Harmonic Production in Ferromagnetic Materials at Low Frequencies and Low Flux Densities

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SYNOPSIS: When a multi-channel communication circuit includes a non-linear element such as a ferromagnetic core coil, distortion of the wave form impressed upon the circuit is produced. In terms of the single frequency components, this distortion is manifested in the appearance of new components. This distortion may give rise to the reduction of quality in any channel, and it may also introduce crosstalk and interference, which consists of new frequencies not present in the impressed wave of any channel under consideration, produced by independent channels. In view of the recent increased use of multi-channel systems, it has become necessary to investigate the effects of this type of distortion, to determine the dependence of this distortion upon the properties of the magnetic materials constituting the cores of inductance coils and transformers, as well as upon the circuit impedances, and to determine those constants of core materials which are significant in the distorting process.

The behavior of magnetic materials to complex waves of magnetizing force is ordinarily a highly involved process, so that a direct correlation between distortion and some of the easily measured constants of materials is a matter of some difficulty. It has been established experimentally, as a confirmation of theoretical speculations, that the third harmonic e.m.f. generated by a sinusoidal wave of magnetizing force may serve as an index of the distortion with a complex wave of magnetizing foice. This relation is valid for low flux densities and for frequencies at which the screening effect of eddy currents is not important. The paper is therefore devoted to an investigation of the third harmonic production in its dependence upon the properties of hysteresis loops. These loop constants in turn are shown to be deducible from AC bridge measurements on a coil of known dimensions having a core of the magnetic material under investigation. The loop constants for a few materials are included in the text. An analogy exists between the treatments of hysteresis loop and of three-element vacuum tube characteristics which enables us to compare simplifying relations introduced by Rayleigh and by H. J. van der Bijl in the two cases.

The theoretical deductions are found to be in general agreement with experiment, and are applied to a number of cases of practical interest. These include the effects of air gaps and dilution, and the choice of core material in third harmonic production by inductance coils and transformers. Finally, the amount of third harmonic current flowing out of long lines is deduced with both lumped and continuous loading.

PART 1. HYSTERESIS LOOPS AND THEIR MATHEMATICAL REPRESENTATION

NEW and improved systems of multi-channel communication which have come into use during the past few years have imposed rigorous requirements on the circuit elements constituting the communicating link, and have made it necessary to investigate the degree of distortion which arises from the use of ferromagnetic apparatus. The distortion introduced may have two general effects: distortion of the signal in any one channel, which is usually the minor effect, and production of crosstalk and interference between the various channels to which the non-linear element is common. Such elements are, in general, ferromagnetic core coils or transformers. The distortion introduced by the non-linear relation between flux density and magnetizing force is therefore of fundamental importance in the design of iron core coils and transformers which carry simultaneously a number of communication channels.

The use of iron core coils and transformers in communication work is confined to comparatively low flux densities in contrast to the ordinary practice in power work, where operation usually occurs above the knee of the normal magnetization curve at a value of the order of a thousand times greater than that used in communication work. There are two main reasons for this restriction: losses are reduced, and the relation between flux density and magnetizing force approaches linearity so that distortion is minimized.

Under actual operating conditions in which the more important crosstalk effects arise, we have a complex wave of magnetizing force acting on the magnetic core. Non-linearity in the magnetic circuit gives rise to new frequencies which normally [1] are related to those impressed upon the circuit, being sums and differences of integral multiples of the originally impressed frequencies. These modulation products are all of odd order [2] when the core is unpolarized. On purely theoretical grounds we would expect to find relations between the amplitudes of the different frequencies resulting from any one order of modulation. To take the third order modulation products of two impressed frequencies (f_1, f_2) as an example, we would expect the amplitudes of the harmonics $3f_1$ and $3f_2$ to be related to the amplitudes of the other third order products: $2f_1 \pm f_2$, $2f_2 \pm f_1$. This has been confirmed by direct experimental test, so that we are enabled to use the generated third harmonic voltage as an index of the generated voltages corresponding to the other third order products. Accordingly, we shall deal in the following with the third harmonic produced by a sinusoidal wave of magnetizing force, and so avoid a more involved analysis.

The fundamental relation in the operation of ferromagnetic apparatus is of course the relation between the flux density B and the magnetizing force H. In contrast to the usual behavior of circuit elements, the relation between the two fundamental quantities—the independent and the dependent variables, H and B in this case—is a function not only of the value of the independent variable, but is also

[1] This is true in the low flux density region. At high densities and with highly reactive circuits as in magnetic modulators, other frequencies are sometimes found which correspond to natural oscillations of the coil and circuit.

[2] *Bell System Technical Journal,* Jan. 1928, pp. 110, 111.

a function of its previous history as manifested in the phenomenon of hysteresis. This complicates matters to an extent far greater than is the case with other circuit elements, and some analysis has been carried out in which the hysteresis loop has been replaced by the normal magnetization curve, or by some such single valued relation between the variables. For some purposes this convenient simplification—it cannot be called a close approximation for our present purposes—is satisfactory, while for others it does not begin to tell the story. Inasmuch as our aim here is to deal with the actual phenomena involved rather than to arrive at some arbitrary procedure for representing the facts, we shall in the following base considerations upon the hysteresis loop.

Fig. 1 will serve to illustrate the effect of previous history upon flux density. The main loop there, which extends between the two

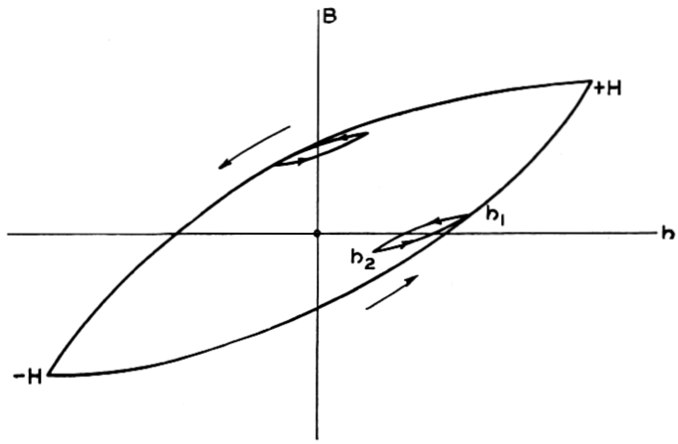

Fig. 1

limits of magnetizing force $-H$ and H, is obtained when the magnetizing force is varied cyclically between those two values in such a way that the magnetizing force has but one maximum and one minimum per cycle. In that case the B–H loop is traversed in the direction shown by the arrow. It is independent of frequency when the eddy-current losses in the iron are small, as we shall suppose them to be, and it is independent of the wave form of the magnetizing force so long as that wave form satisfies the condition we have laid down above. A sinusoidal magnetizing force, for example, satisfies that condition. When the magnetizing force contains components of such magnitude and phase that the wave form has multiple maxima or minima, the simple B–H loop no longer suffices to represent that relation, but auxiliary loops shown in the figure are involved.

Thus suppose a subsidiary maximum to exist at h_1; as the magnetizing force decreases, the flux density no longer follows the main loop but branches off on a subsidiary loop as indicated by the arrows. When the subsidiary minimum at h_2 is reached a new branch is started which completes the subsidiary loop, and which brings the magnetizing force back to the main loop at the point from which it originally diverged, and the main loop is thereafter followed until another maximum (or minimum as the case may be) of the magnetizing force wave is reached. For simplicity in the following we are going to deal solely with a sinusoidal wave of magnetizing force so that subsidiary loops are never called into play. With this understood, the relation between B and H is described by the simple loops of Fig. 2 over a

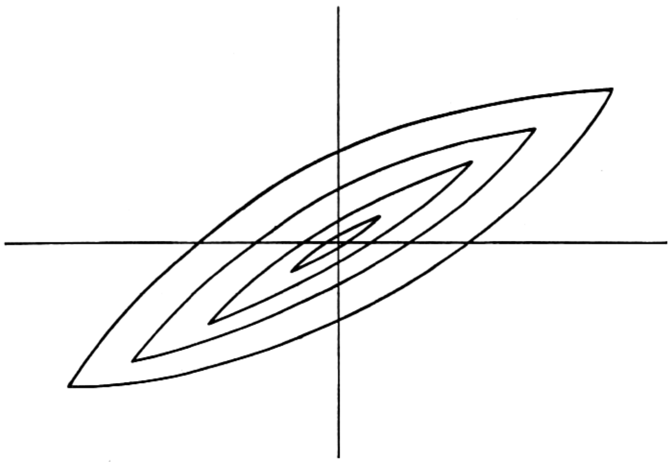

FIG. 2

certain range of magnetizing force, each loop being defined by a particular value of maximum magnetizing force. Each loop may be considered as constituted by two branches which join at the maximum field of the loop.

It is clear, therefore, that with a periodic magnetizing force having but one maximum and one minimum per cycle, the flux density depends upon three properties of the magnetizing force—the maximum value, the instantaneous value, and the sign of dh/dt, being located on the lower branch when dh/dt is positive, and on the upper when it is negative. Now it is of course evident that when a definite loop form is available, a numerical solution by graphical or step-by-step methods may be had. It is further evident that, in the case of a definite impressed magnetizing force, the B–H loop may be broken up into its harmonic components, as was done by S. P. Thompson.[3] Solutions in this form possess all the advantages and disadvantages of numerical ones in which any change of conditions leads to a new problem; the solution desired is a general one which will describe the phenomena in terms of coefficients characteristic of the magnetic material, and this type of solution is the subject of analysis in the following pages.

