\tan\varphi = \omega\sqrt{LC}$.

For example, the solutions for one and two section networks take the form

$$i_1 = \frac{E}{2R}[\cos(\omega t + \theta - 2\varphi) - 2e^{-(t/\sqrt{LC})}\cos\varphi\cos(\theta-\varphi)]$$

$$i_2 = \frac{E}{2R}\left[\cos(\omega t + \theta - 4\varphi) - 2e^{-(t/\sqrt{LC})}\left[2\cos^2\varphi\cos(\theta - 2\varphi)\right.\right. \tag{40}$$

$$\left.\left. - 2\cos\varphi\cos(\theta-\varphi) + \frac{2t}{\sqrt{LC}}\cos(\theta-\varphi)\cos\varphi\right]\right].$$

It appeared desirable to obtain some numerical calculations on the building up of current in this type of network. This calculation has been carried through for two section, four section, and six section networks. The current has been calculated for a constant voltage, for an alternating voltage whose frequency is the resonant frequency of the network, and for one whose frequency is twice the resonant frequency of the network. The current building up for a constant applied voltage is shown for the three networks on Fig. 5. The current building up in a two section network, in which an alternating voltage whose frequency equals the resonant frequency of the network, is shown on Fig. 6. The steady state and the transient terms are shown separately in the dotted lines, and the complete solution in the full line. The applied voltage is of the form $E_0 \cos \omega t$ and hence θ is taken as zero in equation (39). Similarly, curves for two, four,

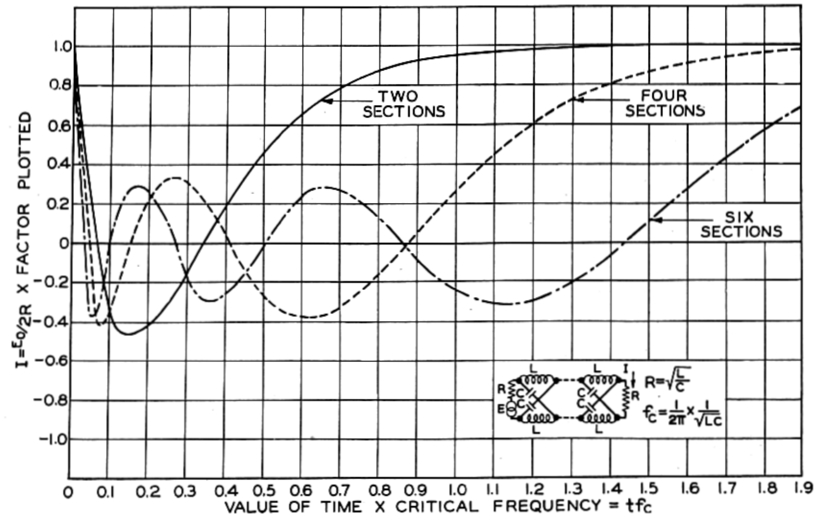


Fig. 5—Current resulting from the application of a constant voltage, E, on several sections of lattice network. The current plotted is the current in the termination of the network.

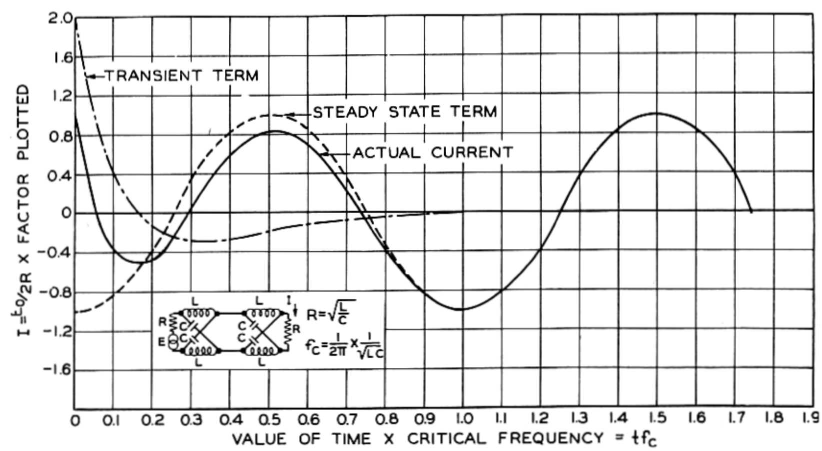


Fig. 6—Current resulting from the application of an alternating voltage, $E = E_0 \cos \omega_c t$, on a two section lattice network. The current plotted is the current in the termination of the network. The frequency of the applied voltage is the resonant frequency, f_c, of the network.

