Distortion Correction in Electrical Circuits with Constant Resistance Recurrent Networks

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SYNOPSIS: Constant resistance recurrent networks, that is, networks whose iterative impedances are a pure constant resistance at all frequencies, form here the basis of a method of distortion correction which is applicable to any electrical circuit. The paper takes up first the general problem of distortion correction, then this method of correction and its application in the following Parts and supplementary Appendices.

PART 1. *Ideal Circuit Characteristics.* Both ideal steady-state attenuation and phase characteristics are formulated and then verified as being necessary and sufficient for the preservation of signal-shape under transient conditions.

PART 2. *Constant Resistance Recurrent Networks.* These networks are of three general types and are made possible by the introduction of inverse networks of constant impedance product. Their propagation characteristics are considered in some detail and various methods of design are indicated.

PART 3. *Arbitrary Impedance Recurrent Networks.* These networks are a generalization of those in Part 2.

PART 4. *Applications.* The large variety of uses to which these networks may be put is illustrated by specific designs made for complementary distortion correcting networks, for a submarine cable circuit, a loaded-cable program transmission circuit, and an open-wire television circuit. In addition, networks are given for the equalization of variable attenuation in carrier telephone circuits, for phase correction in the transatlantic telephone system and for the simulation of a smooth line.

APPENDIX I. *Discussion of Linear Phase Intercept.*

APPENDIX II. *Linear Transducer Theorems.*

Three theorems are proved which relate to the variation with frequency over the entire frequency range of the propagation constants and iterative impedances of certain passive linear transducers.

APPENDIX III. *Propagation Constant and Iterative Impedance Formulæ for General Ladder, Lattice and Bridged-T Types.* This includes an improved formula for $\cosh^{-1}(x + iy)$.

APPENDIX IV. *Propagation Characteristics and Formulæ for Various Lattice Type Networks.* These results can be applied quite readily to many problems arising in the design of distortion correcting networks.

Introduction

EVERY actual electrical circuit or transmission system distorts transmitted signals; that is to say, the received signal, regarded as a time-function, differs in shape from the impressed signal. Heaviside studied in detail the distorting action of the transmission line itself and indicated the necessary electrical properties of the distortionless line.[1] The distortionless line of Heaviside was approximately realized in the loaded line [2] in which similar lumped inductances are inserted in series with the line at uniform intervals. While this loading has the effect of partially correcting distortion of the lower frequency components of the signal, it also tends to increase the distortion of the higher frequency components and so limit somewhat the useful frequency range. More recently the transmission characteristics of some newly installed submarine cables have been greatly improved by means of continuous loading with the new magnetic material permalloy.[3]

The methods mentioned above are directed to rendering the line itself more nearly perfect. The method of distortion correction presented here may be used to supplement them and is that of passive *terminal* networks; more particularly networks whose iterative impedances are a pure constant resistance at all frequencies.[4] These networks are, however, not limited in their use to any particular type of transducer or transmission system but have general applicability. For this reason the general problem of distortion correction by this method resolves itself principally into a study of the transmission properties of these networks together with systematic methods of design to meet specified requirements.

This paper takes up first the characteristics necessary for no distortion in an electrical circuit; then, an extended study of constant resistance networks which can be used for distortion correction; finally, several applications to important practical problems. In addition, Appendix IV gives a considerable number of network structures and

[1] "Electrical Papers," Vol. II, p. 123, 1892; "Electromagnetic Theory," Vol. I, p. 445, 1893, Oliver Heaviside.

[2] U. S. Patent No. 652,230 to M. I. Pupin, dated June 19, 1900. See also "On Loaded Lines in Telephonic Transmission," G. A. Campbell, *Phil. Mag.*, March, 1903. Later a loading system more specifically directed to reducing distortion *per se* was disclosed in U. S. Patent No. 1,564,201 to J. R. Carson, A. B. Clark and J. Mills, dated December 8, 1925.

[3] "The Loaded Submarine Telegraph Cable," O. E. Buckley, *B. S. T. J.*, July, 1925.

[4] The equalization of the attenuation of certain transmission lines has for some time been obtained by means of comparatively simple series or shunt terminal networks. See, for example, U. S. Patent No. 1,453,980 to R. S. Hoyt, dated May 1, 1923. Such networks necessarily produce total terminal impedances which vary with frequency.

corresponding formulæ which will be found useful in further applications.