Perhaps the least difficult method of arriving at an analytical solution without making any assumptions as to the form of the loops is the following. A power series for each branch of any loop is formulated, and the two resultant equations are combined in a trigonometric series. In this way the solutions, each one valid over but half the cycle, are combined to represent the relation of B to H over the entire cycle.

The flux density on each branch of the loop may be defined in terms of two values of magnetizing force, one the maximum value of magnetizing force on the particular branch with which we happen to be concerned and the other the instantaneous value of the magnetizing force on that branch. The equation for either branch may then be expressed as a double power series in these two variables, the instantaneous magnetizing force (h) which will be expressed in gilberts/cm, and the maximum value of the magnetizing force (H), expressed in the same units;

$$B(h, H) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} h^m H^n, \tag{1}$$

where

$$a_{mn} = \frac{1}{m!\,n!} \frac{\partial B(h, H)}{\partial h^m \partial H^n} \bigg]_{0,\,0}. \tag{2}$$

The parallelism of this representation with that for the plate current-grid potential curves of a three electrode thermionic tube is evident,—in both cases double power series are involved.[4] The coefficients a_{mn} are derivatives which are evaluated at the point $h = 0$, $H = 0$, and it will be understood in (2) that these particular values are inserted after the derivatives have been taken, in quite the usual manner.

There is a further interesting parallel here to the vacuum tube case regarding simplification of the general relation. In the case of the vacuum tube a simplification of the double power series due to van der Bijl has been employed which represents the family of tube characteristics by an equation in a single variable. This simplification consists in assuming the amplification factor to be constant, and the important point for us here is that it is equivalent to the assumption

[3] "On Hysteresis Loops and Lissajous' Figures," *Phil. Mag.*, 1910.

[4] Peterson and Evans, "Modulation in Vacuum Tubes used as Amplifiers," *Bell System Technical Journal*, July, 1927.

that the different branches of a family have the same form, so that by suitable change of plate or of grid potential the plate current-grid voltage curves may be superposed. A relation of precisely the same form in the case of magnetic hysteresis loops was stated by Lord Rayleigh [5] upon examination of data obtained by a magnetometric study of the behavior of a single low permeability specimen. Examination of his data enabled Rayleigh to conclude that the branches corresponding to different loops of a family could be superposed when referred to a common loop tip, the branches all having the same parabolic form within the limits of accuracy of the measurements, over the range of magnetizing forces involved.

It is of course evident that even if this relation held at low fields in all materials it would break down at sufficiently high fields. Further, there is no *a priori* reason to expect this relation to hold for magnetic materials other than the one Lord Rayleigh investigated unless we restrict consideration to a very small range of magnetizing force. With these ideas in mind it seems the safer procedure to assume no such simple relation between the different loops of a family, however convenient it might be, and to treat the problem in more general terms; if any such simplifying relations exist they will be made apparent after application to definite materials. We may anticipate matters a bit to state at this point that certain materials seem to obey the relation while certain others seem to violate it within the range of forces involved in communication work, and further light is shed on the significant processes involved in harmonic production by treating the problem in this way.[6]

General Equations for Hysteresis Branches. The equations of both the upper branch family and the lower branch family have the form of equation (1)—we may designate the upper branches by B_1 and the lower branches by B_2—but the coefficients of the two series differ in general.

In order to put the equations in shape so that they may be of practical utility it is now necessary to determine the coefficients of the expressions for B_1 and B_2 so that they apply to a definite loop family, and this is accomplished by reference to some of the more general properties of the loops and of the normal magnetization curve. These properties are as follows:

[5] "Notes on Electricity and Magnetism, III," *Phil. Mag.*, 1887, V. 23.

[6] Unpublished work based on assumptions which include Rayleigh's relation was independently carried out by W. P. Mason of these Laboratories in 1922 and 1923. An account of some applications of Rayleigh's relation is to be found in an interesting paper by Jordan published in the *Elektrische Nachrichten Technik*, B. 1, H. 1, July 1924.

1. When both h and H are zero the flux density on either branch is zero.
2. The flux density on one branch with H and h given is equal and opposite in sign to the flux density on the other branch corresponding to the negative of h and to the same maximum force H.
3. The two branches corresponding to a definite H meet at the normal magnetization curve.

The application of these properties to the power series enables us to deduce relations between the coefficients as demonstrated in Appendix 1:

$$\begin{aligned} a_{01} &= a_{00} = 0, \\ a_{02} &= -a_{20}, \\ a_{03} &= -a_{21}. \end{aligned} \tag{7}$$

We shall find it sufficient to include the third degree terms for our work, so that we need not investigate relations between coefficients of higher degree. The loop equations are simplified by utilizing (7) and we can make a number of interesting deductions. Thus the equation for the normal *magnetization curve* is given by

$$B(H, H) = a_{10}H + a_{11}H^2 + (a_{12} + a_{30})H^3. \tag{6a}$$

In this equation a_{10} will be recognized as the initial permeability usually expressed as μ_0 since upon division by H we have for the *permeability*

$$\mu = a_{10} + a_{11}H + (a_{12} + a_{30})H^2 \cdots.$$

According to this equation the change of permeability with magnetizing force is linear at sufficiently small fields. The above equations are, in all rigor, infinite series, but for our purposes it will be found sufficient to consider only coefficients of the third and lower orders,—in some cases the second order will suffice.

An expression for the *remanence curve* of the loop family may be obtained by setting h equal to zero in the equation for the upper branch. In that case we find from (4a)

$$B(O, H) = a_{02}H^2 + a_{03}H^3 + \cdots, \tag{8}$$

hence for sufficiently small magnetizing forces the remanence increases as the square of the magnetizing force.

The *hysteresis loss per cycle per unit volume* may also be obtained directly from the branch equations (4a) and (5a). The loss in ergs, w, is equal to the area of the hysteresis loop divided by 4π. If now we consider the loop area to be built up of strips of infinitesimal width based on the h-axis, the height of any one of the strips is given as

$$B = B_1(h, H) - B_2(h, H)$$

and we have

$$w = \frac{1}{4\pi} \int_{-H}^{H} B dh \tag{9}$$

so that, by Appendix 1, (9) may now be expressed as

$$w = \frac{2}{3\pi} (a_{02}H^3 + a_{03}H^4 + \cdots). \tag{10}$$

It is clear, therefore, that at sufficiently low fields the hysteresis loss varies as the cube of the magnetizing force, and diverges when the field is made sufficiently large—it seems to be in general agreement with experimental results on ballistic loops and, as will be pointed out later, is verified by impedance change data under alternating excitation. In view of the remanence curve equation, (10) may be rewritten as

$$w = \frac{2}{3\pi} HB(O, H). \tag{11}$$

Various approximations have been made in the past to hysteresis loop forms in order to obtain convenient expressions for the hysteresis loss. Thus if we consider the loop as an ellipse the loss becomes

$$\frac{1}{4} HB(O, H),$$

while if we consider the loop a parallelogram the loss is

$$\frac{1}{\pi} HB(O, H).$$

Both these expressions give too large a result since the coefficient $2/3\pi$ of the exact equation (11) is 0.212.

Branch Equations for Materials Obeying Rayleigh's Relation. Rayleigh's observation enables us to establish relations between the different coefficients involved in our development above when the loops of a family are similar in form. For the sake of completeness a derivation of these relations is given in Appendix 2; their validity may then be judged by test in specific cases. In the derivation we assume a power series expansion for one branch of the largest loop of a family referred to the tip, and assume that the smaller loops are of the same form. Then by referring the equations to the origin instead of to the loop tip, we arrive at the hysteresis branch equations.

By comparing the coefficients of this equation with those of the general equation (1), we can deduce relations between the coefficients. These are, from Equation 16 of Appendix 2,

$$\begin{aligned} &a_{01} = 0, \\ &a_{20} + a_{02} = 0, \qquad && a_{11} = 2a_{20}, \\ &a_{21} + a_{03} = 0, && a_{12} = a_{21} = 3a_{30}, \\ &a_{40} + a_{22} + a_{04} = 0, && a_{31} = a_{13} = 4a_{40} = 6a_{22}. \end{aligned} \tag{17}$$

The left column is in agreement with Equation (7), while the right column furnishes new relations between the coefficients. Thus the coefficients based on loop similarity are not inconsistent with those deduced under no assumptions, but the former involve additional relationships between coefficients which need not be satisfied in the general case.

Families of hysteresis loops obtained by the usual ballistic method have been examined for these relationships in the case of several materials with the following results for coefficients up to and including the second degree:

Hysteresis Loop Coefficients

	Silicon Steel	'B' Dust [7]	'C' Dust [7]
a_{10}	270	35.3	26.2
a_{11}	2600	1.53	0.59
$a_{11}/2$	1300	0.76	0.29_5
a_{02}	967	0.65	0.29
a_{20}	− 951	− 0.71	-0.28_5

The values tabulated were obtained by 'smoothing' data obtained from ballistic loops, which leaves a great deal to be desired from the standpoint of precision. The difference between a_{02} and $-a_{20}$ is an index of this lack of precision. The average deviation of $a_{11}/2$ from the mean of a_{02} and a_{20} (a relation deduced on the basis of similarity) is small for C dust and large for silicon steel while the third order coefficients show much poorer results. The lack of precision in the original data however, is such as to leave unsettled the question of loop similarity for the materials tested at the fields to which the coefficients apply; the experimental error is too great.