and six sections are shown on Fig. 7 and Fig. 8. Fig. 7 shows the transient terms and Fig. 8 the complete solution. Fig. 9 and Fig. 10 show similar curves for a frequency twice the resonant frequency of the network. In addition, the solution for infinite frequency is readily obtained from (39) since for this frequency $\varphi = 90°$. Then

$$i = \frac{E}{2R}[\cos(\omega t + \theta - m\pi)].$$

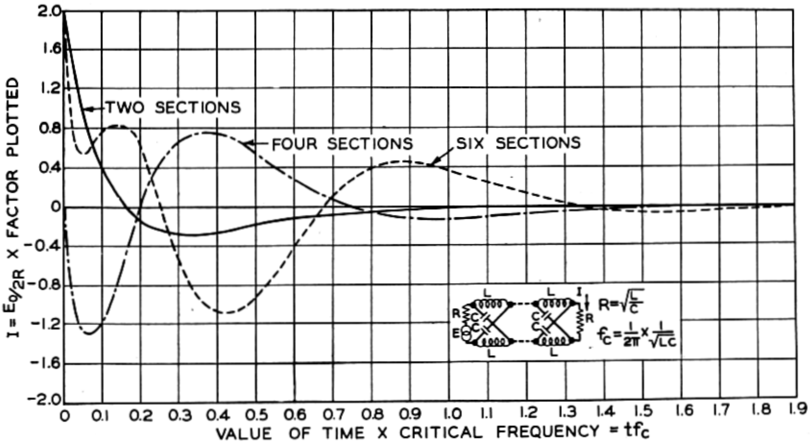


Fig. 7—Transient current resulting from the application of an alternating voltage, $E = E_0 \cos \omega_c t$, for several sections of lattice network. The current plotted is the current in the termination of the network. The frequency of the applied voltage is the resonant frequency, f_c, of the network.

III. Laplacian Infinite Integral Equation and Its Formal Solution

The solution of circuit problems by means of the Laplacian integral equation has been used by Carson [4] to a large extent. It is interesting to note that this integral form can be derived in a simple manner by means of this expansion method, and that this method provides a means for solving the Laplacian integral equation.

Any impedance Z is made up of resistances, inductances and capacities, and hence the current i can be represented by a series

$$i = \frac{E}{Z} = E[a_0 + a_1 e^{-j2\omega D} + a_2 e^{-j4\omega D} + \cdots + a_n e^{-j2n\omega D} + \cdots]. \quad (41)$$

The interpretation of this expansion from a physical standpoint is that the current is Ea_0, for the first interval of time $2D$, $E[a_0 + a_1 e^{-j2\omega D}]$ for the next interval of time $2D$, etc. Hence at the time $t = n(2D)$, the current i will be given by the sum of n terms

[4] See "Electric Circuit Theory and the Operational Calculus," *B. S. T. J.*, October 1925, and following.

of this series. We can, therefore, express the current i at the time t by the integral

$$i = E\left[\int_0^t a(t)e^{-j\omega t}dt + a_0\right], \tag{42}$$

where the value of $a(t)$ for any interval of time $(n - 1)2D$ to $n(2D)$ is the constant of the above series a_{n-1} divided by $2D$. The value of this integral for an infinite time must reduce to the steady state value of $i = E/Z$, hence

