Contemporary Advances in Physics, XVI

The Classical Theory of Light, Second Part[1]

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MEASUREMENT of wave-lengths is the subject which we shall now consider. So entitled, the topic seems unpromising, as some dry exercise in mensuration; but in truth it is distinguished for beauty and variety, and implicated with the whole of modern physics. This is not measurement of the lengths of palpable objects, as pieces of lumber or cloth, which are laid alongside of a yardstick or clamped in the jaws of a gauge. In optics, the methods of measuring wave-lengths are the methods of proving that waves exist, therefore of testing the undulatory theory of light. One could not reasonably ask for evidence of light-waves more convincing than the concord of the values obtained for the wave-length say of sodium yellow light, by all the diverse instruments which act by causing interference or diffraction: Newton's tapering film of air between a lens and a plate, Fraunhofer's grid of iron wires, the tilted mirrors of Fresnel, Michelson's echelon, and all the many gratings and interferometers continually in use in laboratories and classrooms. Wave-lengths of X-rays are computed from the diffraction-patterns imposed on X-ray beams by intercepting crystals, and these patterns were the evidence which showed some fifteen years ago that the rays are of the nature of undulations, though it could not disprove that in some paradoxical way they are also of the nature of corpuscles. From similar diffraction-patterns imposed by crystals on electron-streams it follows that these also are partly of the nature of waves, and again the patterns have supplied the values of the wave-lengths.

Moreover, evidence for waves and values of their lengths are only part of what a grating can supply. Once we are sure that we know the wave-length of a certain kind of light, we can send it against a grating and study the diffraction-pattern with the opposite intent: analyzing not the light but the grating, and deducing the widths and the spacings of the slits, if it is an alternation of slits and stops—the spacing and the shaping of its grooves, if it is an engraving on metal or glass—the arrangement of the atoms and of the electricity within the atoms, if it is a crystal. Therefore the methods for measuring wave-lengths of X-rays are also those for exploring the structures of solids and of the atoms of which these are composed. Remember

[1] Continued from the April, 1928, issue.

also that the instruments efficient in this field are the most delicate and accurate which have ever been made for any purpose; that they may be used to measure ordinary lengths and other physical quantities with an almost unbelievable precision; that the theory of relativity sprang from an experiment performed with one, and the only known way of measuring the diameter of a star involves the use of another. Surely, if this topic is not interesting, nothing in physics is interesting.

Methods of measuring wave-length are sometimes divided into those which operate by interference and those which utilize diffraction. Though to a thorough insight the distinction is only trivial, at the outset it is convenient. In an extreme example of what is specially called "diffraction," a single train of plane parallel waves is sifted through a sieve in the form of a grid or a sequence of slits; and each element of wave-front which passes through a slit evolves and spreads thenceforward according to the law of wave-propagation. Eventually—as a rule, in the focal plane of the lens beyond the grating—a region is reached where the light from the several slits intermingles; and here occur the variations in amplitude which disclose the waves and the wave-lengths. For, as I have said earlier, the eye perceives only the amplitude of the light-waves, and not their phase; therefore, in a plane-parallel beam where the phase is perpetually changing but the amplitude is everywhere the same, the eye receives a uniform impression, with nothing wavelike in it; and to make such a beam reveal that it is undulatory, we must cause the amplitude to vary from point to point. This is what we accomplish by breaking the beam into fragments, or lacerating it with obstacles, preferably with an obstacle having a periodic structure of its own, which is a grating. But it may also be accomplished by causing two plane-parallel beams to intersect one another under proper conditions. The region where they overlap is then a region of varying amplitude—indeed, the variations are as great as one can imagine; for if the beams are equally intense, there is a succession of parallel planes of no vibration and darkness, which separate spaces where there is vibration and light. The widths of these spaces, the *fringes*, may be computed from the wave-length, and reversely the wave-lengths from the widths, very simply and without any knowledge of the law of wave-propagation beyond the familiar expression for plane waves. Therefore this method of measuring wave-lengths by causing *interference* of two parallel beams is much the easiest to grasp; but it does not differ in principle from the method involving a grating, for that acts by interference between the beams from the various slits.

The Diffraction Grating

The ideal diffraction grating of theory is a sequence of equally-wide perfectly vacant slits, separated from one another by strips absolutely opaque and equal in width to one another though not necessarily to the slits. Actual gratings seldom resemble this picture, though Fraunhofer's first—the most important instrument, I suppose, in the story of spectroscopy—was an approximation to it which he made by winding a wire around and around a pair of screws held parallel and wide apart, soldering it in place and cutting away the alternate strands. So were some of his others, composed of gold-leaf mounted on glass and scratched along parallel lines with a diamond. So-called *reflection gratings* would also conform with the picture, if they consisted of bands of perfectly smooth reflecting metal separated by absolutely non-reflecting bands; for then the result would be the same as if the light came through the reflecting strips from a virtual image of the source located behind. Practical reflection gratings are not usually very like this conception, for the entire surface of the metal block is ploughed up into roughly-shaped furrows. In fact one could scarcely define the word "grating" less generally than as a periodically-repeated obstruction, or better yet a periodically-repeated device for perturbing the free onward flow of a beam of light.

Nevertheless the theory of the ideal grating contains most of what is required for the theory of the practical appliance. The reason is, that the action of the grating upon the light can be separated into two factors, each of which produces its own separate effect, each of which may be studied apart from the other. Commonly there is a set of maxima of brilliance in the diffraction-pattern; otherwise expressed, there are certain directions in which the intensity of the diffracted light is exceptionally great. From the locations of these maxima, the wave-length of the light is calculated. Now these locations are determined by the spacing of the units—be they slits and bars, furrows and ridges on a reflecting surface, planes of atoms in a crystal, or what not—whereof the exact repetition in sequence constitutes the grating. Thus if we know that a certain grating is ruled with 1000 "lines" to the inch, we can compute the wave-length of sodium light from the positions of the maxima in its diffraction-pattern, without knowing or caring whether the rulings are slits, grooves, triangular indentations, wavy ripples, or rough-bottomed troughs. If we know that in a crystal a certain grouping of atoms repeats itself one million times in a centimetre, we can calculate the wave-length of an X-ray beam or an electron-beam from the locations of its diffraction-maxima, without knowing anything about the arrangement of atoms in the group.

The contour of the rulings of a grating does, on the other hand, affect the relative intensities of the various diffraction-maxima and the details of the distribution of intensity throughout the diffraction-pattern. If they are grooves or troughs, their profiles in cross-section have influence upon the pattern; if they are slits with bars between, the ratio of width of slit to width of bar must be taken into account. In crystals the arrangement of the atoms in the groups controls the intensity-ratios among the diffraction-maxima and conversely is deduced from observations made on these. Even the distribution of electricity in the separate atoms of a crystal may be read from the details of the diffraction. These effects however can intrude upon the measurement of wave-lengths only in the cases—comparatively rare—in which some of the diffraction-maxima are actually blotted out, so that the uninformed observer may misinterpret those remaining. Except for cases such as these, one may derive the formula for computing the wave-length by assuming any convenient form of grating; and therefore we may think about a grid of slits and bars.

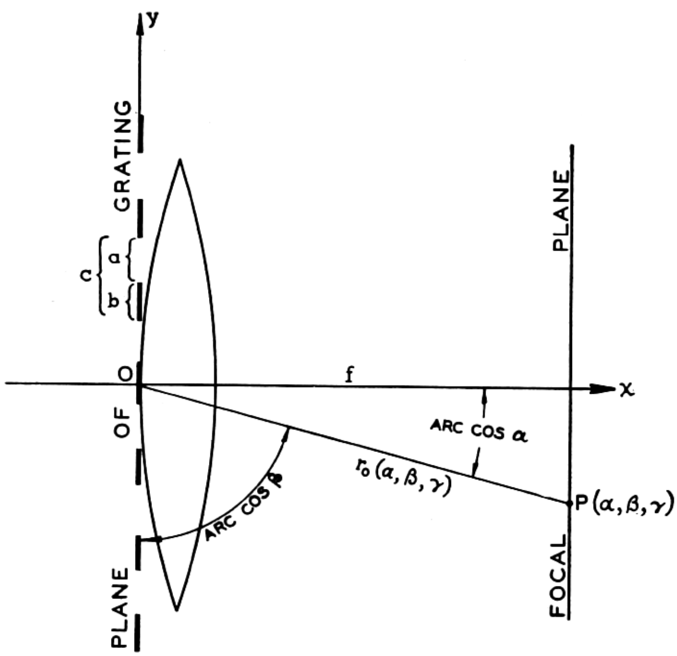

Fig. 1.

Consider then an alternation of slits of width a and bars of width b, occupying the plane $x = 0$. A beam of plane-waves, monochromatic and of the wave-length λ, travelling along the x-direction in the positive sense, shall fall normally upon it from behind. It is our object to determine the amplitude of the waves in the region in front of the grating. Suppose for instance that we select a plane parallel to the grating and at the distance x in front of it, and derive the formula for the amplitude at any and every point (x, y, z) of this plane. This formula is the description of the theoretical diffraction-pattern in the plane in question; and the actual pattern may be observed by setting up a screen or a photographic plate in the corresponding place.