The use of ballistic methods for determining coefficients becomes even less satisfactory for materials with low hysteresis losses and cannot be used for analyses which have any aspirations to precision. It will be shown in the next section, however, that the significant coefficients enter into the impedance of a coil which employs the material under examination as a core, so that the desired coefficients

[7] Speed and Elmen, "Magnetic Properties of Compressed Powdered Iron," *A. I. E. E.*, V. 40.

may be obtained with the aid of *AC* bridge measurements under appropriate conditions for which eddy currents and winding capacities are unimportant. These measurements are of a higher order of precision than those obtained by the ballistic method, and will be treated in detail in the next section.

Part 2. Alternating Magnetization

Sinusoidal Magnetizing Force. The same magnetic characteristics which were the subject of the investigation of the last section are involved in alternating magnetization when the applied field has but one maximum and but one minimum per cycle, and when the eddy losses are not great enough to introduce screening effects. As one of the results of that investigation, we have arrived at equations for both the upper and the lower branch families, so that to obtain an expression for the flux density valid over the entire cycle, it is now necessary to combine the two branches by a Fourier's series in the usual manner.

Before doing this, however, it is necessary to express the branch equations for the flux density in terms of time, rather than as series of powers of the independent variable, the magnetizing force. To do this we substitute the equation for the magnetizing force [8]

$$h = H \cos pt \tag{18}$$

in the branch equations (4*a*) and (5*a*) of Part 1. The result of the substitution is to give the upper and lower branch flux equations in terms of powers of cos pt, which may be expressed in terms of multiple angles. These two equations are then combined in a Fourier series which is valid over the entire cycle:

$$B = \frac{b_0}{2} + \sum_{k=1}^{k=\infty} (b_k \cos kpt + a_k \sin kpt),$$

in which the coefficients are given by the usual expressions, equation 22*a* of Appendix 3. The coefficients b_0, b_2, $\cdots$ b_{2k}, a_0, a_2, $\cdots$ a_{2k} are found to be identically zero on account of the symmetry of the loop family about the origin, while the fundamental and third harmonic coefficients are found to be as follows:

$$\begin{aligned} a_1 &= \frac{8}{3\pi}(a_{02}H^2 + a_{03}H^3), \\ b_1 &= a_{10}H + a_{11}H^2 + (a_{12} + \tfrac{3}{4}a_{30})H^3, \\ a_3 &= -\frac{8}{15\pi}(a_{02}H^2 + a_{03}H^3), \\ b_3 &= a_{30}H^3/4. \end{aligned} \tag{25}$$

[8] The purely sinusoidal magnetizing force may be obtained, despite the varying reaction of the iron core coil, by connecting a generator through a low pass or band pass filter having a high impedance outside the pass band, or by connecting a pure sine wave generator to the coil through a high impedance.

Details of the derivation are given in Appendix 3. Inasmuch as we assume the applied field to vary as cos pt, the b's are in phase with the applied field and the a's are in quadrature. The two a's, it is observed, are connected by a constant of proportionality ($a_1 = -5a_3$) and depend upon the remanence; or upon what is the same thing, the hysteresis loss divided by the magnetizing force.

If we expanded the loop equations to higher powers we should find that a_1 and a_3 cease to be linearly proportional. The coefficient b_1 is observed to have its first three terms identical with those of the normal magnetization curve, but the fourth term differs by precisely the amount b_3.

The voltage existing across a coil enclosing a core characterized by the above coefficients is

$$E = nA10^{-8}\frac{dB}{dt}, \qquad (26)$$

where n is the number of turns enclosing the core, A is the core area in cm², B is the total flux density and E is the total generated potential. Carrying out this operation we have

$$E = nA10^{-8}(pa_1 \cos pt + 3pa_3 \cos 3pt - pb_1 \sin pt - 3pb_3 \sin 3pt) \qquad (26a)$$

in which the voltage components in phase with the current depend on the hysteresis coefficients, and the quadrature components depend only on the coefficients of the normal magnetization curve. The dependence of each of these components upon the applied field is clear from Equation (25). To take the two third harmonic components it will be observed that a_3 starts to vary with the square, while b_3 starts to vary with the cube, of the applied field. It follows therefore that at sufficiently small amplitudes the third harmonic is produced by the a_3 term and not by the b_3 term, which means that *under the conditions noted, harmonic production is due to hysteresis and not primarily to permeability change.*

From the two fundamental components of voltage across the coil an expression for the inductance and resistance offered to the flow of alternating current may be deduced. The inductance, it is easy to see, is obtained directly from the magnetization curve,—the d.c. permeability of the normal magnetization curve therefore coincides with the a.c. permeability at small fields. The resistance may be obtained from the expression derived above for hysteresis loss per cycle per cc. If we multiply that value by the volume of iron in the coil, by the frequency, and by 10^{-7} to convert ergs to watts we have the loss per second, and this may be equated to the square of the effective fundamental current multiplied by the hysteresis resistance, or

$$wfA\pi d10^{-7} = I^2R_h/2.$$

Here d is the diameter in cm, I is the peak value of the fundamental current, f is the frequency, and R_h is the hysteresis resistance. This may be solved for the hysteresis resistance in terms of the r.m.s. value of fundamental current $\bar{I}$, which is at low fields,

$$R_h = 1.2\cdot 10^{-8}n^3Af\bar{I}a_{02}/d^2. \tag{27}$$

The hysteresis resistance at small fields thus varies linearly with the applied current.

It is easy to derive the relation between the hysteresis resistance and the generated third harmonic voltage at low fields since they involve the same constants a_{02} and a_{03}; from (26*a*) and (25) we have for the r.m.s. third harmonic voltage

$$\bar{E}_3 = 3pnA10^{-8}a_3/\sqrt{2} = 0.72\cdot 10^{-8}fn^3A\bar{I}^2a_{02}/d^2.$$

Comparison with (27) shows that

$$\bar{E}_3 = 0.6R_h\bar{I}. \tag{28}$$

This simple relation is valuable in obtaining an idea of the harmonic production in a specific coil through resistance measurements, and more than that, it enables us to determine the coefficients significant in the distorting process for the material under test, so that the harmonic production in a coil of any dimensions enclosing the same core material may be calculated. The degree of precision ordinarily attainable is brought out for two materials in Fig. 3, and the agreement is observed to be within the experimental error in those two representative cases.

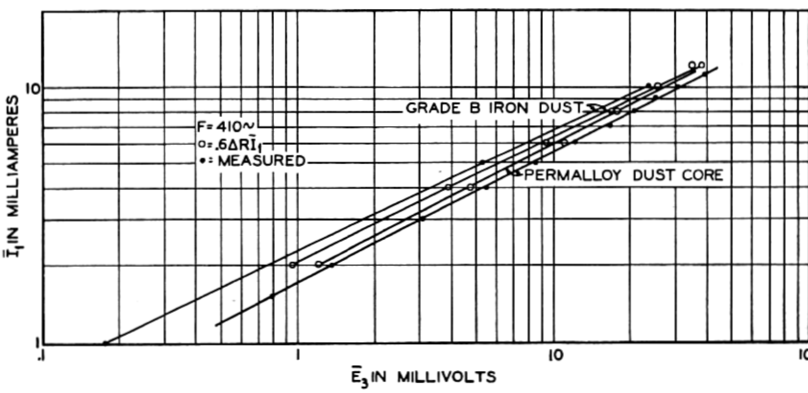

FIG. 3

These relations provide us with an expression for the ratio a_{11}/a_{02} which may be obtained from impedance measurements. It will be recalled that Rayleigh's relation would give this ratio the value two, and a.c. bridge measurements may be used to check this relation. The change of resistance with current is given by (27), and the change of reactance with current may be calculated from (25) and (26), since we have

$$\Delta X = a_{11} H^2 pnA 10^{-8}/\bar{I}\sqrt{2}$$
$$= 1.42 \cdot 10^{-8} n^3 Af\bar{I} a_{11}/d^2.$$

Hence if we denote R_h by ΔR, we may write

$$\frac{a_{11}}{a_{02}} = 0.85 \frac{\Delta X}{\Delta R}.$$

The results obtained on some dust cores of different composition and two solid materials may be tabulated as follows:

Material	Number of Specimens	a_{11}/a_{02}
Grade 'B' Iron Dust	10	2.65
Grade 'C' Iron Dust	1	2.63
Permalloy Dust [9] 'A'	3	2.10
" " 'B'	3	1.81
Iron Dust No. 1	1	2.80
Iron Dust No. 2	2	2.55
Perminvar [10]	1	3.0
Alloy 'D'	2	1.84

The method of analysis and the results obtained above may be summarized as follows. Starting with the general development of the hysteresis branches in a double power series and making use of only the most fundamental experimental observations, equations have been derived for the normal magnetization curve, the remanence characteristic, and the hysteresis loss characteristic in terms of the original hysteresis branch coefficients by combining the equations for the two branches in a trigonometric series. The series for the normal magnetization curve is found to start with the first power of the magnetizing force, the remanence starts with the second power, and the hysteresis loss starts with the third power, the curves varying in accordance with their respective first terms when H is small. These results seem to be in general agreement with experience. For large values of H the shapes of the different curves depend of course upon the size and sign of the coefficients involved, about which nothing can be said until the coefficients have been evaluated from experimental data.

[9] Shackelton and Barber, "Compressed Powdered Permalloy," A. I. E. E. Convention, February 1928.
[10] Elmen, "Magnetic Properties of Perminvar," *J. F. I.*, September 1928.