$$\frac{E}{Z} = E\left[\int_0^\infty a(t)e^{-j\omega t}dt + a_0\right].$$

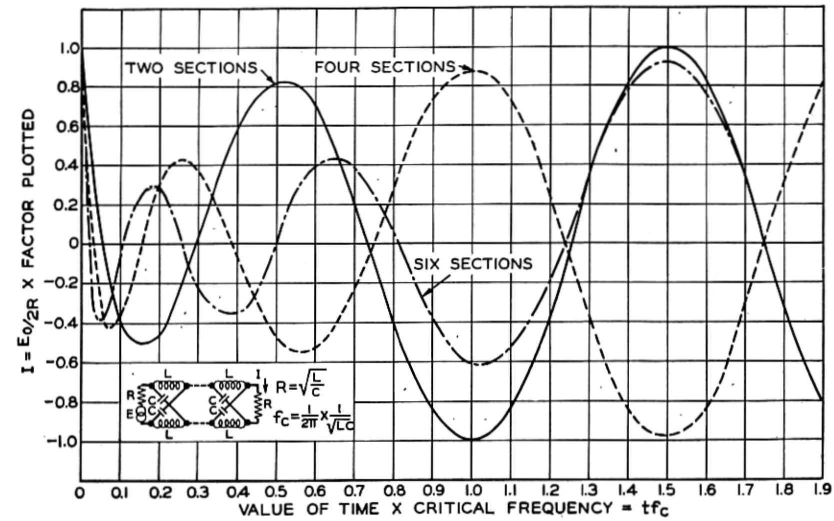


Fig. 8—Current resulting from the application of an alternating voltage, $E = E_0 \cos \omega_c t$, on several sections of lattice network. The current plotted is the current in the termination of the network. The frequency of the applied voltage is the resonant frequency, f_c, of the network.

Cancelling out the common factor E, we have the infinite integral equation

$$\frac{1}{Z(j\omega)} = \left[\int_0^\infty a(t)e^{-j\omega t}dt + a_0\right]. \tag{43}$$

The physical interpretation of the quantity $a(t)$ is readily obtained by reference to equation (42). If we set $\omega = 0$ and $E = 1$ in this equation we have

$$i = \int_0^t a(t)dt + a_0 = \int_0^t a(t)dt + h(0),$$

where $h(0)$ is a constant denoting the current when t is zero. Now i at any time t is the indicial or direct current admittance, designated by $h(t)$, hence $a(t)$ is

$$a(t) = \frac{d}{dt}(h(t)).$$

The infinite integral equation (43) takes the form

$$\frac{1}{Z(j\omega)} = \left[\int_0^\infty \frac{d}{dt}(h(t))e^{-j\omega t}dt + h(0)\right]. \tag{44}$$

This integral equation does not have quite the same form as Carson's integral equation but can be readily put into that form by means of

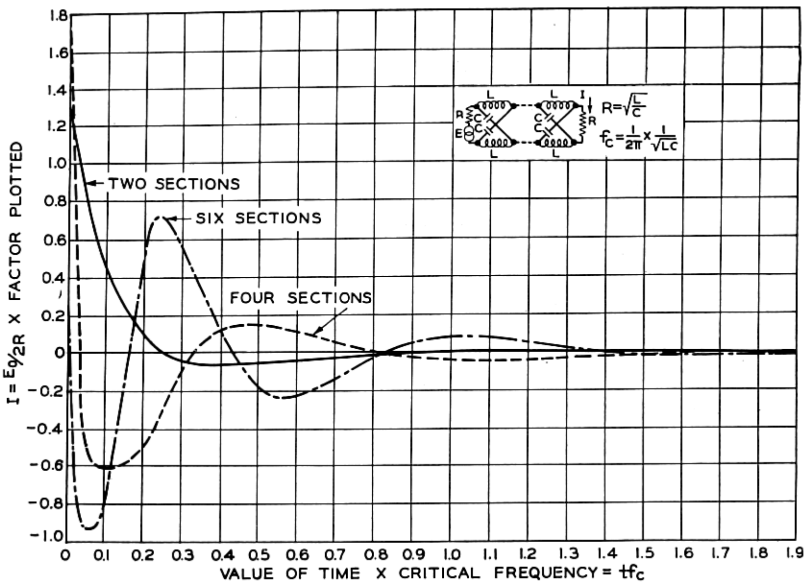


Fig. 9—Transient current resulting from the application of an alternating voltage, $E = E_0 \cos 2\omega_c t$, on several sections of lattice network. The current plotted is the current in the termination of the network. The frequency of the applied voltage is twice the resonant frequency, f_c, of the network.