We went through this process for a single aperture in the first part of this article; and there we found that the pattern is simpler (or, at least, more calculable) the farther the plane of observation is removed from the plane of the slit, being simplest when the two are infinitely far apart. To realize this case in practice we have only to set a lens immediately before the grating. Then, the diffraction-pattern appropriate to the plane at infinity—the so-called "Fraunhofer diffraction-pattern"—is transposed into the focal plane of the lens, where we must place the photographic plate in order to record it. Naturally it is reduced in scale and augmented in intensity when it is thus transposed, and for this as well as other reasons we had better express it, not in terms of the coordinates (x, y, z) of the points in the focal plane, but in terms of the direction-cosines ($\alpha = x/r$, $\beta = y/r$, $\gamma = z/r$) of the lines drawn to these points from the origin of coordinates.[1] Our formulæ for the diffraction-pattern in the infinitely-distant plane are in fact naturally expressed in terms of α, β and γ; and the lens may be regarded as an agency whereby that value of amplitude, which otherwise would have existed infinitely far away upon the line with direction-cosines (α, β, γ), is amplified by a constant factor and shifted inward along this line to the point where it intersects the focal plane.

We wish, then, to determine the vibration produced by a regular sequence of slits, all over the plane which is either infinitely distant or else the focal plane of the lens, according as the lens is absent or present.

Now we already have a formula for the vibration produced in that plane by any slit individually. It is the formula (93) of the first part of this article; to wit:

$$\begin{aligned} s &= \text{const. } (1 + \alpha)[C \sin (nt - mr_0) - S \cos (nt - mr_0)] \\ &= \text{const. } (1 + \alpha)\sqrt{C^2 + S^2} \sin (nt - mr_0 - \epsilon). \end{aligned} \tag{1}$$

[1] The origin should coincide both with the centre of the lens and with some point in the plane of the diffracting apertures. This is impracticable; but the error apparently does not make any trouble in practice.

Here the symbol s stands for the amplitude of the vibrating entity—whatever that may be—at the various points ("field-points") where the plane in question is intersected by the lines drawn from the origin with direction-cosines (α, β, γ); the symbols n and m for 2π times the frequency and 2π over the wave-length of the vibration, respectively; and the symbols C, S, r_0 and ϵ for various functions of (α, β, γ). In particular, C and S denote certain integrals extended over the slit, so that they involve the breadth of the slit as well as the variables (α, β, γ); this last is true of ϵ also; but r_0 denotes the distance from the origin of coordinates to the field-point, and thus involves the variables but not the breadth of the slit. As for the nature of the vibrating entity which is designated by s, I am keeping it intentionally vague. Suffice it to say that s is something of which the phase cannot be detected in any known way, but the amplitude controls the intensity of the light; the observed intensity being, according to the classical theory of light, proportional to the square of the amplitude.[2]

If therefore we were studying the diffraction-pattern of a single slit, we should be concerned only with the factor $(1 + \alpha)\sqrt{C^2 + S^2}$ in the expression for s. It would be short work to develop the expressions for C and S for a single rectangular aperture, finite or infinite in length; and having developed them, we should have solved the problem of the single slit; but in respect of our present purpose, it would be a detour. Remarkable as it may seem, the pattern of the single slit is only of secondary importance in determining that of a regular sequence of slits. When we undertake to sum up expressions such as (1) in order to compute the diffraction-pattern of such a sequence, we find the emphasis violently shifted. A new set of diffraction-maxima appear, and their positions are determined by the variation of the phase $(nt - mr_0 - \epsilon)$ from one slit to the next—in more general language, the variation which the phase undergoes in passing over one complete *period* of the grating-structure. Meanwhile the influence of the coefficients C and S, and that of the breadth of the slit which they involve, recede into the background. Not the features of the individual slit, but the interval at which one follows another, is now the dominant factor. This is the situation foreshadowed in the introductory pages.

To bring this out, let us orient the z-axis in the plane of the grating so that it runs parallel to the slits, which are of width a and are separated by bars of width b so that the period c of the grating is equal to $(a + b)$. The x-axis is to run, as heretofore, perpendicular to the surface of the grating and through the centre of the lens, so that it

[2] Or rather to the sum of the squares of the amplitudes of several quantities, any one of which separately satisfies the same equations as s.

intersects the focal plane (or the plane at infinity) at the point which is the centre of the diffraction-pattern. The y-axis is to lie in the plane of the grating perpendicular to the slits. The light comes up to the grating normally from behind, and therefore follows the x-direction. For reasons which will presently appear, it will suffice to calculate the diffraction-pattern over not the entire focal plane, but only the line where this is intersected by the xy-plane. For any field-point on this line $\gamma = 0$ and $\alpha = \sqrt{1 - \beta^2}$.

To the total vibration at any field-point, each slit now makes a contribution given by the expression (1). Numbering them in order, we may write for the contribution of the kth slit:

$$s_k = \text{const. } (1 + \alpha) A_k \sin \varphi_k, \tag{2}$$

in which A_k stands for the value of $\sqrt{C^2 + S^2}$, and φ_k for the value of $(nt - mr_0 - \epsilon)$, appropriate to the kth slit.

Now A_k has the same value for all the slits. This may be proved directly from the formulæ [3] for C and S, or indirectly by the following chain of reasoning. The function $\sqrt{C^2 + S^2}$ describes the diffraction-pattern formed by the single slit on an infinitely-distant screen, when there is no lens. Two similar slits a finite distance apart would produce two such patterns, one displaced by the same finite amount relatively to the other. But on the infinitely-distant screen the fringes and other details of the patterns are themselves infinitely broad, so that a finite displacement of one with respect to the other leaves them still practically—and, in the limit, exactly—in coincidence. This remains true when the patterns are transposed to the focal plane of the lens; those produced by a slit in one place coincide exactly with those which would be produced by an exactly similar slit lying anywhere else.[4] Therefore $\sqrt{C^2 + S^2}$ must be the same function of (α, β, γ) for every slit.

At first glance this argument seems to prove that the diffraction-pattern for the grating is merely that of the individual slit, multiplied manyfold; but that conclusion would in general be false, for we have not to add amplitudes but to compound vibrations with due regard to their relative phases. The phase φ_k which figures in equation (2) differs from slit to slit; and if these follow one another at equal intervals, φ_k changes from one to the next in equal steps.

[3] By operating on the expressions (presently to be derived) for C and S in the case of a rectangular aperture, one may show that, while each separately varies when the position of the rectangle with reference to the origin is changed, the sum of their squares remains the same. As any finite aperture may be regarded as a collection of finite or infinitesimal rectangles, the theorem is general. I am indebted to Mr. L. A. MacColl for working this out.

[4] The practical limitation to this statement would be set by the impossibility of making an ideally perfect lens of indefinitely great size.

To prove this, and to find the magnitude of these equal steps, one may proceed as follows. Omitting the lens again, consider in the grating any two consecutive slits k and $(k + 1)$, and on the very

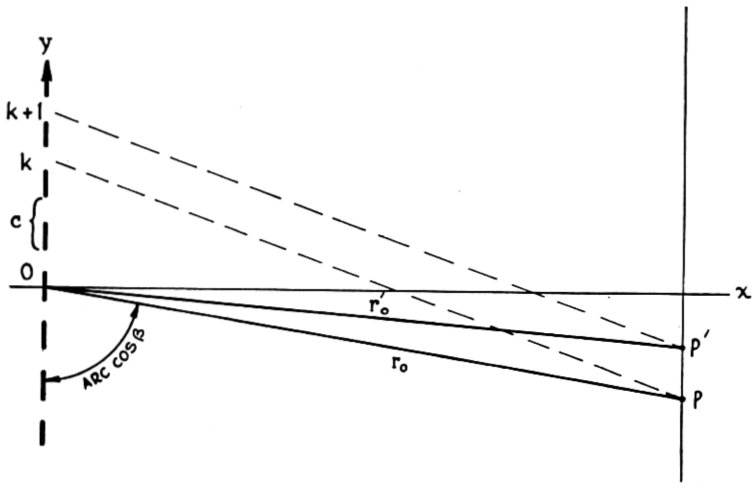

Fig. 2.

distant screen two field-points P and P' separated by the same distance c as separates corresponding points of the two slits—that is, the period of the grating. Write down successively the formulæ for the vibrations produced by k at P and by $(k + 1)$ at P'. They are, respectively:

$$A \sin (nt - mr_0 - \epsilon_k); \qquad A \sin (nt - mr_0' - \epsilon_{k+1}).$$

Since P lies in the same direction from k as P' from $(k + 1)$, these two are equal; hence:

$$\epsilon_{i+1} - \epsilon_i = m(r_0' - r_0). \tag{3}$$

Here the factor $(r_0' - r_0)$ on the right is the difference between the distances from the origin to P' and to P. In the limit when these distances become infinitely great, all the lines from the origin and the slits to P and P' become parallel and inclined to the plane of the grating by the angle of which the cosine is β; and the difference between the paths to P' and to P from the origin attains the limiting value $c\beta$. Hence in the limit:

$$\epsilon_{k+1} - \epsilon_k = mc\beta. \tag{4}$$

This is the "step" or difference in phase between the contributions of successive slits to the vibration at the field-point. The expression looks more familiar if we put θ for the angle between the normal to the grating and the direction from the grating to the field-point, so that $\beta = \sin\theta$; then:

$$\epsilon_{k+1} - \epsilon_k = mc \sin\theta = \frac{2\pi}{\lambda} c \sin\theta. \tag{5}$$

Thus we have arrived at the conclusion—indeed almost self-evident—that the consecutive slits of the grating supply to the total vibration at the field-point contributions which are exactly equal in magnitude and follow one another at equal intervals of phase.