In the case of a sinusoidal magnetizing force the fundamental and third harmonic flux densities were derived with the aid of the trigonometric series development. These were shown to depend in a simple way at low fields upon the normal magnetization and hysteresis loss characteristics. The in-phase fundamental voltage component depends directly upon the hysteresis loss per unit current, while the fundamental component in quadrature varies with the magnetization curve at low forces. The in-phase third harmonic varies with the in-phase fundamental at low forces, and the third harmonic in quadrature comes in only with larger forces in a manner which depends upon the magnetization characteristic. The range of forces over which our equations are valid depends simply upon the number of terms taken in the development of the branch equations, and is not necessarily restricted to small forces. The hysteresis loop coefficient enters into the impedance offered to the fundamental frequency by a coil enclosing the core material in question in such a way that the third harmonic produced may be deduced from the change of resistance with current. Further, the ratio of the change of reactance with current to the change of resistance with current may be used to provide a test of Rayleigh's relation, which is found to hold for some materials, while it is invalid for others. The precision obtainable in the evaluation of hysteresis loop coefficients is much greater by the a.c. bridge measurement under proper experimental conditions than by the analysis of ballistic loops. Incidentally attempts have been made to obtain *B–H* loops by *AC* methods with the aid of the Braun tube, for example,[11] but the precision attainable is not sufficiently high for our purpose.

Complex Magnetizing Force. Our preceding analysis has furnished us with the fundamental voltage drop across a coil enclosing the iron core under consideration, together with the third harmonic voltage generated in the coil winding due in general partly to the non-linear *B–H* relation and partly to the effect of hysteresis. The generated voltages corresponding to the fifth, seventh, and higher orders are also calculable by the same methods but will not be specifically considered since they are smaller than the third harmonic at low fields. This generated third harmonic voltage exists in its entirety across the coil winding only when the impedance of the external circuit is much higher than that of the coil at the harmonic frequency, a condition not usually satisfied in telephone circuits. A current of the third harmonic frequency then flows in the circuit, its amplitude and phase depending evidently upon the generated third harmonic voltage—that is, the coil structure and core material—as well as upon the total

[11] Peterson, *Phys. Rev.*, Vol. 27, No. 3, p. 320.

circuit impedance to the third harmonic which includes that of the coil. We have now to determine the coil impedance to a complex magnetizing force constituted by the fundamental and the third harmonic. It is evident at the outset that if we restrict consideration to a very small third harmonic, the impedance to the fundamental cannot be very materially altered.

In general the fundamental and third harmonic may exist in any phase but for our present purpose we shall take the magnetizing force to be

$$H = H_1 \cos pt + H_3 \cos 3pt, \tag{29}$$

in which the phase angle is assumed zero. This seems to be an arbitrary assumption, but it may be shown that the results are not materially different for other phase displacements, and there is some gain toward simplicity of treatment by taking the phase angle zero. The equations of Part 1 may then be carried over without change, and details of the analysis are given in Appendix 4.

The ratio of the resistances to the third harmonic and to the fundamental is 0.77, from the relations deduced in the appendix. Some experimental work has been done to test the validity of the expressions derived above for the reactance and resistance components of the third harmonic, the results of which are given in Fig. 4. It is seen that the check for the inductance to the third harmonic is very good but that the resistance component appears to be substantially that of the fundamental rather than 77 per cent of it, as the analysis indicates.

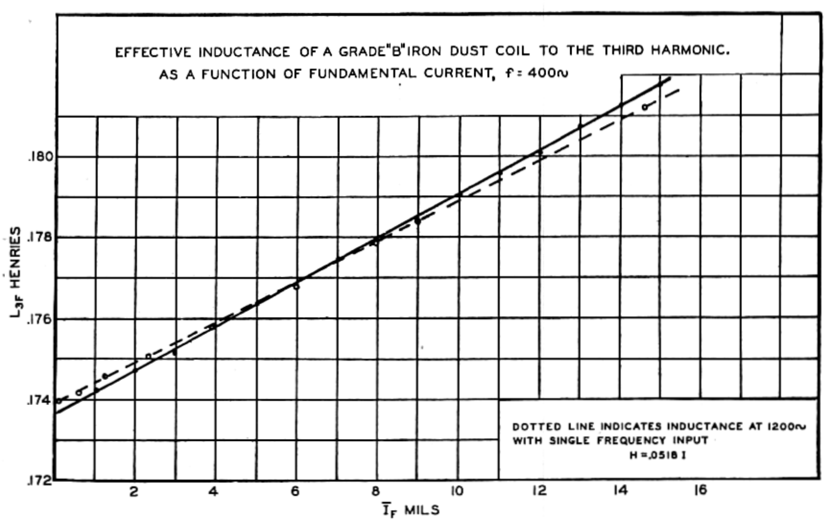

FIG. 4*A*

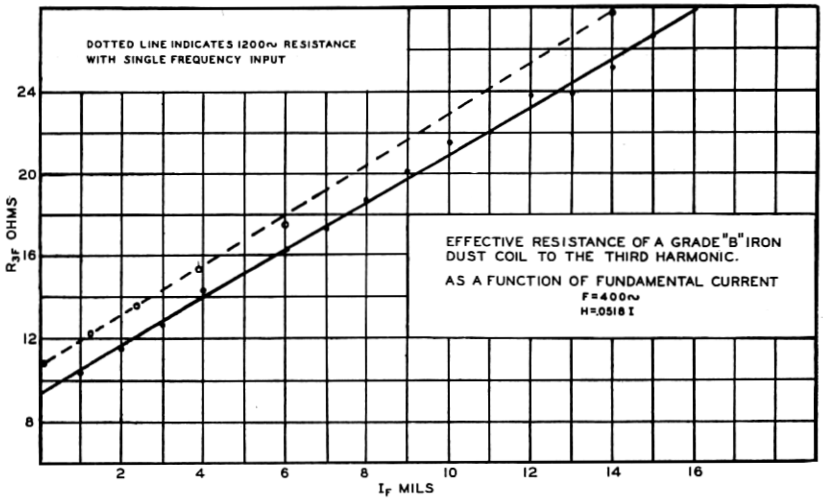

FIG. 4*B*

The experimental method for obtaining this result—the impedance to a small third harmonic in the presence of a relatively large fundamental—is illustrated in Fig. 5. The method consists in measuring the third harmonic current by means of a current analyzer [12] for a number of circuit conditions in which the fundamental current is

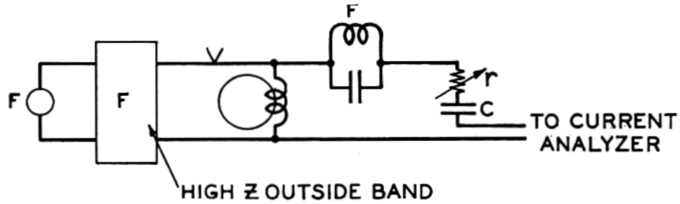

FIG. 5

maintained constant. The circuit is first tuned to the third harmonic by varying the capacity C in the third harmonic circuit, and the current is then measured for a series of values of the series resistance r. A shunt resonant circuit tuned to the fundamental is inserted in the third harmonic path so as to separate effectively the third harmonic circuit from that of the fundamental. With this precaution taken to avoid harmonic production in the analyzer and to maintain the fundamental current constant while r and C are varied, the inductance to the third harmonic is obtained from the resonating capacity, and the resistance is determined as that value of r for which the third harmonic current falls to half its maximum.

[12] For details of current analysis see the paper by A. G. Landeen, *B. S. T. J.*, April 1927.

PART 3. APPLICATION OF THE ANALYSIS

Air-Gaps and Dilution. Under certain conditions improvement in the operation of iron core coils toward freedom from harmonic production may be attained by inserting air-gaps in the magnetic path. The expressions which we have derived up to this point are valid for a material having the constants assigned, and the question now arises as to the parameters which characterize the operation of the iron core including air-gaps, and their relation to the parameters for the original core without air-gaps. With these relations given, our previous work may be applied to cores with air-gaps.

In establishing the correspondence between the parameters for the two cases, it is instructive to use two methods—one a direct attack,[13] the other resting on an analogy with non-linear vacuum tube circuits.[14] We may determine the effects sought for by the direct method on consideration of a single branch of a hysteresis loop, which is expressed by equations (4*a*) or (5*a*) of Part 1,

$$B(h, H) = \sum\sum a_{rs} h^r H^s. \tag{36}$$

Now with an air-gap in the magnetic circuit, the magnetomotive force effective is that applied, reduced by the drop across the air-gap, or

$$\begin{aligned} m' &= m - \rho\varphi, \\ M' &= M - \rho\Phi, \end{aligned} \tag{37}$$

in which mM, $\varphi\Phi$ are instantaneous and maximum values, respectively, of the impressed m.m.f. and flux, and in which ρ represents the air-gap reluctance

$$\rho = \lambda/A, \tag{38}$$

λ being the length of air-gap and A the core cross-section.

In order to apply (37) we re-express (36) in terms of magnetomotive force and flux as follows:

$$\varphi(m, M) = \sum_r \sum_s \frac{A a_{rs}}{l^{r+s}} m'^r M'^s, \tag{39}$$

where l is the length of magnetic circuit in the iron. Equations (37) may now be substituted in (39) to yield

$$\varphi(m, M) = \sum_r \sum_s \frac{A a_{rs}}{l^{r+s}} (m - \rho\varphi)^r (M - \rho\Phi)^s. \tag{40}$$

[13] Due to Mr. H. P. Evans.