Borel's theorem [5] which is given below. Suppose that $1/Z'$ and $1/Z''$ are two admittances, which when multiplied together give the admittance $1/H$. The admittances $1/Z'$ and $1/Z''$ have the expansions

$$\frac{1}{Z'} = [a_0' + a_1'e^{-2j\omega D} + a_2'e^{-4j\omega D} + \cdots];$$

$$\frac{1}{Z''} = [a_0'' + a_1''e^{-2j\omega D} + a_2''e^{-4j\omega D} + \cdots].$$

[5] See *B. S. T. J.*, October 1925, page 722.

The expansion $1/H$, then takes the form

$$\frac{1}{H} = \frac{1}{Z'Z''} = [a_0 + a_1 e^{-2j\omega D} + \cdots + a_n e^{-2nj\omega D} + \cdots]$$
$$= [a_0'a_0'' + [a_0'a_1'' + a_0''a_1']e^{-2j\omega D} + \cdots$$
$$+ [a_0'a_n'' + \cdots + a_k'a_{n-k}'' + \cdots + a_n'a_0'']e^{-2nj\omega D} + \cdots].$$

We have then the relation

$$a_n = [a_0'a_n'' + \cdots + a_k'a_{n-k}'' + \cdots + a_n'a_0''].$$

Now a_n'' is the value of a'' when $t = n(2D)$, hence the above relation can be put into the form of an integral

$$a(t) = \int_0^t a''(\tau)a'(t - \tau)d\tau,$$

where $\tau = k/2D$, and the complete relation is

$$\frac{1}{H} = \frac{1}{Z'Z''} = \int_0^\infty a(t)e^{-j\omega t}dt + a_0$$
$$= \int_0^\infty \left[\int_0^t a''(\tau)a'(t - \tau)d\tau\right] e^{-j\omega t}dt + a_0'a_0''.$$

Suppose now that we let $Z' = j\omega$; $Z'' = Z(j\omega)$. We know from Heaviside's rule and from Section I that $1/j\omega$ has the direct current or indicial admittance solution

$$\frac{1}{j\omega} = t = h(t).$$

Hence

$$a'(t) = \frac{d}{dt}h'(t) = \frac{d}{dt}t = 1 \qquad \text{and} \qquad a_0' = 0,$$

and the infinite integral equation

$$\frac{1}{H} = \frac{1}{j\omega Z''(j\omega)} = \int_0^\infty \left[\int_0^t a''(\tau)a'(t - \tau)d\tau\right] e^{-j\omega t}dt + a_0'a_0''$$

takes the form

$$\frac{1}{j\omega Z''(j\omega)} = \int_0^\infty \left[\int_0^t a''(\tau)d\tau\right] e^{-j\omega t}dt = \int_0^\infty h''(t)e^{-j\omega t}dt.$$

Dropping the primes, we have the Laplacian integral equation

$$\frac{1}{j\omega Z(j\omega)} = \int_0^\infty h(t)e^{-j\omega t}dt. \tag{45}$$

Hence (43) is equivalent to Carson's integral equation, if $(j\omega)$ is replaced by p.

It will be noted that in deriving this equation use is made only of the general form of the expansion of admittances. For particular admittances, the values of the a's in equation (43) or the h's in equation (45) can be derived directly from an expansion of the admittance function, as shown in the foregoing work. Hence, if the solution of

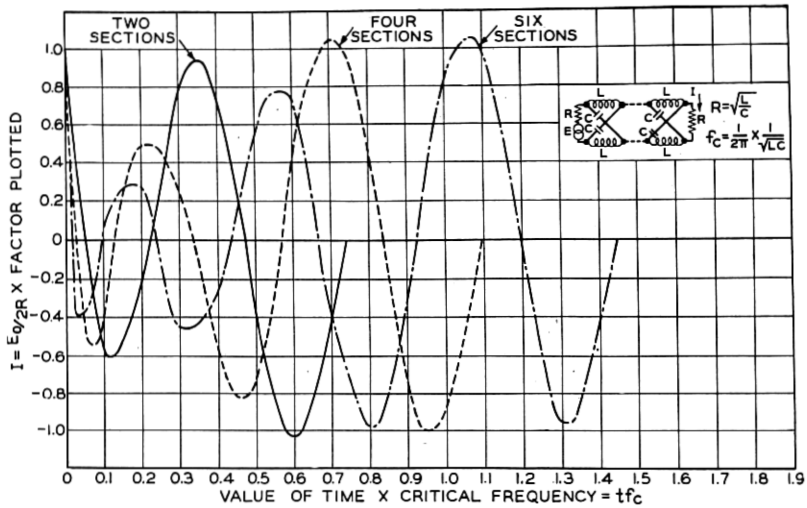


Fig. 10—Current resulting from the application of an alternating voltage, $E = E_0 \cos 2\omega_c t$, on several sections of lattice network. The current plotted is the current in the termination of the network. The frequency of the applied voltage is twice the resonant frequency, f_c, of the network.

the integral equation is not known from a table of integrals, one method for obtaining its solution is the expansion method developed above. This method may then have some application as a method for solving integral equations.