Part 1. Ideal Circuit Characteristics

There is no distortion in the transmission of an impressed signal over an electrical circuit or network when the shape of the received signal, considered as a time-function with usually a time-of-transmission, is identical with that of the impressed signal. A uniform decrease in magnitude only is not distortion, and it can be restored to its original value by means of a distortionless amplifier.

Let us assume in the general case that the e.m.f. impressed on the circuit is E, and that the circuit is always terminated by a receiver of resistance, R, across which is the received voltage, v, in which we are interested. The received current is then directly proportional to the received voltage.

The necessary and sufficient conditions for distortionless transmission can be stated quite simply in terms of the steady-periodic transfer voltage ratio of the circuit which will be written as

$$\frac{v(i\omega)}{E(i\omega)} = e^{-a-ib}, \tag{1}$$

with the terminology $a + ib$ = *the transfer voltage exponent of the circuit*, or concisely, *the transfer exponent*. Here a represents attenuation in napiers and b phase difference in radians, omitting in the latter any constant integral multiple of 2π, and assuming the two voltages to have zero phase difference at zero frequency. That is, the origin of phase difference is so chosen that the phase intercept at zero frequency is zero.

For ideal transmission characteristics the steady-periodic transfer exponent of the circuit should have an attenuation independent of frequency and a phase proportional to angular frequency, ω, whose slope is the time-of-transmission of the circuit.

In mathematical terms these ideal characteristics, represented by primes, are

$$a' = \text{constant (napiers)},$$

and

$$b' = \tau\omega \text{ (radians)}, \tag{2}$$

where

$$\tau = \text{time-of-transmission (seconds)}.$$

To show this, consider first what the indicial voltage, $g(t)$, would be under these assumptions. By *indicial voltage* is meant the received voltage as a time-function per unit constant e.m.f. impressed at the sending end at time $t = 0$. With (1) and (2) in the integral equation of electric circuit theory [5] we obtain

$$\frac{e^{-a'-\tau p}}{p} = \int_0^\infty e^{-pt} g(t) dt, \tag{3}$$

whose solution is

$$g(t) = 0, \quad t < \tau,$$

and

$$g(t) = e^{-a'} = \text{constant}, \quad t > \tau. \tag{4}$$

Thus, a constant voltage, which has been attenuated by the circuit an amount a' napiers, arrives suddenly at the receiving end after a time $\tau = (b'/\omega)$ seconds, and there is no distortion with respect to the unit constant e.m.f. impressed on the circuit at time $t = 0$.

If now any type of e.m.f., $E(t)$, is impressed on this circuit which is specified by the steady-state characteristics (2) or the indicial voltage (4), we obtain through a general formula [6]

$$v(t) = \frac{d}{dt}\int_0^t E(t-y)g(y)dy = e^{-a'}E(t-\tau). \tag{5}$$

This received voltage has the same shape as the impressed e.m.f., there being an attenuation, a', and a time-of-transmission, τ. Hence, *a circuit specified as above is distortionless to any type of impressed e.m.f.* A further discussion involving the phase intercept is taken up in Appendix I.

It may be stated that Heaviside's theoretical distortionless smooth line was that in which the line constants R', L', G' and C' per unit length had the relation

$$R'/G' = L'/C', \tag{6}$$

giving attenuation and phase constants per unit length, respectively,

$$\alpha = \sqrt{R'G'} \text{ napiers,}$$

and

$$\beta = \sqrt{L'C'}\omega \text{ radians;}$$

also an iterative (or characteristic) impedance

$$k = \sqrt{\frac{R'}{G'}} = \sqrt{\frac{L'}{C'}} \text{ ohms,}$$

which is a constant resistance at all frequencies. A circuit made up

[5] "Electric Circuit Theory and the Operational Calculus," John R. Carson.
[6] L.c.

of such a line of length l terminated by a resistance $R = k$ is readily seen to satisfy the conditions (2) above for no distortion. It would have an attenuation $a' = \sqrt{R'G'}l$ napiers and time-of-transmission $\tau = \sqrt{L'C'}l$ seconds.