[14] See Appendix 5.

This equation may be solved for φ by identifying coefficients when we substitute the solution

$$\varphi(m, M) = \sum_r \sum_s \frac{A a_{rs}'}{l^{r+s}} m^r M^s,$$

which is equivalent to

$$B(h, H) = \sum_r \sum_s a_{rs}' h^r H^s. \tag{41}$$

Carrying out the operations, the primed coefficients are determined in terms of the original coefficients and the core structure as follows:

$$\begin{aligned} a_{10}' &= a_{10} \bigg/ \left(1 + \frac{\lambda}{l} a_{10}\right), \\ a_{02}' &= - a_{20}' = a_{02} \bigg/ \left(1 + \frac{\lambda}{l} a_{10}\right)^3, \\ a_{11}' &= a_{11} \bigg/ \left(1 + \frac{\lambda}{l} a_{10}\right)^3. \end{aligned} \tag{42}$$

It is clear that the effect of an air-gap is to diminish the higher order coefficients to a greater extent than those of lower order; no new coefficients are introduced. If we refer to a constant flux density, then the impressed force is

$$H' = H\left(1 + \frac{\lambda}{l} a_{10}\right) \tag{43}$$

and the modulation voltage becomes proportional to

$$a_{02}' H'^2 = \frac{a_{02} H^2}{1 + \frac{\lambda}{l} a_{10}},$$

so that the harmonic e.m.f. has been reduced in the ratio of $1 + \frac{\lambda}{l} a_{10}$ to unity, which is precisely the ratio of initial permeabilities.

Dilution. If the magnetic material is diluted with a non-magnetic substance (insulation) so that the effective air-gap length is λ, the above equations for the flux density still hold. The effective cross-sectional area, however, is reduced to

$$A' = A\left(\frac{l - \lambda}{l}\right)^2, \tag{44}$$

which is ordinarily of small account compared to the other factors involved.

Choice of Core Material. The ideal characteristics for a core material in respect to its freedom from harmonic production may be determined by a consideration of the circuit problem. The third harmonic driving e.m.f. is obtained from the last section as

$$E_3 = pnAa_{02}H^2 10^{-8}, \qquad \doteq \frac{n^3A}{d^2}a_{02}I_1^2, \tag{45}$$

in which n represents the number of turns enclosing the core of area A, d is the mean diameter of the toroidal core and a_{02} is the hysteresis coefficient.

Suppose that we have to design a coil of fixed inductance in which the harmonic is to be a minimum, so that

$$L = K\frac{n^2\mu A}{d} = \text{const.} \tag{46}$$

We proceed to the consideration of a number of special cases subject to condition (46).

Case 1—L **Fixed,** *n* **Variable.** From (46)

$$n = \text{const.}\left(\frac{d}{\mu A}\right)^{1/2},$$

which may be substituted in (45) to give

$$E_3 \doteq \frac{1}{\sqrt{A}d}\frac{a_{02}}{(\mu)^{3/2}}. \tag{47}$$

Hence in a coil of fixed inductance, fixed core area, and fixed core diameter, minimum harmonic voltage is produced with a material for which $a_{02}/\mu^{3/2}$ is minimum.

Case 2—L **Fixed,** *A* **Variable.** From (46)

$$A = \text{const.}\ d/n^2\mu,$$

which, upon insertion in (45), yields

$$E_3 \doteq \frac{n}{d}\frac{a_{02}}{\mu}, \tag{48}$$

in which the modulated voltage varies linearly with a_{02}/μ.

Case 3—L **Fixed,** *d* **Variable.** By a similar procedure, we obtain an expression for the generated third harmonic voltage

$$E_3 \doteq \frac{1}{nA}\frac{a_{02}}{\mu^2}. \tag{49}$$

For further work we suppose the coil reactance considerably greater than the circuit resistance at the fundamental frequency, and consider the case of an input transformer in which the secondary is practically open-circuited.

Case 4—**Input Transformer.** In accordance with the above assumption the fundamental current through the coil is determined by the coil reactance x or

$$I_1 = \frac{E}{x} \doteq \frac{d}{n^2 \mu A}.$$

Putting this in (45), we find

$$E_3 \doteq \frac{1}{nA} \frac{a_{02}}{\mu^2}, \tag{50}$$

which is identical with Equation (49).

In the four cases treated above it has appeared that there are three quantities which characterize the modulating properties of a ferromagnetic coil under different conditions—a_{02}/μ, $a_{02}/\mu^{3/2}$, a_{02}/μ^2. The values of these ratios have been determined for a number of materials and are tabulated below.

	a_{02}/μ	$a_{02}/\mu^{3/2}$	a_{02}/μ^2
B dust	18×10^{-3}	3.1×10^{-3}	0.52×10^{-3}
C dust	11	2.2	0.42
Permalloy dust [15]	17	2.1	0.24
Perminvar [15]	2.1	0.1	0.0045
Silicon Steel	2500	210	16

These results apply only when eddy currents are negligible; as the frequency is raised eddy currents effectively screen the core and the harmonic flux is reduced to a greater extent than is the fundamental. This effect, as is well known, depends upon the specific resistivity of the core material and will not be evaluated here.

It has been pointed out that under different circuit conditions an index of the modulation is given by the ratios a_{02}/μ, $a_{02}/\mu^{3/2}$, a_{02}/μ^2, depending upon specific conditions. For diluted materials, these ratios referred to the undiluted material become

$$\frac{a_{02}}{a_{10}\left(1 + \frac{\lambda}{l} a_{10}\right)^2}, \qquad \frac{a_{02}}{\left[a_{10}\left(1 + \frac{\lambda}{l} a_{10}\right)\right]^{3/2}}, \qquad \frac{a_{02}}{a_{10}^2\left(1 + \frac{\lambda}{l} a_{10}\right)},$$

and the utility of air-gaps toward the reduction of harmonic is made evident quantitatively in every case.

In order to test these relations a comparison was made of the

[15] Laboratory specimens.

coefficients for specimens of grade "B" and grade "C" dust which represent different dilutions of electrolytic iron. The coefficients used were those tabulated in Part 1. In accordance with equations (42) above we may write

$$\begin{aligned} a_{10}\Big/\left(1+\frac{\lambda_1}{l}a_{10}\right) &= 26.2, & a_{10}\Big/\left(1+\frac{\lambda_2}{l}a_{10}\right) &= 35.3, \\ a_{11}\Big/\left(1+\frac{\lambda_1}{l}a_{10}\right)^3 &= 0.59, & a_{11}\Big/\left(1+\frac{\lambda_2}{l}a_{10}\right)^3 &= 1.53, \\ a_{02}\Big/\left(1+\frac{\lambda_1}{l}a_{10}\right)^3 &= 0.29, & a_{02}\Big/\left(1+\frac{\lambda_2}{l}a_{10}\right)^3 &= 0.65. \end{aligned} \tag{51}$$

From these, we should have equality of the ratios

$$0.59/1.53, \qquad 0.29/0.65, \qquad (26.2/35.3)^3,$$

or

$$0.37, \qquad 0.45, \qquad 0.41,$$

which are evidently in fair agreement. It is clear then that the properties of diluted materials may be calculated from the characteristics of the original material, at least when the process of dilution does not change the intrinsic properties of the magnetic material involved.

Applications to Transformers. The third harmonic flux component may be obtained when the fundamental magnetizing force is given, and the latter is easily obtained in toroidal core inductance coils from the relation

$$h = 0.4ni/d, \tag{52}$$

where both h and i refer to the fundamental frequency. In a transformer, however, the net magnetizing force is obtained as the sum of two components, one due to the primary and the other produced by the secondary, so that some further investigation is required before the net magnetizing force in the core may be calculated in terms of the transformer constants and the primary current.

Net Magnetizing Force. To obtain the net magnetizing force which determines uniquely the generated flux components we are required to obtain the primary and secondary currents ($i_1 i_2$), to multiply each current by the number of times it encircles the core, to add the two products, and finally to multiply the sum by $0.4/d$:

$$H_{\text{net}} = 0.4(n_1 i_1 + n_2 i_2)/d. \tag{53}$$

To evaluate (53) then we shall solve for the primary and secondary fundamental currents in terms of the circuit constants and the applied potential.

The transformer circuit equations from which the currents may be evaluated are the familiar ones

$$\begin{aligned} E &= Z_1 i_1 + jpMi_2, \\ O &= Z_2 i_2 + jpMi_1, \end{aligned} \tag{54}$$

in which we assume the transformer to be an idealized structure—the capacity and leakage effects of actual transformers will be assumed combined with the connected impedances for simplicity—they will therefore not be dealt with explicitly. From the second of (54) is obtained the relation between the two currents

$$i_2 = -jpMi_1/Z_2 \tag{55}$$

which may be substituted in the first of (54) to give the usual expression for the primary current

$$i_1 = E \Big/ \left[R_1 + \frac{p^2M^2}{Z_2^2} R_2 + j\left(X_1 - \frac{p^2M^2}{Z_2^2} X_2 \right) \right]. \tag{56}$$

An expression for the net magnetizing force in terms of i_1 may be had putting (55) and (56) in (53)

$$H_{\text{net}} = \frac{0.4n_1 i_1}{d}\left(1 - \frac{jpMK}{Z_2}\right), \tag{57}$$

in which K represents the turns ratio:

$$K = \frac{n_2}{n_1} = \frac{\sqrt{X_2}}{\sqrt{X_1}}. \tag{58}$$

If we make the convenient assumption, which is closely approximated under working conditions, that the coupling is perfect (no leakage) we have

$$pM = \sqrt{X_1 X_2} \tag{59}$$

and (57) may be simplified to the form

$$H_{\text{net}} = \frac{0.4n_1 i_1}{d} \frac{R_2}{Z_2}. \tag{60}$$

This equation shows that the net magnetizing force may be obtained from that of the primary alone by applying the reduction factor R_2/Z_2. Now in well designed transformers the winding reactances are much greater than the resistances to which they are connected, and in this case

$$X^2_{1,2} \gg R^2_{1,2}. \tag{61}$$

Applying this further simplification to (60) we have finally for the net

50

magnetizing force

$$H_{net} = \frac{0.4 n_1 i_1}{d} \frac{R_2}{X_2}, \tag{60a}$$

in which the reduction factor applied to the force due to the primary alone is seen to be the ratio of resistance to reactance in the secondary circuit.