Our problem therefore is to sum up the series of these contributions. The process is an easy one; but we shall be able to foresee the major feature of a diffraction-spectrum without even writing down the summation. For it is evident that there must be maxima of vibration-amplitude, maxima of diffracted intensity, at the field-points or in the directions where the contributions of all of the slits agree in phase—that is to say, differ in phase by integer multiples of 2π. Counting outward from the centre of the diffraction-pattern, or normal to the grating, the first of these maxima must lie in the direction for which the "step" in phase of equation (5) is equal to 2π; hence for this "first-order maximum":

$$\sin\theta = \lambda/c. \tag{6}$$

The second lies in the direction for which the step in phase is equal to twice 2π; the third in the direction for which the step is thrice 2π; and in general there is a sequence of maxima, the general formula for the nth of which is the celebrated "plane-grating formula":

$$\sin\theta_n = n\lambda/c, \qquad n = 1, 2, 3, 4 \cdots. \tag{7}$$

The symbol n stands customarily, and in this article henceforth shall stand, for the *order* of the maximum; from now on I will write $2\pi\nu$ for the quantity which before was denoted by n.

These are the great principal maxima of the spectrum cast by a diffraction-grating. There are others between, but in practice they are inconspicuous or invisible. Those of which I have just derived the locating formula are the maxima from which wave-lengths are computed. Let it be emphasized again that the formula was derived without taking into account the ratio of slit-width to bar-width, and that it does not involve the width of the individual opening, but only the spacing between corresponding points of consecutive slits. Indeed, if one examines the deduction, it will be seen that really nothing peculiar to a slit enters into it at all. All that is preassumed is that the grating sends to the field-point a series of component vibrations, equal in amplitude and stepped off equally in phase. Such is indeed the case when the grating is a series of windows letting light through towards the camera, separated by walls which intercept the light. Such is also the case when the grating is a series of mirrors reflecting light towards the camera, separated by windows which let it escape or by absorbing surfaces which swallow it up. Such is the case when the instrument is a surface of metal ploughed into furrows, so that the reflecting-power towards any assigned direction varies from point to point across the furrow, and varies periodically as one moves across the system of furrows. Such is the case if the waves traverse or are reflected from all points of the grating equally, but with phase-retardations which vary periodically across the grating-surface. Such is the case when the instrument is a surface containing oscillators able to vibrate in unison with the incident waves and able to radiate new waves because of their vibration, these oscillators being evenly spaced or else clustered in identical groups which themselves are evenly spaced. Such in fact is in general the case whenever the "grating" is any object with a periodic structure, the details of which are able in any manner known or unknown to perturb the passage of the waves; for anything which confuses or impedes the even onward progress of a train of waves, whether it be a vibrator which they set into oscillation or merely an inert impenetrable obstacle in their way, becomes thereby the source of a new system of undulations.

Wave-lengths of light therefore are determined by setting up in the path of the light-stream something which has a periodic structure, of which the period is known; locating the diffraction-maxima, if such there be; and using the formula (7), provided that the object is plane (another, which we shall eventually derive, is used if the object is three-dimensional and the waves travel across its structure). Location of one maximum would not as a rule suffice, for without further knowledge its order could not be identified. One must measure sufficiently many maxima to infer from the ratios of their values of $\sin \theta$ what their orders are. On the other hand, understanding of the precise mode and mechanism of the action of the elements of the grating upon the light is not required, desirable as it may be. Perhaps we do not properly understand how the atoms of a crystal scatter even X-rays; and certainly the founders of the wave-mechanics did not foresee that crystals scatter electron-waves. Yet Davisson and Germer determined the wave-lengths of these latter in 1927 with the same equation wherewith Fraunhofer in 1821 had ascertained the wave-lengths of the lines of the solar spectrum—the very equation,

$$\lambda = \frac{c}{n} \sin \theta_n,$$

taking for c the spacing between consecutive lines of atoms in the surface-layer of the nickel crystal which diffracted the electrons, where Fraunhofer had taken the spacing between the wires of the grid which was his primitive grating.

Next it is important to discover how distinct these maxima are; whether, when the amplitude of the vibration in the focal plane is plotted against θ, the peaks are broad and flattish or narrow and sharp. This too can be foretold without the labour of a complete solution of the problem. Taking any of the principal maxima—say, that of nth order, which is located at the angle $\theta_n = \arcsin(n\lambda/c)$—let us inquire how near to it the amplitude will sink to zero.

Now the nth of the principal maxima is located by the condition that the phase of the contribution of every slit is $2n\pi$ in arrear of that of the slit preceding. If there are $2M$ slits altogether (it is convenient to suppose the total number to be even, though whether it is even or odd makes no appreciable difference), then at θ_n the contribution of the last slit is $(2M - 1)n \cdot 2\pi$ behind that of the first. Estimate now the vibration at the point in the focal plane —call its direction-angle $(\theta_n + \Delta)$—where the contribution of the last slit is $(2M - 1)(n + 1/2M)2\pi$ behind that of the first. It is readily shown that here the component vibration due to the last or $2M$th slit is exactly equal in magnitude and opposite in phase to that which is produced by the Mth of the slits; and in the same way every ruling of one-half of the grating may be paired off with the corresponding ruling of the other half, their effects destroying each other pair by pair. At the angle $(\theta_n + \Delta)$, therefore, there is darkness; and likewise at the angle $(\theta_n - \Delta')$, where in the focal plane or the infinitely distant plane the contributions from the first and the last slit arrive with a phase-difference of $(2M - 1)\left(n - \frac{1}{2M}\right)2\pi$.

The entire peak culminating in the nth principal maximum is consequently bounded by the directions $(\theta_n - \Delta')$ and $(\theta_n + \Delta)$; and it is easy to see that the greater the number of rulings (the spacing being supposed to remain the same) the narrower and sharper is the peak, and the more accurately can the location of its summit and therefore the wave-length be determined. Its breadth, in fact, varies inversely as the number of rulings or "lines." This is shown by writing down the formulæ for the angles corresponding to the minima which bound it. We have:

$$\sin(\theta_n + \Delta) = \left(n + \frac{1}{2M}\right)\frac{\lambda}{c}; \quad \sin(\theta_n - \Delta') = \left(n - \frac{1}{2M}\right)\frac{\lambda}{c},$$

whence, approximately,

$$\Delta = \frac{\lambda}{2Mc}\frac{1}{\cos\theta_n} = \frac{1}{2Mn}\tan\theta_n, \tag{8}$$

so that the narrowness of the peak, as one might say, is proportional to the order thereof as well as to the total number of lines in the grating.[5] If the grating were infinitely wide, an unlimited sequence of perfectly-evenly-spaced identical units, the peaks would be infinitely narrow; light of a definite wave-length would be diffracted only in certain perfectly definite discrete directions.

Alikeness, closeness, and multitude of rulings are therefore the desiderata of a grating; alikeness, because without it the first condition for the formation of sharp diffraction maxima would be lacking—closeness, so that the maxima of lower orders, the only ones sufficiently intense to be perceived, may be spread out widely enough for convenience of observation—multitude, so that the diffracted beams shall be narrow and sharp, easy to set upon and easy to discriminate from one another. The degree of closeness which is required depends upon the spectral range which is to be explored. Ordinary "optical" gratings ruled with a diamond on metal or on glass are acceptable throughout the visible spectrum and the range to which the title "ultra-violet" is commonly restricted, extending from the visible down to wave-lengths of the order of one hundred Angstroms. They are however too fine for the remoter infra-red, for the study of which coarse lattices of wire have been used; *a fortiori* they are much too fine for Hertzian or radio waves, for which it is no exaggeration to say that a colonnade might operate as a grating; and they are commonly considered much too coarse for X-rays, though during the last two or three years several men of science have achieved the great technical feat of forcing optical gratings to measure wave-lengths which formerly were thought accessible to crystals only. Crystals are too fine for the visible spectrum, and too coarse for certain of the gamma-rays which proceed from the collapsing nuclei of atoms undergoing transmutation. Crystals with spacings of unusual width from atom-plane to atom-plane are

[5] These facts are usually expressed as statements about the "resolving power" of a grating; for if the incident light contains two not very different wave-lengths, they will form two peaks of each order not very far apart, and the possibility of distinguishing these two—of "resolving" them, to use the technical term—will depend upon the narrowness of each. If arbitrarily one says that two such peaks are just distinguishable when the summit of one falls upon the minimum adjacent to the other—in which circumstance the difference $\delta\lambda$ between their wave-lengths may readily be proved equal to the quotient of the mean of their wave-lengths, λ, by $2Mn$—then by this criterion a grating is able in its nth order to discriminate two adjacent lines of the spectrum, if their wave-lengths differ by more than that amount; and by definition the resolving power of the grating in its nth order is $\lambda/\delta\lambda = 2Mn$, the product of the number of rulings by the order.

chosen for work upon the longer X-rays, as optical gratings are ruled with lines unusually far apart for work in the near infra-red.