Having seen above what constitutes ideal transmission characteristics, the problem of distortion correction in any practical distorting circuit is that of altering the circuit in some way so as to approach this ideal. In most circuits it is impossible to obtain these ideal characteristics throughout the entire frequency range. More or less satisfactory transmission results will be had, however, if this ideal is approached over the range of frequencies most essential to the composition of the impressed e.m.f., as shown by its Fourier integral analysis.

How accurately an ideal attenuation characteristic has been met in any case depends upon how nearly constant the attenuation is in the frequency range. A simple practical measure of the degree of approach to an ideal phase characteristic at the frequencies in this range is furnished by a consideration of the time-of-phase-transmission in the steady state,

$$\tau_p = b/\omega \text{ seconds,} \tag{7}$$

in which b is defined as in (1) for the complete circuit. The more nearly constant τ_p is in the frequency range, the closer it approaches equality with τ, the time-of-transmission of the circuit for those frequencies.

In many cases approximately ideal phase characteristics already exist in the desired frequency ranges so that corrections need be made for attenuation only. In others, such as those in which the steady-periodic state is of most importance and where the phase relations between the components are immaterial, it is satisfactory to obtain uniform attenuation at the desired frequencies. The method of altering circuit transmission characteristics to be shown in this paper follows in Part 2.

Part 2. Constant Resistance Recurrent Networks

2.1. *Fundamental Basis of Distortion Correction*

The general transmission circuit of Fig. 1 is shown as having a resistance, R, at the receiving end, as in the case where the energy is absorbed. Usually the circuit characteristics at this resistance with respect to the sending terminals show distortion in the required frequency range. If so, an ideal method of correcting the distortion would appear to be that of interposing between the circuit and the receiving resistance a transducer having the requisite corrective propagation constant and an iterative impedance, R. By so doing, the transfer exponent at the end of the circuit proper would remain un-

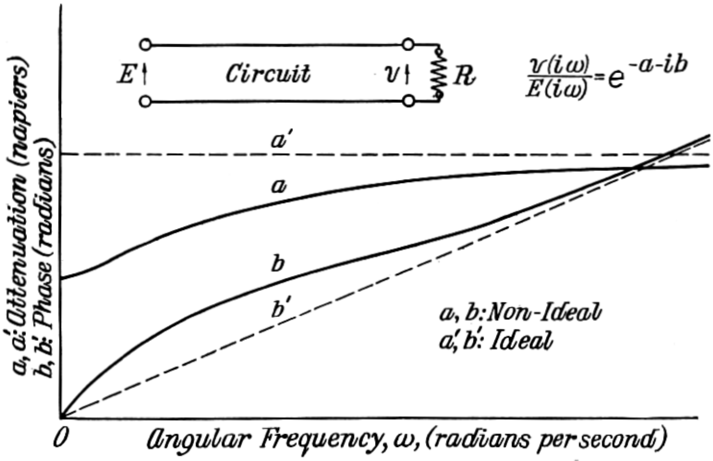


Fig. 1—Non-ideal and ideal transfer exponents of circuits.

altered, *irrespective of the exact nature of the network beyond*, since the latter has the impedance R; but the total transfer exponent would become ideal through the addition of the complementary propagation constant of the transducer. Stated analytically,

let $a + ib$ = transfer exponent of the distorting circuit at a terminating resistance R,

$A + iB$ = propagation constant of the correcting transducer of iterative impedance R,

and $a' + ib'$ = resultant ideal transfer exponent at the receiving resistance R.

Then the correcting transducer must be so designed that A and B satisfy over the required frequency range the conditions

$$a' = a + A = \text{constant},$$

and

$$b' = b + B = \tau\omega,$$

where τ is a positive constant. Or, explicitly,

$$A = a' - a, \text{ positive},$$

and

$$B = b' - b = \tau\omega - b. \tag{8}$$

The total attenuation, a', and time-of-transmission, τ, are somewhat at our disposal; it will be found that their best choice is usually guided by experience. The transducer will often consist of a number of sections, not necessarily alike. This distortion correcting process may be called "equalizing both the attenuation and the time-of-phase-transmission."