This equation with the aid of (56) may now be put in terms of the primary voltage—a fixed quantity—rather than in terms of the primary current which is variable according to the circuit resistances used. Applying the assumptions (59) and (61), Equation (56) may be written

$$i_1 = E/(R_1 + R_2/K^2), \tag{56a}$$

which may be put in (60a) to express the net force explicitly in terms of the circuit constants:

$$H_{net} = \frac{0.4 n_1 E}{d X_1}\left(\frac{1}{1 + K^2 R_1/R_2}\right) \tag{62}$$

In view of (58). When the transformer is terminated in its normal resistances the expression within parentheses reduces to the value one-half. It is of interest in this connection to compare the field here with that produced with the secondary open; in the latter case it is clearly

$$0.4 n_1 E/d X_1,$$

which is twice as great as the field in the properly terminated transformer. This result is otherwise evident, for in the properly terminated transformer but half the generator voltage is applied across the primary.

Third Harmonic. According to Equation (28) the amplitude of third harmonic voltage generated in a coil of n turns encircling a core subjected to a magnetizing force of H_{net} gilberts/cm is

$$E_3 = \frac{8}{5\pi} 10^{-8} a_{02} p n A H^2_{net} \tag{63}$$

and the generated primary or secondary voltage is obtained by attaching appropriate subscripts to n. Because of the coupling, the two circuits react on one another at the third harmonic frequency as they did at the fundamental frequency, and to find the two third harmonic currents we must go through a procedure analogous to that followed for the fundamentals except that here we have an independent generator voltage effective in each of the two windings.

Indicating quantities referring to the third harmonic frequency by the subscripts p, s for primary and secondary circuits, respectively, we have the circuit equations for the third harmonic currents:

$$\begin{aligned} e_p &= Z_p i_p + j3pMi_s, \\ Ke_p &= Z_s i_s + j3pMi_p, \end{aligned} \tag{64}$$

and by equating the two expressions for e_p we obtain a relation between the two currents:

$$i_s = \frac{KZ_p - j3pM}{Z_s - j3pMK} i_p. \tag{65}$$

Putting i_p in terms of e_p, and using (59) and (61) we find

$$i_s = e_p KR_p/R_s X_p(1 + K^2 R_p/R_s). \tag{66}$$

Suppose now that we are dealing with pure resistance loads so that

$$R_1 = R_p, \qquad R_2 = R_s.$$

We may then substitute (63) for e_p in (66) to obtain the third harmonic current in the secondary.

From (63)

$$e_p = \frac{8}{5\pi} 10^{-8} a_{02} p n_1 A \frac{0.16 n_1^2 E^2}{d^2 X_1^2} \left(\frac{1}{1 + K^2 R_1/R_2} \right)^2, \tag{67}$$

and by substitution in (66)

$$i_s = \frac{E^2 K R_1 \alpha}{3 X_1 R_2 \left(1 + K^2 \dfrac{R_1}{R_2} \right)^3}. \tag{68}$$

If we consider the factor containing the resistances,

$$F = \frac{R_1}{R_2} \frac{1}{(1 + K^2 R_1/R_2)^3}, \tag{69}$$

it is zero for R_1/R_2 zero or infinite, and reaches a maximum under the condition

$$\frac{\partial F}{\partial(R_1/R_2)} = 0. \tag{70}$$

The third harmonic secondary current is maximum when

$$K^2 = \frac{R_2}{2R_1}. \tag{71}$$

Now K^2 is fixed by the turns ratio (58), so that we may say the secondary third harmonic current is maximum when the primary resistance is made half its nominal value, or when the secondary resistance is made twice its nominal value. This is well borne out by Fig. 6 due to A. G. Landeen taken under conditions to which the above discussion applies.

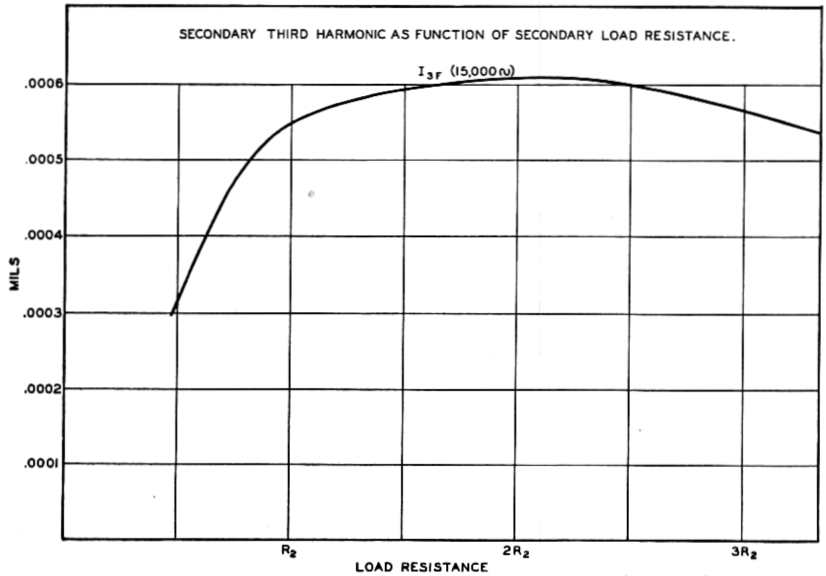

Fig. 6

At first sight the result obtained, in which the secondary harmonic current increases as the secondary resistance is increased for all values of secondary resistance below twice the nominal secondary resistance, seems a bit strange since the fundamental secondary current decreased under the same conditions. A little thought however shows that the increase of secondary resistance acts in two ways; first to reduce the harmonic current directly, and second to increase the net magnetizing force in the core which in turn increases the generated harmonic. Thus with zero secondary resistance the net magnetizing force is zero and no harmonic flux can be produced, while with large secondary loads the flux density in the core or the net magnetizing force reaches an asymptotic maximum so that the generated third harmonic increases slowly if at all while the reduction in secondary current by the resistance is continuously increasing. Under the assumptions noted Equation (68) is valid for any circuit condition and it will be observed that i_s decreases with the primary resistance below the optimum point. The value of α is explicitly

$$\alpha = 2.72 \times 10^{-10} \frac{a_{02}}{p^2 L_1^3} \left(\frac{n_1^3 A}{d^2} \right), \tag{72}$$

in which the bracketed expression coincides with the variable factor in equation (45). Hence when L_1 is constant the choice of core material follows the rules laid down for inductance coils. When the inductance is not constant, however, the factor becomes proportional to

$$\alpha = \text{const.}\frac{da_{02}}{n_1^3\mu^3A^2} \tag{73}$$

with the understanding that the turns ratio, resistances and frequency are fixed. Thus the secondary harmonic current with a given material is reduced by increasing the turns and core area and reducing the diameter. As far as core materials go, we have for the significant ratio a_{02}/μ^3 the values:

Material	a_{02}/μ^3
Silicon Steel	130×10^{-4}
Permalloy Dust [16]	2.4×10^{-4}
B Dust	5.3×10^{-4}
C Dust	4.3×10^{-4}
Perminvar [16]	0.05×10^{-4}

The great superiority of perminvar indicated above is restricted of course to fields of the order of less than 0.1 gilbert/cm. above which it tends to become smaller; at a field of 0.7 for example the above factor for perminvar would be multiplied by the factor three. Permalloy dust is observed to be approximately twice as good as the iron dust cores.

A very important question to answer with regard to transformer cores is this—what benefit can be gained as to harmonic production by inserting air-gaps, or by diluting the core material, leaving everything else unchanged. This is evidently to be answered by an investigation of the ratios a_{02}/μ^3 and a'_{02}/μ'^3, the primes referring as before to the diluted material. These relations are given by Equation (42) in which μ and a_{10} are interchangeable:

$$a_{10}' = a_{10} \Big/ \left(1 + \frac{\lambda}{l} a_{10}\right) = \mu',$$

$$a_{02}' = a_{02} \Big/ \left(1 + \frac{\lambda}{l} a_{10}\right)^3.$$

These permit us to evaluate a'_{02}/μ'^3:

$$\frac{a_{02}'}{\mu'^3} = \frac{a_{02}}{\left(1 + \frac{\lambda}{l} a_{10}\right)^3} \frac{\left(1 + \frac{\lambda}{l} a_{10}\right)^3}{a_{10}^3} = \frac{a_{02}}{\mu^3}, \tag{74}$$

[16] Laboratory specimens.

which clearly indicates that no change is produced in the secondary third harmonic current by introducing air-gaps or by diluting the core material, the change in core area being negligible. This relation has been verified experimentally.