A. Illustrative Example

As an illustration of the use of this method in solving integral equations we will consider the equation

$$\frac{1}{\sqrt{(j\omega)^2 + 2\lambda j\omega}} = \int_0^\infty a(t)e^{-j\omega t}dt. \tag{46}$$

The expression on the left can be written

$$\frac{1}{\sqrt{j\omega}\sqrt{2\lambda + j\omega}}. \tag{47}$$

Noting that the square of the first factor has the form of an inductance and the second the form of a resistance and inductance in series we replace

$$j\omega \to R_{0_1}\left(\frac{1 - e^{-2P_1}}{2}\right) \qquad \text{where} \qquad R_{0_1} \to \infty;\ P_1 \to 0$$

$$R_{0_1}P_1 = j\omega$$

and $2\lambda + j\omega \to R_{0_2}\left(\frac{1 - e^{-2P_2}}{2}\right)$ where $R_{0_2} \to \infty$; $P_2 \to 0$; and $R_{0_2}P_2 = 2\lambda + j\omega$. We note that P_1 has the form $j\omega D$ where $R_{0_1}D = 1$, while P_2 has the form $A + j\omega D$ where $R_{0_2}A = 2\lambda$ and $R_{0_2}D = 1$. Substituting these values in (47), we have

$$\frac{1}{\sqrt{\frac{R_{0_1}R_{0_2}}{4}(1 - e^{-2j\omega D})(1 - e^{-2(A+j\omega D)})}}.$$

Expanding these two factors by the binomial theorem, we have

$$\frac{2}{\sqrt{R_{0_1}R_{0_2}}}\left[1 + \frac{1}{2}e^{-j(2\omega D)} + \frac{1/2 \times 3/2}{2!}e^{-j(4\omega D)} + \cdots + \frac{(2n)!e^{-j(2n\omega D)}}{2^{2n}(n!)^2} + \cdots\right]$$
$$\times\left[1 + \frac{1}{2}e^{-2(A+j\omega D)} + \cdots + \frac{(2n)!e^{-2n(A+j\omega D)}}{2^{2n}(n!)^2} + \cdots\right].$$

At this point we make use of Stirling's theorems on factorials which states that when K is large

$$K! = \left(\frac{K}{e}\right)^K \sqrt{2\pi K}.$$

The typical terms of the above expression become

$$\frac{(2n)!e^{-j(2n\omega D)}}{2^{2n}(n!)^2} = \frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{4\pi n}\, e^{-j(2n\omega D)}}{2^{2n}\left(\frac{n}{e}\right)^{2n} 2\pi n} = \frac{e^{-j(2n\omega D)}}{\sqrt{\pi n}}.$$

Inserting this value in the above expression and multiplying, there results

$$\sqrt{\frac{4}{R_{0_1}R_{0_2}}}\Big[1 + e^{-i(2\omega D)}[1/2e^{-2A} + 1/2] + \cdots + e^{-i(2n\omega D)} \left(\frac{e^{-2nA}}{\sqrt{\pi n}} + \frac{e^{-2(n-1)A}}{2\sqrt{\pi(n-1)}} + \cdots + \frac{e^{-2(n-k)A}}{\sqrt{\pi(n-k)}\sqrt{\pi k}} + \cdots + \frac{1}{\sqrt{\pi n}}\right) + \cdots\Big]. \tag{48}$$