The ruling of good gratings is an art; and those who have practiced it with conspicuous success are fewer far than those who have attained pre-eminence in music or in painting. Amateurs, mechanics, and professors figure upon the list, the first of all being Fraunhofer, who from a glazier's apprentice evolved into the founder of spectroscopy. After his gratings of wires and of scratches in a foil of gold-leaf, he invented the method of engraving with a diamond-point upon a surface of metal or of glass (he used the latter) which is followed to this day. He met and grappled with all the difficulties which were later to beset his followers, and described them in language which now sounds strangely modern. Then, as now, it was possible to rule tens of thousands of rough grooves roughly to the inch; the trouble lay, as still it lies, in making them identical and spacing them equally.

Equality of spacing depends upon a screw, which is turned through a prearranged angle and is expected to advance through a definite distance carrying the future grating with it, whenever the diamond has completed one ruling and is waiting to begin the next. Screws as manufactured are not good enough; and anyone who aspires to be a maker of gratings must first of all procure the best available, and then devote a long and tedious time—literally years—to making it still better. Primacy in the art passed to America in the eighties of the last century, because Rowland of Johns Hopkins developed with much labour a process for removing, or at least for mitigating, the imperfections of a screw. The greater the number of rulings to be laid down side by side, the longer the portion of the screw which must be made, as nearly as humanly possible, perfect; and Michelson has testified, from unrivalled experience of many years, that the time required for the process varies as the cube of the length of the screw and width of the planned-for grating. Increase of resolving-power thus is bought at an enormous price in patience and in perseverance. A research institute is as proud of a notable grating by Rowland or Michelson or Wood, as a picture gallery of an authentic Titian or Velasquez; and the promise of a new talent is not more joyfully received, than a rumour that someone is working to perfect a yet longer screw to make a yet wider grating.

Alikeness of successive rulings depends on the endurance of the diamond. The ruling-engine is sequestered in a well-insulated room, and after the temperature has settled down to constancy is set in motion by some device worked from outside, and left to do its task in solitude. If the diamond breaks, or suffers any great change in shape during the operation, the grating is good for nothing. This cannot be foreseen, it is not even known when it happens; to stop the process to see how things are going would be like digging up a seed to see how it is sprouting. The chance of such an accident is naturally greater, the more numerous the lines—another obstacle to the successful ruling of many-lined gratings of high resolving power.

A grating having been completed, it is removed from the engine and examined, to learn not merely whether it has been impaired by deformations of the diamond, but how—assuming it to have escaped that peril—the intensities of the various diffraction-maxima of different orders compare with one another. This is something which, as I have intimated, is controlled by the shape of the groove; this is the feature in which the individual units of the periodic structure manifest their quality. One shaping might obliterate all diffraction-maxima of even order; another might make the maxima on one side of the normal to the grating-surface stand out much more prominently than their companions on the other; still another could concentrate most of the diffracted light into one single beam. The maker of the grating cannot foresee, or can at best foresee only in part, what distribution of intensities he is going to get; for he cannot control the shape of the diamond-point, nor find it out by examination.[6] Having observed the distribution of intensities, however, he can deduce from it some facts about the shape of the grooves. This I suppose would be classified in most cases as useless knowledge; but the problem happens to be very nearly the same as that of determining the finer details of the arrangement of atoms in a crystal from the relative intensities of the various diffraction-beams which it produces when acting on an X-ray beam; and so I will devote a few paragraphs to it.

We return, then, to the grating of alternate slits and bars, to determine the influence of the ratio of slit-width to bar-width on the diffraction-pattern. Before making any calculations whatever, one striking prediction can be made directly. I have said that diffraction-maxima occur in every direction θ_n for which

$$\sin \theta_n = n\lambda/c, \qquad n = 0, 1, 2, 3, 4 \cdots,$$

because in every such direction the component vibrations arrive at the focal plane from the various slits with identical phase. But if for any of these directions the component vibrations are themselves

[6] He can control the result to a slight extent by varying the pressure with which the diamond bears upon the plate, ruling "with a light touch" or reversely; if he guesses the force just right, he may approach the condition of grooves separated by unbitten bands of smooth metal as wide as they, which resembles the theoretical case of an alternation of slits and bars of equal width.

non-existent, evidently the maxima in question are blotted out. This will happen, for example, to every maximum of even order, if bars and slits are equally wide. For, taking the direction $\theta_2(\sin\theta_2 = 2\lambda/c)$ as an instance: the contribution made to the total vibration by the upper half of each slit will be equal in magnitude and opposite in phase to that made by the lower half, and the total contribution of the slit will be zero. If in the spectrum produced by a grating the even orders are missing, or if—to say what would actually be noticed—the values of $\sin\theta$ for the present maxima stand in the ratios $1:3:5:7\cdots$ instead of $1:2:3:4\cdots$, the inference is that the grating has been so ruled that over half of every period the phase of the emerging (transmitted or reflected) light is constant, and over the other half no light comes forth at all; as for instance would be the case if half of every period were the unmarred surface of the metal, and the diamond had made the other half perfectly black. The reader may work out for himself what it must mean if every third, or every fourth, or every nth of the maxima is absent.

We return now to the expression (equation 2) for the contribution of a single slit or period of the grating and rewrite it, taking due account of our subsequently-gained knowledge that A_k is constant and φ_k increases by equal steps $mc\sin\theta = mc\beta$ from slit to slit:

$$s_k = \text{const.}\,(1+\alpha)\,A\sin(nt - mr_0 - \epsilon_0 - kmc\beta). \tag{9}$$

For convenience number the slits from 0 to $N-1$, representing by N their total number (formerly called $2M$, but now there is no reason for supposing it even), and locate the origin so that $mr_0 = \epsilon_0$. Gathering all the factors of the sine-function under a single symbol B, and writing out the expression for the summation of s_k from $k=0$ to $k=(N-1)$, we find for the resultant vibration in the direction θ:

$$\begin{aligned} s &= B\sum_{k=0}^{N-1}\sin(nt - kmc\beta) \\ &= B\sin nt(1+\cos a+\cos 2a+\cdots\cos(N-1)a) \\ &\qquad\qquad - B\cos nt(\sin a+\sin 2a+\cdots\sin(N-1)a) \\ &= B\textstyle\sum_c \sin nt - B\sum_s \cos nt, \end{aligned} \tag{10}$$

in which a stands for $mc\beta$ and $\sum_c$ and $\sum_s$ for the finite series of cosines and sines which are indicated.

For the amplitude of the vibration—the only thing which matters—we then have

$$D = B\sqrt{\textstyle\sum_c^2 + \sum_s^2}. \tag{11}$$

Now, as may easily be proved [7]:

$$\sqrt{\Sigma_c^2 + \Sigma_s^2} = \sin\left(\tfrac{1}{2}Na\right) : \sin\left(\tfrac{1}{2}a\right) \tag{12}$$

so for the amplitude of the vibration in the direction θ we have:

$$D = B\,\frac{\sin\left(\tfrac{1}{2}Nmc \sin\theta\right)}{\sin\left(\tfrac{1}{2}mc \sin\theta\right)}\,. \tag{13}$$

Here we have that product of two factors which was foreshadowed in the early pages of this article—one factor (the second) depending on the periodicity of the grating, and controlling the location of the diffraction-maxima; the other depending on the structure of the individual slit or groove or atom-row, and controlling their intensity.

The second factor displays the qualities which have already been deduced by simpler means, and others. It vanishes whenever $\tfrac{1}{2}Na$ is an integer multiple of π, except when simultaneously $\tfrac{1}{2}a$ is an integer multiple of π, in which exceptional cases the great principal maxima occur. These are not the only maxima, for between any two of them there are $(N - 1)$ equally-spaced minima (directions where $\tfrac{1}{2}Na$ is an integer multiple of π but $\tfrac{1}{2}a$ is not) and between these in turn there are $(N - 2)$ maxima of which the locations may be found by the usual method. These so-called "secondary" maxima are however faint and inconspicuous, having, according to Wood, but 1/23 the intensity of the principal peaks, unless the grating is composed of only half-a-dozen lines or fewer.