The discussion above considers a constant potential applied to a transformer circuit and the relations derived are somewhat changed when we consider constant current or constant potential transformers. We seldom have to do with constant current transformers but constant potential transformers are met with frequently in vacuum tube circuits as input transformers or interstage transformers. The above discussion may be applied to this case by taking the primary generator resistance much lower than its nominal value—a condition, incidentally, which works toward the suppression of harmonic.

The conclusions are also somewhat altered when we have networks offering different impedances to the harmonic and the fundamental. Thus it is evident that the third harmonic current can be eliminated from both primary and secondary circuits by inserting a high series impedance to the harmonic frequency, or by encircling the core with a winding connected to a network which has a very low impedance to the third harmonic and high impedance to the fundamental. Further, in single frequency transmission the result may be equally well obtained by shunting a series tuned circuit around the primary or secondary winding.

Applications to Harmonic Production in Loaded Lines

If distortion takes place at any one point of a loaded line, the amount of distorted current received at the far end depends upon the amplitudes of the currents producing it, and upon the line attenuation from the point of origin to the far end. The phase similarly depends upon the phase of the fundamentals, and upon the phase shift of the line. When we have a number of distorting sources at different points along the line, the net distorted output is obtained by combining vectorially the currents due to the individual centers of distortion, since it may be assumed that no interaction exists. In a uniformly loaded line we may think of these sources of distortion as being uniformly distributed, but the amount of distortion introduced is, on the contrary, not distributed uniformly on account of the line attenuation which reduces the distortion generated at the far end of the line.

It is apparent, then, that if we are to calculate the net distortion introduced by the line, a complete specification of the phase shift and attenuation is required, together with a knowledge of the law of production of the distortion. This last also requires a specification of amplitude and phase angle in their relation to the fundamentals producing the distortion.

A situation somewhat analogous to the one under discussion which is less involved, however, is found in the diffraction of light, in which the illumination at a point proceeds from a number of coherent sources at various distances from the point. An important difference in the two cases is the attenuation of the transmitting medium which is ordinarily small in the optical case, so that the results of the optical investigations cannot be taken over directly.

In the following we propose to calculate the third harmonic output currents with the aid of our previously established relations. The law of harmonic production has been quite definitely established in the low frequency-low flux density range. The driving third harmonic voltage of frequency *3f* produced by a fundamental current of *rms* value $\bar{I}$ is given by (25) and (26*a*) as

$$E_3 = 0.72 \times 10^{-8} a_{02} \frac{n^3 A}{d^2} f \bar{I}^2 = M\bar{I}^2, \tag{75}$$

while the phase angle of the third harmonic generated potential is related to that of the fundamental current by the expression

$$\theta_3 = 3\theta_1. \tag{76}$$

The above data are sufficient for a solution of the problem of distortion in continuously loaded lines when the propagation constant is known as a function of frequency.

The fundamental current at any point distant x from the sending end of the continuously loaded, properly terminated line is

$$I = I_0 \epsilon^{-Px}, \tag{77}$$

where

$$P = \alpha + j\beta.$$

The third harmonic driving e.m.f. dE_3 generated in a length dx of the line may be written with the aid of (75), (76), and (77) as

$$dE_3 = MI_0^2 \epsilon^{-(2\alpha + j3\beta)x}. \tag{78}$$

Writing the line impedance as Z_0, the resulting current at x is

$$dI_3 = \frac{MI_0^2}{2Z_0} \epsilon^{-(2\alpha + j3\beta)x}. \tag{79}$$

With the line parameters a function of frequency we may write for the propagation of third harmonic current

$$i_3 = I_3 \epsilon^{-(\gamma + j\delta)x}. \tag{80}$$

The distortion at x produces a third harmonic current at the receiving end

$$di_3 = \frac{MI_0^2}{2Z_0}\epsilon^{-(2\alpha+j3\beta)x}\epsilon^{-(\gamma+j\delta)(l-x)}. \tag{81}$$

The total third harmonic current at the receiving end may now be obtained by integrating (81) over the length of the line l, which gives us

$$i_3 = \frac{MI_0^2}{2Z_0}\frac{\epsilon^{-(\gamma+j\delta)l}}{[(2\alpha-\gamma)+j(3\beta-\delta)]}(1-\epsilon^{-[2\alpha-\gamma+j(3\beta-\delta)]l}). \tag{82}$$

Inasmuch as we are concerned with the output amplitude, the above expression may be put in a somewhat more convenient form:

$$|i_3|^2 = \left(\frac{MI_0^2}{2Z_0}\right)^2\frac{\epsilon^{-2\gamma l}}{(2\alpha-\gamma)^2+(3\beta-\delta)^2} \times [1+\epsilon^{-2(2\alpha-\gamma)l}-2\epsilon^{-(2\alpha-\gamma)l}\cos(3\beta-\delta)l]. \tag{83}$$

Thus when l is zero the harmonic vanishes as it should, and as l increases the current passes through maxima and minima determined by the cosine term. If the attenuation is not very great, the maxima occur approximately at the line lengths

$$l = \frac{(2n-1)\pi}{3\beta-\delta},$$

where n is a positive integer. These distances correspond to odd half wave-lengths, as is true of the optical case. As l is increased the bracketed expression approaches unity and the current falls exponentially. Before this point is reached, however, the current increases in certain regions as the line length is increased. The fact that in an actual line the parameters vary with frequency means that these maxima and minima will vary in position according to the frequency, so that a maximum for one frequency may well coincide with a minimum for another frequency.

In the case of lumped loading, the integrations used for continuous loading are replaced by summations, as was done by Mason in the unpublished investigation previously cited.[4] If the spacing between coils is x_1, and if we have n coils for which $nx_1 = y$, the received third harmonic current at the end of a properly terminated line may be written

$$i_3 = \frac{MI_0^2}{2Z_0}\frac{\epsilon^{-(\gamma+j\delta)l}\{1-\epsilon^{-[2\alpha-\gamma+j(3\beta-\delta)]y}\}}{(\epsilon^{[2\alpha-\gamma+j(3\beta-\delta)]x_1}-1)}, \tag{84}$$

which is to be compared with (82) for the continuously loaded line.

Appendix 1: Simplification of Loop Equations

From the first of the three properties mentioned we have $a_{00} = 0$, and from the second property

$$B_1(h, H) = -B_2(-h, H) \tag{3}$$

whence, if we write

$$\begin{aligned} B_1(h, H) = {} & a_{10}h + a_{01}H \\ & + a_{20}h^2 + a_{11}hH + a_{02}H^2 \\ & + a_{30}h^3 + a_{21}h^2H + a_{12}hH^2 + a_{03}H_3 \cdots, \end{aligned} \tag{4}$$

then

$$\begin{aligned} B_2(h, H) = {} & a_{10}h - a_{01}H \\ & - a_{20}h^2 + a_{11}hH - a_{02}H^2 \\ & + a_{30}h^3 - a_{21}h^2H + a_{12}hH^2 - a_{03}H^3 \cdots. \end{aligned} \tag{5}$$

From the third property, the two branches meet at the loop tip which lies on the magnetization curve for which $h = H$, or

$$B_1(H, H) = B_2(H, H). \tag{6}$$

From the two equations (4) and (5) we have by virtue of this relation

$$a_{01}H + (a_{20} + a_{02})H^2 + (a_{21} + a_{03})H^3 = 0$$

and since this relation holds for every value of H, the coefficients of each power of H must be zero so that

$$\begin{aligned} a_{01} &= 0 = a_{00}, \\ a_{02} &= -a_{20}, \\ a_{03} &= -a_{21}. \end{aligned} \tag{7}$$

The final expressions for the branches are then evidently obtained by putting (7) in (4) and (5) which become

$$\begin{aligned} B_1(h, H) = {} & a_{10}h \\ & - a_{02}h^2 + a_{11}hH + a_{02}H^2 \\ & + a_{30}h^3 - a_{03}h^2H + a_{12}hH^2 + a_{03}H^3, \end{aligned} \tag{4a}$$

$$\begin{aligned} B_2(h, H) = {} & a_{10}h \\ & + a_{02}h^2 + a_{11}hH - a_{02}H^2 \\ & + a_{30}h^3 + a_{03}h^2H + a_{12}hH^2 - a_{03}H^3. \end{aligned} \tag{5a}$$

Appendix 2: Rayleigh's Relation

If we suppose the lower branch of any loop, when referred to the tip of the largest loop considered, to be given by the equation

$$B_1 = \mu_0 h_1 + \nu h_1^2 + \lambda h_1^3 + \omega h_1^4 \tag{12}$$

we may refer the family of branches to the origin by the transformation

$$\begin{aligned} h &= h_1 - H, \\ B &= B_1 - B_m, \end{aligned} \tag{13}$$

if B_mH refer the midpoint of the largest loop to the tip. Then

$$B_m = \mu_0 H + 2vH^2 + 4\lambda H^3 + 8\omega H^4. \tag{14}$$

Putting (13) in (12)

$$B + B_m = \mu_0(h + H) + v(h + H)^2 + \lambda(h + H)^3 + \omega(h + H)^4,$$

whence, subtracting (14),

$$\begin{aligned} B' = \mu_0 h + v(h^2 + 2hH - H^2) + \lambda(h^3 + 3h^2H + 3hH^2 - 3H^3) \\ + \omega(h^4 + 4h^3H + 6h^2H^2 + 4hH^3 - 7H^4), \end{aligned} \tag{15}$$

which represents the hysteresis branch equation referred to the origin, on the basis of loop similarity.