Since the value of $a(t)$ is given by the factor multiplying the term $e^{-j2n\omega D}$ divided by $2D$ we can write

$$a(t) = \frac{2}{\sqrt{R_{0_1}R_{0_2}}2D}\left[\frac{e^{-2nA}}{\sqrt{\pi n}} + \frac{e^{-2(n-1)A}}{2\sqrt{\pi(n-1)}} + \cdots + \frac{e^{-2(n-k)A}}{\sqrt{\pi(n-k)}\sqrt{\pi k}} + \cdots + \frac{1}{\sqrt{\pi n}}\right]. \tag{49}$$

This expression can be written as the sum

$$a(t) = \sum_{k=0}^{k=n} \frac{e^{-2(n-k)A}}{\pi\sqrt{(n-k)k}}.$$

We introduce now the value $n = t/2D$ and define a new variable τ by $k = \tau/2D$. Inserting these values in the above summation and noting that $A/D = 2\lambda/R_{0_2}/1/R_{0_2} = 2\lambda$, we have

$$a(t) = \sum_{k=0}^{k=n} \frac{e^{(t-\tau)2\lambda}2D}{\pi\sqrt{(t-\tau)\tau}}. \tag{50}$$

But $2D = d\tau$, the element of time, so that the summation can be written as the integral

$$a(t) = \frac{e^{-2\lambda t}}{\pi}\int_0^t \frac{e^{2\lambda\tau}d\tau}{\sqrt{(t-\tau)\tau}}. \tag{51}$$

The value of the integral is $\pi e^{\lambda t}I_0(\lambda t)$, where $I_0(\lambda t)$ is the Bessel's functions $J_0(i\lambda t)$, when $i = \sqrt{-1}$. To show this we can expand the exponential and integrate the series giving

$$\pi\left[1 + \lambda t + \frac{1.3(\lambda t)^2}{(2!)^2} + \frac{1.3\cdot 5(\lambda t)^3}{(3!)^2} + \cdots\right].$$

This can be recognized as the series expansion of $e^{\lambda t}I_0(\lambda t)$. Hence the value of (51) is

$$a(t) = e^{-\lambda t} I_0(\lambda t), \tag{52}$$

which is the solution of the integral equation.

IV. Other Types of Boundary Conditions

The solutions obtained before are all on the assumption that no energy exists in the network before the voltage is applied. Other types of boundary conditions are sometimes desirable, but these can all be derived from the above solutions.

The next most important case is the case where the network has come to its equilibrium value and the voltage is suddenly taken off. This condition is the same as would result if a negative voltage E were suddenly applied to the circuit, and hence the solution is the steady state value of the current minus the current which flows on application of the voltage E.

Another type of boundary condition sometimes occurring is the one where energy exists in the network when t is zero. This may be taken account of by assuming that the voltage is applied before t equals zero. To take account of this condition analytically, examine the expansion

$$i = \frac{E}{Z} = E[a_0 + a_1 e^{-2j\omega D} + a_2 e^{-4j\omega D} + \cdots + a_n e^{-2nj\omega D} + \cdots].$$

According to the above assumption, the voltage is applied when $t = -t_0$, hence for n we substitute

$$n = \frac{t}{2D} + \frac{t_0}{2D} \cdot$$

The above series is then replaced by the integral

$$i = E\int_{-t_0}^{t} a(t + t_0) e^{-j\omega(t+t_0)} dt. \tag{53}$$

Another boundary condition of interest occurs when the voltage is taken off before an equilibrium value has been reached. If we count time as starting when the voltage is taken off, or what amounts to the same thing, when a negative voltage is applied, the symbolic solution takes the form

$$i = E[\int_{-t_0}^{t} a(t + t_0) e^{-j\omega(t+t_0)} dt - \int_0^t a(t) e^{-j\omega t} dt]. \tag{54}$$

Appendix

The expression

$$\left[\frac{4R}{R_0}\left(\frac{1}{1-e^{-2(R/R_0+P)}}\right)-1\right]^m$$

can be expanded into the form

$$\left[\frac{4R}{R_0}\left(\frac{1}{1-e^{-2(R/R_0+P)}}\right)\right]^m - m\left[\frac{4R}{R_0}\left(\frac{1}{1-e^{-2(R/R_0+P)}}\right)\right]^{m-1}+\cdots + (-1)^m. \quad (55)$$

and hence the general solution depends only on the solution of the general form

$$\left[\frac{4R}{R_0}\left(\frac{1}{1-e^{-2(R/R_0+P)}}\right)\right]^m. \quad (56)$$