The first factor consists essentially of that function

$$(1 + \alpha)\sqrt{C^2 + S^2}$$

mentioned in equation (1) and earlier, which describes the diffraction-pattern of the single slit (or groove, or atom-row). Wherever that diffraction-pattern has a zero of intensity—the "centre of a black fringe," to use the common language—the intensity in the pattern of the grating is likewise forced to vanish. When the slit occupies half

[7] One method is based on the fact that Σ_c and $i\Sigma_s$ are respectively the real and imaginary parts of Σe^{ika}, so that

$$\Sigma_c^2 + \Sigma_s^2 = \left(\sum_{k=0}^{n-1} e^{ika}\right)\left(\sum_{0}^{n-1} e^{-ika}\right).$$

Further, by a well-known formula

$$\begin{aligned}\sum_{0}^{N-1} e^{ika} &= 1 + e^{ika} + (e^{ika})^2 + \cdots (e^{ika})^{N-1}\\ &= (1 - e^{iNa})/(1 - e^{ia})\end{aligned}$$

and there is a corresponding expression for e^{-ika}, multiplying the two of which together and taking the square root one arrives directly at the stated result.

the width of the period (slit plus bar) of the grating, the first of its black fringes falls square upon the second-order principal maximum of the grating spectrum, which is obliterated. This is a new way of expressing the fact already mentioned, that when the slits are as wide as the bars the diffraction-maxima of even order are absent.

More generally, the intensity at any of the principal maxima is proportional to the value of $(C^2 + S^2)$ appropriate to that direction—proportional to the intensity, in that direction, of the diffraction-pattern of the single slit; and from this we can understand how, from the relative intensities of the maxima of various orders, it is possible to deduce the breadth of the slit or something about the shape of the groove. If we had only a single slit, and could send through it light of known wave-length sufficiently intense to form a measurable diffraction-pattern, we could trace the curve representing observed relation between diffracted intensity and angle θ, and compare it with the predicted curves for various values of slit-breadth; the actual width of the slit would be the value for which the agreement was perfect. If instead we had a multitude of such slits equally spaced, the observed intensities of the diffraction-maxima would supply us, not indeed with the entire continuous curve of intensity-versus-angle for the single slit, but with as many points upon that curve as there were principal maxima within our range of observation; and these—if we had two or more—would be sufficient for the comparison with the theoretical curve for the single slit, out of which the width would be deduced. From this aspect, the function of the grating is to *enhance* the intensity, at certain discrete points, of the diffraction-pattern for the single slit. Of course, when we are interested in the breadth of the single slit or the shape of the single groove, we should prefer to observe the entire continuous diffraction-pattern produced by one alone. But it may be impossible to separate one from the rest; or if we could isolate one, it might be too small to transmit or scatter any perceptible amount of light. Such is the case with atoms.

The natural gratings which atoms form in crystals are three-dimensional, and to them the reasonings which are valid for plane gratings cannot be applied without some change; but the resemblance is very close. A beam of X-rays or electron-waves falling upon a crystal is spread out into a diffraction-pattern with strong maxima, of which the relative intensities depend upon the qualities of the individual diffracting units, the atoms or the groups of atoms which are repeated over and over again to form the crystal; while their directions depend upon the spacings between these identical groups, the periodicity of the crystal. From the directions of the principal diffraction-beams one may determine the spacings within the crystal if one knows the wave-length of the waves, or the wave-length if one knows the spacings. From the relative intensities of the beams one may deduce the distribution of the atoms within the groups, or rather the distribution of that which scatters the waves—commonly supposed to be mobile negative electricity, when the scattered waves are light; I do not know whether anyone has yet conjectured what it is that scatters the electron-waves.

The diffraction-beams proceeding from a crystal large enough to be manageable are very sharp, for the rows of atoms are far more numerous than the lines of the largest optical grating which can be made or hoped for. However, there is a limitation on their sharpness set by something to which an artificial grating is quite indifferent—the thermal agitation of the atoms, which has the same effect as though the widths of successive periods were variable and fluctuating. This effect is naturally more pronounced, the higher the temperature of the crystal; but the measurements show—for from the breadth of the diffraction-maxima it is possible to determine the mean amplitude of the temperature-agitation, another service of the crystal grating—that even at absolute zero it would not disappear, the atoms retaining a certain minimum amount of energy of vibration which apparently can never be taken from them, so long as they remain bound together in a crystal.

A few words, before leaving the subject of gratings, about the diffraction-pattern of a multitude of gratings oriented at random.

On an earlier page I said that, in computing the diffraction-pattern of a sequence of slits, we need determine it not for the entire focal plane, but only for a single line thereof—the line for which $\gamma = 0$, which is the line of intersection of the focal plane with the plane running normal to the slits and containing the infinitely-distant point-source of the parallel waves of light. The reason can now be stated. If we work out the expression $C^2 + S^2$ for a single long and narrow rectangular slit with its long sides are parallel to the z-axis, we find that the brighter parts of the diffraction-pattern form a long narrow band (criss-crossed with dark lines) with its length parallel to the y-axis and its breadth parallel to the z-axis. If the length of the rectangle grows infinitely long, the breadth of this band shrinks to zero; we have a single line of varying brightness parallel with the y-axis, which is the diffraction-pattern of the infinite slit. If instead of a single slit we have a regular sequence, their diffraction-pattern is still concentrated upon this line; it is the pattern which has just been computed, a function of the single variable β, or y, or θ; away from the line, the intensity is everywhere zero. Spectroscopists broaden this linear pattern in practice by using as source of light not a point, but a luminous line—an incandescent filament, for instance, or a slit backed by a flame—made parallel to the slits or rulings of the grating. Then the diffraction-pattern is spread into a band. If the light is monochromatic, one sees in the focal plane, at the positions of the principal maxima, not a sequence of brilliant points as the foregoing theory implies, but a sequence of brilliant lines—the lines of the spectrum.

Instead of these lines one will obtain circles, if one uses a point-source of light and a mosaic of gratings all lying side by side in a single plane and oriented every way. Each piece of the mosaic forms its own linear diffraction-pattern, perpendicular to the direction of its own rulings; and if the pieces are numerous enough, all of these are fused into a single circular pattern, each of the principal maxima standing forth as a brilliant ring. I am not sure whether this has been done with plane optical gratings; but the analogous method with X-rays and crystals is the familiar procedure known by the names of Debye and Scherrer and Hull, or as the "powder method." Being a case of diffraction in three dimensions, it is not entirely like my imaginary case of a mosaic of plane gratings. The resemblance however extends so far, that from the broadness of the rings one may infer the size of the tiny crystals which make up the three-dimensional mosaic, the "powder"; for the smaller these are, the fewer rows of atoms each contains, and the wider their diffraction-maxima must be. But it requires very fine grinding indeed, or the dispersion of the crystals as a colloid in solution, to make them so small that the broadening of the rings is noticeable.

What would be observed, if individual slits or apertures or atoms were dispersed completely at random over the plane or throughout space? If there were many apertures all alike and all similarly oriented, but with no regularity whatever in arrangement, the diffraction-pattern would be the same as that of any singly, though more intense. The water-droplets in misty air act thus in forming haloes. If atoms were truly spherical and could be crowded together into a dense mass without any regularity, the diffraction-pattern of the mass would be that of the individual atom, and would disclose the radial distribution of its scattering-power—whether that be negative electricity, or something else. Even if atoms are not spherical, one might expect to learn in this way the average distribution of scattering-substance over all the orientations. Experiments have been conducted for this purpose; but it is difficult to find a piece of matter in which the arrangement of the atoms is entirely irregular, that is, a perfectly "amorphous" substance; perhaps not even liquids satisfy this requirement.

Interference

When a pair of beams of light are projected together upon a screen, it is usually observed that the illumination resulting from them jointly is the simple sum of the illuminations which each produces by itself when the other is shut off. One may easily go through life without ever once finding this rule in default. Yet by intelligent design it is possible to contrive conditions in which the rule does not prevail; and actually two rays of light directed upon the same point may counteract one another and cause total darkness, and two perfectly uniform wide beams falling together upon a surface of frosted glass may decorate it with a pattern of dark fringes separated by light, dark circles alternated with bright, black networks upon a background of color—arabesques of shadow and light, more delicately shaded than anything achievable in pigment or stained glass. The brilliant and versatile Thomas Young, he who was the first to read the Egyptian hieroglyphics upon the Rosetta stone, was also the first to discover some of these lovely phenomena; a pair of exploits, which for eminence and diversity will probably never be surpassed. It happened that the first disclosure of the phenomena which demand the wave-theory of light coincided as accurately with the advent of the nineteenth century as the first realization of the necessity of quanta came at the dawn of the twentieth; for Young discovered the interference of light in 1800.