The idea of altering circuit transmission characteristics by means of one or more sections of constant resistance recurrent networks forms the fundamental basis of the method of distortion correction presented here. It is, of course, dependent for its application upon the physical possibility of designing recurrent networks whose iterative impedances are a constant resistance at all frequencies and whose propagation constants have the desired characteristics.

Another method by which distortion correction has sometimes been obtained is by means of terminal thermionic distortion circuits wherein networks of particular frequency characteristics are placed in the plate circuits of successive thermionic tubes. In it any reaction of one stage upon a preceding stage or upon the original circuit is prevented by the unilateral property of the tubes, whereas in the method given here this same result is obtained by the property of a constant resistance iterative impedance and the use of a resistance termination. While from the standpoint of the original circuit both methods give the resultant effect of a terminal unilateral device, one very practical advantage of the constant resistance method over the thermionic tube method appears to be that it corrects distortion before any amplification is added and hence with it there would be less tendency to cause tube distortion or modulation. Another advantage is that the distortion correcting networks can be designed independently of the amplifying device. A description of this other method appeared in the last number of the *Journal*.[7]

Before taking up specific types of constant resistance structures, let us consider some of the inherent limitations of certain transducers as are brought out by the following theorems.

2.2. *Linear Transducer Theorems*

These theorems relate to the variation with frequency over the entire frequency range of the iterative parameters, that is, the propagation constants and iterative impedances, of certain passive linear transducers. In symmetrical transducers we could as well employ the image parameters which are of such utility in a study of electric wave-

[7] "Phase Distortion and Phase Distortion Correction," Sallie Pero Mead, *B. S. T. J.*, April, 1928.

filters and which, together with iterative parameters, were discussed generally by the writer in a previous number of this *Journal*.[8] But since here in the ladder type networks some dissymmetrical sections are also considered, I shall use the iterative parameters throughout this paper.

Theorem I: Any symmetrical transducer whose attenuation constant is zero at all frequencies has a phase constant which increases with frequency and an iterative impedance which is a constant resistance throughout the frequency range.

Theorem II: Any transducer whose iterative impedance is real at all frequencies has a constant resistance iterative impedance, and if in addition its phase constant is proportional to frequency, it has a uniform attenuation constant.

Theorem III: Any symmetrical transducer whose attenuation constant is independent of frequency and whose iterative impedance is a constant resistance at all frequencies has a phase constant which is zero or increases with frequency.

The theorems, whose proofs are given in Appendix II, may be represented by the following table. The variations with frequency of the network parameters shown apply to the entire frequency range and in each theorem the parenthesis designates the dependent property, where A is the attenuation constant, B the phase constant, and K the iterative impedance.

TABLE I

LINEAR TRANSDUCER THEOREMS

Theorem	A	B	K
I..........	0	(Increases)	(Constant)
II..........	(Constant)	$\tau\omega$	Real (Constant)
III..........	Constant	(Zero, or increases)	Constant

That part of Theorem I which relates to the iterative impedance explains why there is no physical ladder type network having zero attenuation throughout the frequency range. For, the ladder type, when non-dissipative and having zero attenuation, requires a mid-series or mid-shunt iterative impedance which varies with frequency.

[8] "Transmission Characteristics of Electric Wave-Filters," O. J. Zobel, *B. S. T. J.*, October, 1924. The term "characteristic impedance" used in that paper for a recurrent or iterative parameter with dissymmetrical transducers is replaced here by "iterative impedance." Thus, the same term "iterative" applies to the structure, to the corresponding impedances, and to the kind of parameters. The use of the term "characteristic impedance" will be limited to smooth lines, or sometimes to symmetrical recurrent structures. In symmetrical structures the "characteristic," "iterative," and "image" impedances are identical.

2.3. *Inverse Networks of Constant Impedance Product*

We have already seen that the fundamental advantage of using constant resistance networks for distortion correction lies in the fact that when they are placed ahead of the receiving resistance, R, they present this same impedance to the circuit proper and hence do not alter the transfer exponent at that point. They can be designed to have, in addition to the impedance R, a propagation constant which complements this exponent and produces a resultant transfer exponent at the receiving resistance which is approximately ideal.

The possibility of physically realizing recurrent networks having a constant resistance iterative impedance at all frequencies rests, as will be seen, upon that of obtaining pairs of two-terminal networks the product of whose impedances is constant, independent of frequency. Such pairs [9] I have defined as *inverse networks of impedance product* R^2, or more concisely, *inverse networks*.