The coefficients obtained by the two methods may now be compared. Thus identifying coefficients of (15) with those of the general equation (1)

$$\begin{array}{llll} a_{10} = \mu_0, & a_{11} = 2v, & a_{02} = -v & a_{12} = 3\lambda, \\ a_{20} = v, & a_{21} = 3\lambda, & a_{03} = 3\lambda, & a_{13} = 4\omega, \\ a_{30} = \lambda, & a_{31} = 4\omega, & a_{04} = -7\omega, & a_{22} = 6\omega. \\ a_{40} = \omega, & & & \end{array} \tag{16}$$

Appendix 3: Alternating Magnetization, Sinusoidal Magnetizing Force

The resulting expression is simplified if we make the following substitutions

$$\begin{aligned} \alpha &= a_{02}H^2 + a_{03}H^3 = B(O, H), \\ \beta &= a_{10} + a_{11}H + a_{12}H^2, \\ \delta &= a_{30}H^3. \end{aligned} \tag{19}$$

It may be noted that α is the remanence and that β is an approximation to the permeability, in fact the permeability is given as the sum of β and δ. With (18) and (19) inserted in the branch equations, then, we have

$$\begin{aligned} B_1(H \cos pt, H) &= \alpha + \beta \cos pt - \alpha \cos^2 pt + \delta \cos^3 pt, \\ B_2(H \cos pt, H) &= -\alpha + \beta \cos pt + \alpha \cos^2 pt + \delta \cos^3 pt. \end{aligned}$$

For convenience we shall express these relations in terms of multiple angles, and we have for the equation of the upper loop family

$$B_1(H \cos pt, H) = -\alpha/2 + (\beta + 3\delta/4) \cos pt - \alpha/2 \cos 2pt + \delta/4 \cos 3pt.$$

If we write

$$\begin{aligned} A &= \alpha/2 = (a_{02}H^2 + a_{03}H^3)/2, \\ B &= \beta + 3\delta/4 = a_{10} + a_{11}H + a_{12}H^2 + 3a_{30}H^3/4, \\ C &= \quad = -A, \\ D &= \delta/4 = a_{30}H^3/4, \end{aligned} \tag{20}$$

the final form for the loop equations is

$$\begin{aligned} B_1(H\cos pt, H) &= \quad A + \mathrm{B}^{17}\cos pt + C\cos 2pt + D\cos 3pt, \\ B_2(H\cos pt, H) &= -A + B\ \cos pt - C\cos 2pt + D\cos 3pt. \end{aligned} \tag{21}$$

We are now in position to combine the two equations of (21) in a Fourier series valid over the entire cycle as

$$B = \frac{b_0}{2} + \Sigma(b_k \cos kpt + a_k \sin kpt), \tag{22}$$

where

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_0^{2\pi} f(pt)\sin kpt\, d(pt), \\ b_k &= \frac{1}{\pi}\int_0^{2\pi} f(pt)\cos kpt\, d(pt). \end{aligned} \tag{22a}$$

For our particular case we have, since $B = B_1$ for the first half of the cycle, and $B = B_2$ for the second:

$$\begin{aligned} \pi a_k &= \int_{-\pi}^{0} B_2(h, H)\sin kpt\, d(pt) + \int_0^{\pi} B_1(h, H)\sin kpt\, d(pt), \\ \pi b_k &= \int_{-\pi}^{0} B_2(h, H)\cos kpt\, d(pt) + \int_0^{\pi} B_1(h, H)\cos kpt\, d(pt). \end{aligned}$$

These integrals may be simplified considerably when we take advantage of the fact that both B_1 and B_2 are even functions of the time as given by (21). Thus

$$\begin{aligned} \pi a_k &= \int_0^{\pi} [B_1(h, H) - B_2(h, H)]\sin kpt\, d(pt), \\ \pi b_k &= \int_0^{\pi} [B_1(h, H) + B_2(h, H)]\cos kpt\, d(pt). \end{aligned}$$

Referring to (21) we may then write

$$\begin{aligned} a_k &= \frac{2}{\pi}\int_0^{\pi} (A + C\cos 2pt)\sin kpt\, d(pt), \\ b_k &= \frac{2}{\pi}\int_0^{\pi} (B\cos pt + D\cos 3pt)\cos kpt\, d(pt). \end{aligned} \tag{23}$$

[17] This coefficient is not to be confounded with the general expression for flux density.

Upon integration of (23) the coefficients for the fundamental and third harmonic flux components are found to be as follows:

$$a_1 = \frac{4}{\pi}(A - C/3), \qquad b_1 = B,$$
$$a_3 = \frac{4}{\pi}\left(\frac{A}{3} + \frac{3C}{5}\right), \qquad b_3 = D, \tag{24}$$

which, by reference to (20), may be put in terms of the branch coefficients.

Appendix 4—Impedance Reaction to a Small Third Harmonic in the Presence of a Large Fundamental

We have for the two hysteresis branch equations from Eqs. (4a), (5a)

$$\begin{aligned} B_1(h, H) &= B(O, H) + \beta h + \gamma h^2 + a_{30}h^3 + \cdots, \\ B_2(h, H) &= -B(O, H) + \beta h - \gamma h^2 + a_{30}h^3 + \cdots, \end{aligned} \tag{30}$$

in which

$$\begin{aligned} \beta &= a_{10} + a_{11}H + a_{12}H^2, \\ \gamma &= -(a_{02} + a_{03}H). \end{aligned} \tag{31}$$

Putting (29) in (30) we get

$$\begin{aligned} B(h, H) = A &+ B\cos pt + C\cos 2pt + D\cos 3pt + F\cos npt \\ &+ G[\cos(n+1)pt + \cos(n-1)pt] \\ &+ J[\cos(n+2)pt + \cos(n-2)pt] \end{aligned} \tag{32}$$

in which the coefficients have the following significance

$$\begin{aligned} A &= B(O, H) + H_1^2\gamma/2, & F &= \beta H_3 + 3a_{30}H_1^2H_3/2, \\ B &= \beta H_1 + 3a_{30}H_1^3/4, & G &= \gamma H_1H_3, \\ C &= \gamma H_1^2/2, & J &= 3a_{30}H_1^2H_3/4. \\ D &= a_{30}H_1^3/4, \end{aligned} \tag{33}$$

The coefficients of the Fourier Series for the output wave may now be obtained as before by combining the two equations (30) since each one is operative during one-half the cycle. There results an expression similar to the one obtained in the single frequency case, and since we have

$$I = b_0/2 + \Sigma a_k \sin kpt + \Sigma b_k \cos kpt$$

the coefficients are evaluated from the expressions

$$
\begin{aligned}
a_k &= \frac{2}{\pi}\int_0^{2\pi} (A + C\cos 2pt + G(\cos 4pt + \cos 2pt)) \sin kpt\, d(pt), \\
b_k &= \frac{2}{\pi}\int_0^{2\pi} (B\cos pt + (D + F)\cos 3pt \\
&\qquad + J(\cos 5pt + \cos pt)) \cos kpt\, d(pt).
\end{aligned} \tag{34}
$$

Upon integration we find

$$b_1 = B, \qquad b_3 = F. \tag{35}$$

Comparing these two coefficients we see that at low amplitudes we may write

$$b_1 = \mu H_1, \qquad b_3 = \mu H_3,$$

in which the permeability is the same to the two components, and is determined by the fundamental amplitude. For the dissipative terms we find

$$
\begin{aligned}
a_1 &= \frac{4}{\pi}(A - C/3 - 2G/5), \\
a_3 &= \frac{4}{\pi}(A/3 - 3C/5 + 6G/35),
\end{aligned} \tag{35'}
$$

but some care is required in interpreting these expressions. Inasmuch as we are primarily interested here in determining the dissipative component to a third harmonic magnetizing force of amplitude H_3, we are required to select from a_3 only those terms containing H_3, which means the single term $24G/35\pi$. The other terms take care of the harmonic producing properties of the core and do not affect the impedance to the third harmonic. The impedance term for the third harmonic comes down to

$$\frac{24}{35\pi}(a_{02}H_1H_3 + a_{03}H_1^2H_3),$$

which may be written as

$$\frac{24}{35\pi}H_3\frac{B(O, H)}{H_1}.$$

This may be compared with the corresponding term for the fundamental given by (25).

Appendix 5. Effect of Air-Gap by Vacuum Tube Analogy

In the elementary treatment of non-linear two element vacuum tube circuits, approximate solutions are obtained in the form

$$J = J_1 + J_2 + \cdots J_n,$$

where J represents the variable part of the space current on the basis that J_n represents the nth approximation, but that the series converges rapidly so that we need consider only the first two terms to arrive at a substantially accurate result. The expressions derived for the first and second approximations are known to be

$$J_1 = a_1E/(1 + a_1R),$$
$$J_2 = a_2E^2/(1 + a_1R)^3,$$

where the 'a' coefficients describe the tube characteristic

$$J = a_1v + a_2v^2,$$

R being the external plate circuit resistance, v the tube potential, and E the circuit e.m.f.

Turning now to the equations for the hysteresis loop branches, we have from (4a)

$$B = a_{10}h - a_{02}h^2 + a_{11}hH + a_{02}H^2.$$

Hence by the analogy between flux and current, and between reluctance and resistance, the first order terms are reduced by the factor $1/(1 + a_1R)$ which corresponds to $1/(1 + \lambda a_{10}/A)$, and the second order terms are reduced by the cube of this factor, which yields the same results (42) as the laborious direct method.