"Interference" is a name which Young selected; he said that in the conditions of his experiments beams of light interfere with one another. For the observer this was not, on the whole, an ill-chosen word, since the visible effect of the two lights conjointly is not the mere sum of the visible effects of each separately. True, it implies that the lights destroy or diminish one another, whereas in fact they are as likely to cooperate as to conflict, two equal beams combining into one of intensity as much as fourfold that of either. This is not serious; we are all accustomed to using the word *addition* to cover subtraction; and here the analogy is very close. The so-called "interference" is simply the necessary result of adding two vibrations with due regard to their direction and their phase. This is the method which was used to calculate diffraction-patterns; and in fact a diffraction-pattern is nothing but a special case of interference-pattern—not usually a simple one, for the vibrations which must be summed are very numerous, demanding integrations and long summations. The simplest interference-pattern occurs when two plane-parallel beams of light of equal amplitude intersect one another; and this we will now consider.

Designate by 2θ the angle at which the two beams are inclined to one another, and draw the x-axis to bisect it; then the two wave-functions are

$$s' = A \sin (nt - mx \cos \theta - my \sin \theta),$$
$$s'' = A \sin (nt - mx \cos \theta + my \sin \theta)$$

and their sum [8] is

$$s' + s'' = s = 2A \cos (my \sin \theta) \sin (nt - mx \cos \theta). \quad (14)$$

We see immediately that this is a situation in which the wave-theory of light predicts a peculiar and characteristic variation of amplitude from point to point in space, which can be tested in detail, and of which a favorably-resulting test has evidential value; whereas in either beam separately the amplitude is constant, and nothing is observable which demonstrates that there are waves. Here, in the region where the beams overlap, the amplitude varies sinusoidally between zero and the maximum value $2A$; the distance between two consecutive loci of zero amplitude, which are planes perpendicular to the axis of y, being

$$d = \pi/m \sin \theta = \tfrac{1}{2}\lambda/\sin \theta. \quad (15)$$

The presence of a series of equally-spaced planes of darkness, their separation varying inversely as the sine of the angle between the beams, is then to be taken as evidence that light is undulatory; and from their separation and the angle between the beams one may compute the wave-length of the light. A more thorough test, made by measuring the distribution of light-intensity between two such planes, would lead (anyway it ought to lead) to the conclusion already known, no doubt, to all the readers of this paper: that the intensity of the light varies as the square of the amplitude of the waves.

To produce this effect of interference, the two intersecting beams must have started from the same source of light, and at very nearly the same instant—that is to say, the optical paths from the source along the two beams to the region of overlapping must be the same within a few millions of wave-lengths, or a few hundreds of centimetres. By the wave-theory, this is easily understood. We must think that

[8] To add them thus implies that the quantity denoted by s is either a scalar, or a vector perpendicular to the xy-plane. Since light is not adequately described by either assumption, we must anticipate defects in the theory, more prominent the larger the angle θ. In practice θ is evidently always so small that there is no trouble from this source.

a beam of light from a flame or an arc consists of myriads of feeble beams each proceeding from a single atom. Each is divided—the methods of division are the methods of producing interference-fringes—and the separate parts are then caused to overlap. Each pair which came originally from a single atom produces a set of interference-fringes, and the fringes for all these pairs coincide in space. Each fraction of a divided beam may also interfere with a fraction of another, proceeding from another atom; but owing to the uncontrolled and uncontrollable phase-differences between the beams of a pair so formed, the fringes for these pairs do not coincide, and on the whole they efface one another. By the quantum-theory the explanation—not indeed of the fact that interference occurs only under these special conditions, but of the fact that it ever occurs at all—is not so easy. Indeed the fact commonly expressed by saying that light from a source is "coherent" with itself, has been regarded as the most difficult of all for the quantum-theory to explain.

To produce interference, then, we must divide a beam of light and cause its parts to cross each other's paths. The simplest of the devices which effect this were invented by Fresnel; a pair of prisms which turn two portions of the beam towards one another, and a pair of mirrors which reflect two portions across each other's routes. A single mirror indeed suffices; standing acoustic waves are produced thus, in Kundt's tube and otherwise, with values of the angle 2θ sometimes as great as 180°; but light-waves are so short that with so great an angle the distance between dark fringes would be too small to measure, if not indeed to perceive; and we must use the facility for expanding them which the factor $\sin\theta$ in equation (15) offers us.

The Interferometer

In the devices which I have thus far mentioned, the interference of overlapping wave-trains oblique to one another causes the formation of alternate zones of darkness and light in space; and the visible fringes are the cross-sections of these zones upon a screen set up to intersect them. There are however other instruments in which the overlapping beams are parallel to one another, the region which they occupy is not traversed by bands of light and shade, and a screen thrust across it shows uniform illumination; and yet when the eye or the camera is located in that region, fringes are produced upon the retina or on the plate by the action of the lens of either. These are not so easily understood as the earlier devices, and yet it is important to comprehend them, for the striking applications of interference have been made by means of such as these. Among them are the interferometer of Fabry and Pérot, and that of Michelson.

Imagine, at the outset, a pair of perfectly plane and parallel mirrors, onto which wave-trains of extended plane wave-fronts are falling from every direction. The mirrors must of course be semi-transparent, so that part of the light which falls first upon one—say, the upper—is reflected from it at once, and part goes on to meet and be reflected by the lower. Thus (as Fig. 3 shows more clearly than words) the

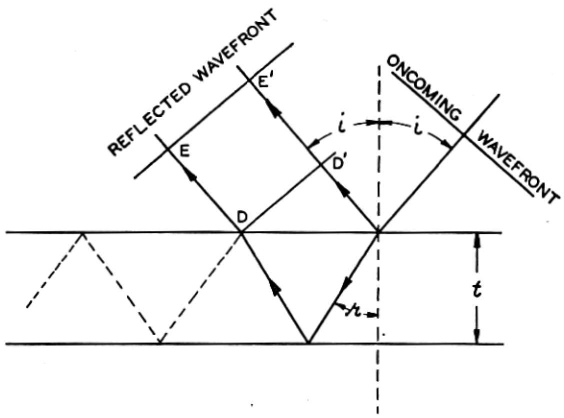

Fig. 3.

mirrors form out of each incident wave-train a first and a second reflected beam, which travel back through the space above the mirrors in the same direction, making according to the law of reflection the same angle i with the normal as the incident wave-train did. In truth there are not merely two reflected beams derived from each incident one, but an infinity thereof, owing to the multiple reflections which are indicated in the sketch. We need not however (as I shall presently show) take account of more than two; by combining the second reflected beam with the first we can predict the most important features of the interference.

It is necessary to be somewhat more precise about the nature of the mirrors. As good an example as any to begin with is that of the "thin plate"—a slab of some transparent substance, glass for instance, embedded in a transparent medium which I will take to be empty space. The mirrors, then, are the upper and lower sides of the plate. Denote by μ the ratio of the speeds of light in the environing medium and in the substance of the plate, by i the angle of incidence of any wave-train and by r the angle of refraction of its transmitted part; then as heretofore we have

$$\sin i = \mu \sin r. \tag{16}$$

The ratio A_2/A_1 of the amplitudes of the first and second reflected beams, and in general the ratios A_n/A_1 of the amplitudes of any of the reflected beams and the first, are determined altogether by μ and i. An important consequence of this will presently appear. One can however alter these ratios, e.g., by half-silvering the sides of the plate; and the formulæ which I am about to quote may be applied to the case of two half-silvered mirrors facing each other in air, by setting $\mu = 1$.

Isolate then in mind a single incident wave-train. Denote by i its angle of incidence upon the upper surface of the glass; by t the thickness of the plate. A wave-front of the oncoming wave-train is divided into two. During the time while the part which entered the glass is advancing to the lower side, being reflected, returning to and re-emerging from the upper side, the part which was first reflected goes on to the level EE' of Fig. 3. The emerging wave coincides with the first-reflected part of a new wave-front which was following along after the old one at the interval $E'D'$. In general, there is a phase-difference φ between these two. The condition for interference is ideally satisfied; two wave-fronts coincide. The amplitude A of the resultant light travelling away from the mirrors is obtained by the prior formula from the amplitudes A_1 and A_2 of the first and second reflected beams and their phase-difference φ:

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos \varphi. \tag{17}$$

It is now our affair to deduce the value of this difference in phase. This is a simple task, for the angle φ, expressed in radians or in degrees, is 2π or 360 times the number of wave-lengths intervening between the wave-front DD' and the wave-front EE' which, so to speak, has gone on ahead leaving part of itself behind to combine with DD'. We have therefore only to multiply $2\pi/\lambda$ or $360/\lambda$ into the distance $D'E'$; which distance is found [9] by very easy manipulations of trigonometry, aided by the equation (16), to be $2\mu t \cos r$; so that for the phase-difference in radians we have

$$\varphi = \frac{2\pi}{\lambda} 2\mu t \cos r + \varphi_0. \tag{18}$$

The constant φ_0 is inserted to leave room for the possibility that abrupt changes of phase may occur at the instant of reflection, unequal for the two reflecting surfaces. Experience shows that in this case of

[9] For the distance $D'E'$ is the difference between BD' and BE'. The latter is evidently $BD \sin i$, which is $2t \tan r \sin i$, which is $2\mu t \tan r \sin r$. The former is the distance cT traversed by light in vacuo during the time T while the beam which entered the glass is traversing its zigzag path BCD; this time is $(\mu/c)BCD$, which is $2(\mu/c)t \sec r$.

the plate the value π must be assigned to φ_0; as though the phase were unaltered at the first reflection, but reversed at the second. This is however a point of minor importance.