If equation (56) is expanded by the binomial theorem, there results the expression

$$\left(\frac{4R}{R_0}\right)^m\left[1+me^{-2(R/R_0+P)}+\frac{m(m+1)}{2!}e^{-4(R/R_0+P)}+\cdots + \frac{(m+n-1)!e^{-2n(R/R_0+P)}}{n!(m-1)!}+\cdots\right]. \quad (57)$$

For any value of m, we can write the typical term of (57) as

$$\frac{(m+n-1)!}{n!(m-1)!}=\frac{\left(\frac{m+n-1}{e}\right)^{m+n-1}\sqrt{2\pi(m+n-1)}}{\left(\frac{n}{e}\right)^n\sqrt{2\pi n}(m-1)!}$$
$$=\frac{e^{-(m-1)}\left(\frac{m+n-1}{n}\right)^n(m+n-1)^{m-1/2}}{(m-1)!\sqrt{n}}. \quad (58)$$

by Stirling's theorem on factorials. Now n for any finite value of time approaches infinity, while m for any finite term in the series is finite. Hence (58) can be written as

$$\frac{e^{-(m-1)}\left(1+\frac{m-1}{n}\right)^n n^{m-1}}{(m-1)!}.$$

The limit of $\left(1+\frac{m-1}{n}\right)^n$ as $n\to\infty$ is $e^{(m-1)}$.[6] Hence

$$\frac{(m+n-1)!}{n!(m-1)!}=\frac{n^{m-1}}{(m-1)!}.$$

[6] See "Probability and Its Engineering Uses," T. C. Fry, page 107.

The value of (57), then reduces to

$$\left(\frac{4R}{R_0}\right)^m \sum_{t=0}^{t=n} \frac{n^{m-1}}{(m-1)!} e^{-2n(R/R_0+P)}. \qquad (59)$$

If now we substitute $n = t/2D$, $P = j\omega D$; $R_0 P = j\omega L$ (59) reduces to the integral

$$\left(\frac{4R}{R_0}\right)^m \times \frac{1}{(2D)^m} \int_0^t \frac{t^{m-1} e^{-[(1/\sqrt{LC})+j\omega]t}}{(m-1)!} dt$$
$$= \left(\frac{2}{\sqrt{LC}}\right)^m \times \frac{1}{(m-1)!} \int_0^t t^{m-1} e^{-t[(1/\sqrt{LC})+j\omega]} dt. \qquad (60)$$

If we integrate (60) by parts, successively, there results the series

$$2^m \left\{ \left(\frac{1}{1+j\omega\sqrt{LC}}\right)^m - e^{-t(1/\sqrt{LC}+j\omega)} \left[\frac{1}{(1+j\omega\sqrt{LC})^m} + \frac{\left(\frac{t}{\sqrt{LC}}\right)}{(1+j\omega\sqrt{LC})^{m-1}} + \cdots + \frac{\left(\frac{t}{\sqrt{LC}}\right)^{m-1}}{(m-1)!\,(1+j\omega\sqrt{LC})} \right] \right\}. \qquad (61)$$

The complete solution of (55) is then obtained by adding terms of the kind given in (61). For example the steady state term is given by the series

$$\frac{2^m}{(1+j\omega\sqrt{LC})^m} - \frac{m2^{m-1}}{(1+j\omega\sqrt{LC})^{m-1}} + \frac{\frac{m(m-1)}{2!}2^{m-2}}{(1+\omega\sqrt{LC})^{m-2}} + \cdots + (-1)^m$$
$$= \frac{2^m - m2^{m-1}(1+j\omega\sqrt{LC}) + \frac{m(m-1)}{2!}2^{m-2}(1+j\omega\sqrt{LC})^2 + \cdots + (-1)^m(1+j\omega\sqrt{LC})^m}{(1+j\omega\sqrt{LC})^m}.$$

This is readily seen to be the binomial expansion of

$$\left[\frac{2-(1+j\omega\sqrt{LC})}{1+j\omega\sqrt{LC}}\right]^m = \left[\frac{1-j\omega\sqrt{LC}}{1+j\omega\sqrt{LC}}\right]^m. \qquad (62)$$

The other terms given in equation (38) follow directly by addition and reference to equations (55) and (61).