The equation (18) is one of the most important in optics; we shall encounter it repeatedly, even so far along as in the X-ray range.

By comparing (17) and (18) one sees that, if wave-trains of equal intensity fall upon a glass plate from all directions, those which depart in the various directions are not equally intense; their brightnesses depend upon their angle of reflection which is their angle of incidence, i. However they are not separated in space, and hence there are no bands of light and darkness. But if a lens be placed in the region above the plate, it will direct each of the beams to a separate point in its focal plane; and since the illumination at the point where a wave-train converges is proportional to the intensity of that wave-train, there will be fringes in that focal plane. If the lens is set with its axis normal to the planes of the mirrors, as when one looks straight at them with the eye, then the points where the wave-trains reflected at any angle i converge lie all upon a circle, its radius depending on i. The fringes are therefore circular; looking vertically down upon a thin plate, or photographing it with a camera pointed directly towards it, one sees or registers a system of concentric rings, their centre wherever the perpendicular dropped from the lens reaches the plate. These are said to be localized at infinity; but the term is not a good one, for the fringes are not at infinity; they are on the retina or on the camera film, formed by the lens in the focal plane thereof.

The values of i for which the amplitudes of the reflected beams are least or greatest may be determined by differentiating (17). If we may neglect the variation of A_2/A_1 with i (as usually but not always we may), the result is the expected one: least intensity and blackest point of a fringe corresponds to a phase-difference of π or an odd-integer-multiple thereof, greatest intensity and brightest point of a fringe occurs with a phase-difference of zero or any even-integer multiple of π. Thus one may compute the actual size of the circular rings to be produced when a given lens sorts out the rays reflected by a given plate, and test the predictions by experiment; or rather, to say what is really done, one may determine the wave-length of the light by measuring the diameters of the rings and comparing them with the formulæ. But this is not a customary way of determining wave-lengths.

At this point it is expedient to remark that the size of the fringes is not affected by the presence of those third, fourth, fifth and indefinitely many reflected beams which I excluded from the computation. For the phase-difference between each of these and the one reflected once-less-often is given by the first term on the right in equation (18), and hence is greater by 180° than that between the second and the first; and when i is so chosen that the second and the first are in opposite phase, then all the beams of higher order are in the same phase as the second and reinforce it, the reinforcement being just so great—in the case of the transparent plate—that the resultant of all these beams together is of the same amplitude as the first reflected beam. Therefore the minima, the centres of the dark fringes, are not shifted by the extra beams, but are rendered absolutely black.

The width of the fringes is greater, the thinner the plate—other things, naturally, being left unchanged. This is the reason why one needs quite a thin lamina of glass to see them well, and cannot get them at all with a windowpane. If the thickness of the plate is changing continually, one sees them narrowing or widening; one sees also a phenomenon much more striking, the generation of rings one after the other out of the centre of the fringe-system—if the plate is growing thicker; the reverse, if it is shrinking. Glass plates capable of shrinking or thickening at will are not as yet available; but at times the former case is realized by a soap-bubble on the verge of dissolution. Where the soap-film is about to give way, the fringes rapidly dwindle and are swallowed up into the central spot of the interference-system, which alternately turns dark and light, and finally goes black just at the instant before the bubble bursts. From this final blackness we infer that the value π must be assigned to the constant φ_0 of equation (3). Reflection from water to air and reflection from air to water are accompanied by phase-changes differing from each other by π, since the beams of light formed by two such reflections destroy each other when the reflecting surfaces are immeasurably close. But the bubble is too tender an instrument for practical use. The plate of variable thickness must be a slab of empty space between movable solid mirrors.*

The interferometer of Fabry and Perot is precisely such a thing: two half-silvered plates of glass facing each other across a narrow gap. The like-named instrument of Michelson is the same in principle; but by ingenious use of a third reflector, the two essential mirrors are transposed far apart—a most valuable feature, as we shall see. The incident wave-trains (coming from below, in Fig. 4) are divided by a semi-transparent mirror C inclined at about 45° to their path, and the fractions are sent off at right angles to one another, along the two

* Or else Lummer's device of a pair of wedges so proportioned, that a face of one may slide along a face of the other, while the two sides not in contact remain parallel to one another.

"arms" of the machine, at the ends of which they meet fully-reflecting surfaces A and B set normal to their courses. Reflected straight back along the arms, they are once more divided by the semi-transparent

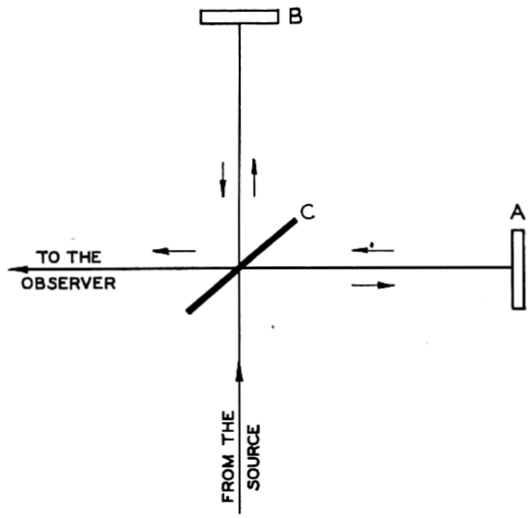

Fig. 4.

mirror, and the fractions which go towards the observer (left, in Fig. 4) are those which interfere. This seems complicated; but in essence it is not so. Everything happens as though the mirrors A and B were at the end of the same arm. Looking from the left through C, one perceives the mirror A and the virtual image in C of the mirror B; and this comes to the same as though C could be removed, and the horizontal arm of the machine swung into coincidence with the vertical arm in spite of the well-known prohibition against two objects occupying the same place at the same time. When the plane of A is accurately normal to the plane of B and the semi-transparent surface C is properly oblique, the virtual image of B is accurately parallel to A; and the observer sees circular fringes, which one by one emerge from a central spot or shrink down into it as either mirror is displaced along its arm.

This clever device of combining one real mirror with the virtual image of another, to form a thin plate of which one surface is impalpable, is of enormous value. One can make the virtual reflector go right through the real one, the fringes being swallowed up into the centre one after another, and then reborn in due order after the whole field of view goes black at the instant of coincidence. By moving one of the reflectors through a chosen distance and counting the fringes which are born or consumed during the motion, one may evaluate the distance in terms of the wave-length of the light, or the wave-length in terms of the chosen distance, according as the one or the other is independently known. For, returning to equation (18) and putting $\mu = 1$ (since the two surfaces of the "thin plate" are separated only by air) and $r = 0$ (since the light is normally incident), we see that φ is changed by 2π when the spacing between the reflectors is changed by $\frac{1}{2}\lambda$; but when φ is changed by 2π, a single ring is added to or subtracted from the system of annular fringes; hence the total number of rings created or destroyed during the motion of the mirror is equal to twice the number of wave-lengths comprised in the distance which it traverses. In this way Michelson counted, as the first step in his determination of the length of the standard metre, the waves of the red cadmium line covering the distance between the two ends of an "intermediate standard" or *étalon*, about half a millimetre long.

In practice the real and the virtual reflector are frequently not quite parallel with one another; and this is sometimes a convenience. If they intersect (another of the things which are not possible with a pair of real reflectors) and the lens of eye or camera is located vertically above the line of intersection, this line stands forth embodied as a fine straight black fringe—the *central fringe*—companioned on either side by a multitude of others.[10] If now either of the reflectors be set

[10] Imagine a pair of mirrors inclined to one another at a very small angle φ. Establish a coordinate-frame such that the z-axis is the line of intersection of the two mirrors, the x-axis lies in the plane of either. Locate the lens of the eye or the camera at any point, say P; drop the perpendicular from P to the zx-plane at P_0; let R stand for its length, and t_0 for the distance between the mirrors at its foot. Consider the pair of reflected wave-trains arriving at P from any direction, making an angle i with the aforesaid perpendicular. Denote by α and β the projections of i upon the xy-plane and the yz-plane respectively. Assume all these angles to be small. The pair of reflected wave-trains arise from the reflection, at first and second mirrors respectively, of a single primary wave-train which fell at the same angle i upon the mirrors at a point where the distance t between them is equal to $(t_0 + R\cdot\tan\alpha\cdot\tan\varphi)$; or, to first approximation, $t = t_0 + R\alpha\varphi$. The phase-difference between them, by equation (3), is equal to $(2\pi/\lambda)2t\cdot\cos i$. To first approximation we have

$$\cos i = 1 - \tfrac{1}{2}i^2 = 1 - \tfrac{1}{2}(\alpha^2 + \beta^2).$$

Hence to this approximation we have for the phase-difference:

$$\delta = \frac{4\pi}{\lambda}(t_0 + R\alpha\varphi)(1 - \tfrac{1}{2}\overline{\alpha^2 + \beta^2}).$$

Rearranging, and dropping the terms in $\alpha^2\beta$ and $\alpha\beta^2$, we get

$$\alpha^2 + \beta^2 - 2(R\varphi/t_0)\alpha - 2 + (\lambda\delta/4\pi)(2/t_0) = 0.$$

The loci of constant phase-difference, and hence the fringes, are circles centred upon the line inclined at angle $\alpha_0 = R\varphi/t_0$ to the perpendicular dropped from the lens to the mirrors. If this perpendicular meets the mirrors along their line of intersection, as assumed in the text, we have $t_0 = 0$; the centre of the circles is infinitely remote, the fringes are straight. If then either the lens or the line of intersection is shifted sidewise, the fringes march sidewise and acquire a curvature.—It should be realized that if the wave-fronts are wide and the mirrors not perfectly parallel, there are fluctuations of intensity along each wave-front; it may be necessary to narrow the aperture of the lens in order to avoid confusion due to these.

into motion, the fringes march off sidewise, growing more curved as they go; and the number passing any fixed marker set up in the field of vision is the double of the number of wave-lengths comprised in the distance through which the mirror moves.

Consider now for a moment what must be observed, if wave-trains of many wave-lengths fall upon the mirrors, instead of pure monochromatic light. Each kind of light produces its own pattern of fringes; but since the breadth of a fringe depends upon the wave-length, those of one kind cannot fall perfectly—light upon light and shade upon shade—upon those of another; and in most parts of the field of view, if not in all, the various patterns must blot one another out. Yet there is one exception; returning to equation (18) one sees that for any value of r the phase-difference φ must vary with λ, *unless* $t = 0$—in which exceptional case $\varphi = \varphi_0 = 180°$ whatever the wave-length.[11] When the real mirror and the virtual one coincide perfectly, the field of view is black, however many wave-lengths are contained in the incident light; and when the real mirror and the virtual one intersect, the line of intersection is marked with a black fringe. Moreover, in the neighborhood of the central fringe there is a brilliant display of colors. Words are too feeble to describe them, but there would be no great scientific advantage to be gained from a description; for the tint observed in any particular direction is not a pure prismatic hue, but results from mixture of the wave-lengths not completely extinguished by interference in that direction, and depends therefore upon the physiology of the eye. What interests us as physicists is the service of the central black fringe and the companioning glory of colors in marking the point and moment when the real and the virtual mirrors intersect or coincide. This service was essential in the measuring of the standard metre, a process which we will now examine.

The interferometer used in the process is sketched as seen from above in Fig. 5. The mirrors M and M′ are at the two extremities of the intermediate standard; they are made parallel with the greatest possible exactness, and the further is elevated above the level of the nearer. As for the other mirrors, D is the movable "reference" reflector; the purpose of N will appear directly; N′ may be ignored.

The instrument is now so adjusted that D is strictly parallel to the virtual image of N, and intersects that of M at a small angle. Red cadmium light being used, a part of the field of view is occupied by the circular fringes due to the cooperation of D and N and part by the

[11] Also φ is independent of λ if $\cos r = 0$, a case which may occur if we have a stratum of air between glass plates, and choose an angle of incidence near the angle of total reflection.

straight fringes due to that of D and M. Among these latter the central fringe marking the line of intersection of D with the virtual image of M could hardly be identified; but when the white light is turned on, it stands forth unique. It is made to coincide with some

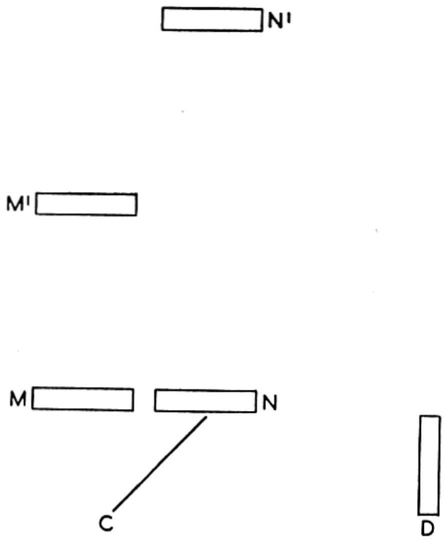

Fig. 5.

fiducial mark in the field of view, and the measurement commences. The red light being restored, the observer transfers his attention to the circular fringes formed by N and D, and counts them as they pass while he moves D along towards the virtual image of M′. Making occasional tests with the white light, he eventually notes the advent of the blaze of colours and the central black fringe which indicate the intersection of this image with D. When the black stripe coincides with the fiducial mark, the reflector D has moved through the length of the standard MM′, and this length has been measured by the count of the circular fringes which meanwhile were passing by in the other part of the field of view. The number of waves is, of course, not as a rule an integer; the fractional excess may be estimated quite accurately in various ways.

In the actual work, this shortest standard was about 0.4 mm. long, and comprised somewhat over 606 waves of red cadmium light. The counting of the 1212 fringes corresponding to its length was the only counting required; the rest of the process consisted of nine stages, the first of which was the type for all the others except the last. This second step was the comparison of the shortest standard with a second, made as accurately as possible to be twice as long; or rather, the shortest standard had been constructed with the deliberate aim of making it as nearly as possible just half so long as the second shortest. It was the office of the interferometer to determine how nearly this ideal had been attained; which was fulfilled by means of coincidences, detected as before through the coloured fringes.

Returning once more to Fig. 5, let M and M′ stand for the mirrors of the shortest standard, N and N′ for those of the second shortest. N and N′ are on a higher level than M and M′, and the field of view may therefore be divided into quadrants, in which appear the fringes—if and when there are any—formed by the collaboration of D with the virtual images of M and M′ and N and N′, respectively. The observer then goes through the following routine: (1) D is made to intersect the images of M and N; (2) D is drawn back till it intersects the image of M′; (3) the shortest standard carrying M and M′ is drawn back till the image of M again intersects D; (4) D is drawn back until it intersects the image of M′. The witness of intersection is always the central black fringe, appearing in the required quadrant or quadrants. Now if the distance NN′ is exactly twice the distance MM′, then when the observer completes the four stages of the routine D will intersect not only the image of M′ but that of N′; at the end of stage 4 the central fringe will appear in each of the upper quadrants, just above the point where at the beginning of stage 1 it appeared in each of the lower quadrants. If NN′ departs by a fraction of a wave from the doubled value of MM′, the central stripe in one of the upper quadrants will lag by a little behind that in the other. From the lag expressed in terms of the fringe-width, one may compute the difference between the length of the second standard and the doubled length of the first, and so obtain the number of waves comprised in the second. Now the second standard is made as nearly as possible of one-half the length of the third, the third of the fourth, and so on up to the ninth, which is made as nearly as possible one-tenth of the length of the standard metre. From each to the next the comparison is made in the same way. To show the remarkable reliability of Michelson's results I quote the three values which he obtained by three independent measurements of twice the number of waves of red cadmium light comprised in the length of the ninth standard:

310678.48, 310678.65, 310678.68.

After this point it remained to compare the ninth standard with a metre-rod, and the metre-rod with The Standard Metre. Returning for the last time to Fig. 5, we take M and M′ as the mirrors at the ends of the ninth standard. The steps are: (1) make M coincide with the end of the metre-rod, and D intersect the virtual image of M; (2) draw D back to intersect with M′; (3) draw MM′ back until M coincides with D; (4) draw D back to intersect with M′—and so forth ten times altogether, until for the last time D intersects M′, and M′ is very near the far end of the metre-rod. The discrepancy is again a fraction of a wave-length.

The result in which all this labour culminated was: **1,553,163.5** wave-lengths of red cadmium light are comprised in the length of the standard metre.

Such was the process of enumerating the millions of light-waves required to make up the length of that standard chosen for the measurements of common life, and so very ill-adapted to magnitudes of the scale of those in light—the distance between two scratches on the bar of platiniridium alloy, conserved in the vault at Breteuil with the care lavished upon a sacred relic. The achievement of Michelson was the bridging of a gap, or let me say a work of translation. Nearly all measurements of wave-lengths to this day are determinations of the ratio of one wave-length to another, as practically all measurements of objects an inch, a metre or a mile long are determinations of the ratios which they bear to metre-sticks. In dealing with tangible objects we use the language of the metre; in dealing with light-waves, we use effectively the language of another scale of measurement. Michelson was the first to make a supremely accurate translation from one to the other of these languages, so making it possible to express a measurement of either realm in the scale familiar for the other. Whether in addition he may be said to have replaced a standard essentially impermanent and transitory by one which in the nature of things is everywhere the same and forever immutable, is a question very likely never to be answered.