In the paper just referred to it was pointed out that one elemental pair of such inverse networks is composed of two resistances R_1 and R_2, and another is composed of an inductance L and a capacity C bearing the impedance product relations at all frequencies

$$R_1R_2 = L/C = R^2. \tag{9}$$

The same paper gave a simple proof of the following theorem relating to series and parallel combinations of networks. *If* z_1' *and* z_2' *are any pair of inverse networks and if* z_1'' *and* z_2'' *are any other pair, such that* $z_1'z_2' = z_1''z_2'' = R^2$, *then* z_1' *and* z_1'' *in series and* z_2' *and* z_2'' *in parallel are a pair; similarly* z_1' *and* z_1'' *in parallel and* z_2' *and* z_2'' *in series are another pair.*

Without much difficulty a theorem relating to simple networks having the form of a general Wheatstone bridge can also be obtained, as follows: *The inverse network corresponding to any given two-terminal bridge network of five distinct branches is also a bridge network, and may be derived by replacing the network in each branch of the given network by its inverse network and then interchanging the networks in either opposite pair of branches.* By successive applications of these relations, beginning with the elemental pairs, very complicated inverse networks can be built up. Only reactance networks were considered in the paper referred to above. Ordinarily the series and parallel

[9] An extensive use of inverse networks of pure reactance types was made in the paper, "Theory and Design of Uniform and Composite Electric Wave-Filters," O. J. Zobel, *B. S. T. J.*, January, 1923. Also in U. S. Patents No. 1,509,184, September 23, 1924; Nos. 1,557,229 and 1,557,230, October 13, 1925; and No. 1,644,004, October 4, 1927.

combinations are most useful, since the bridge structures require at least five elements in each network. Some networks may have other equivalent structures, as well.

If $z_{11} = r_{11} + ix_{11}$ and $z_{21} = r_{21} + ix_{21}$ are inverse networks such that

$$z_{11}z_{21} = R^2, \tag{10}$$

a number of simple relations exist among their impedance components; namely,

$$\frac{x_{21}}{r_{21}} = -\frac{x_{11}}{r_{11}},$$

$$\frac{r_{21}}{|z_{21}|^2} = \frac{r_{11}}{R^2}, \tag{11}$$

and

$$\frac{x_{21}}{|z_{21}|^2} = -\frac{x_{11}}{R^2}.$$

In a smooth line the condition (6) which makes it distortionless is actually the one making the series and shunt impedances per unit length inverse networks of impedance product $R^2 = R'/G' = L'/C'$.

2.4. *Types of Constant Resistance Recurrent Networks and Their Propagation Constants*

The types of recurrent networks considered in this paper are the three simplest ones, the ladder, lattice, and bridged-T types whose general structures are shown in Fig. 2. Propagation constant and iterative impedance formulæ for these types in terms of general impedance elements are given in Appendix III for possible future reference.

By introducing in each of these types the use of inverse networks with z_{11} and z_{21} satisfying relation (10), and assuming various relations in the general formulæ, it is possible to derive general network structures whose iterative impedances are a constant resistance, R, at all frequencies.[10] The structures are of such general nature as to permit a very wide range of propagation constants. Any one of them when closed by a resistance, R, presents at the other terminals the impedance R at all frequencies. They will now be considered.

The networks of the *ladder type* are shown in Fig. 3 as six complete sections, each designated by the termination at which it has the iterative impedance R; one at full-series, one at full-shunt, and two

[10] See U. S. Patent No. 1,603,305 to O. J. Zobel, dated October 19, 1926. Also British Patent Specification No. 236,189, dated July 8, 1926.

each at mid-series and mid-shunt. The first two sections are dissymmetrical as regards the two pairs of terminals. The two mid-series sections are symmetrical and identical except for the structure of their shunt branches, which, however, are equivalent impedances. Similarly, the symmetrical mid-shunt pair have different series branches of equivalent impedance. It may be of interest to point out that if each of these sections is closed by a resistance, R, to form a two-terminal network, then three pairs of these networks are seen directly from series and parallel rules to be inverse networks; namely,

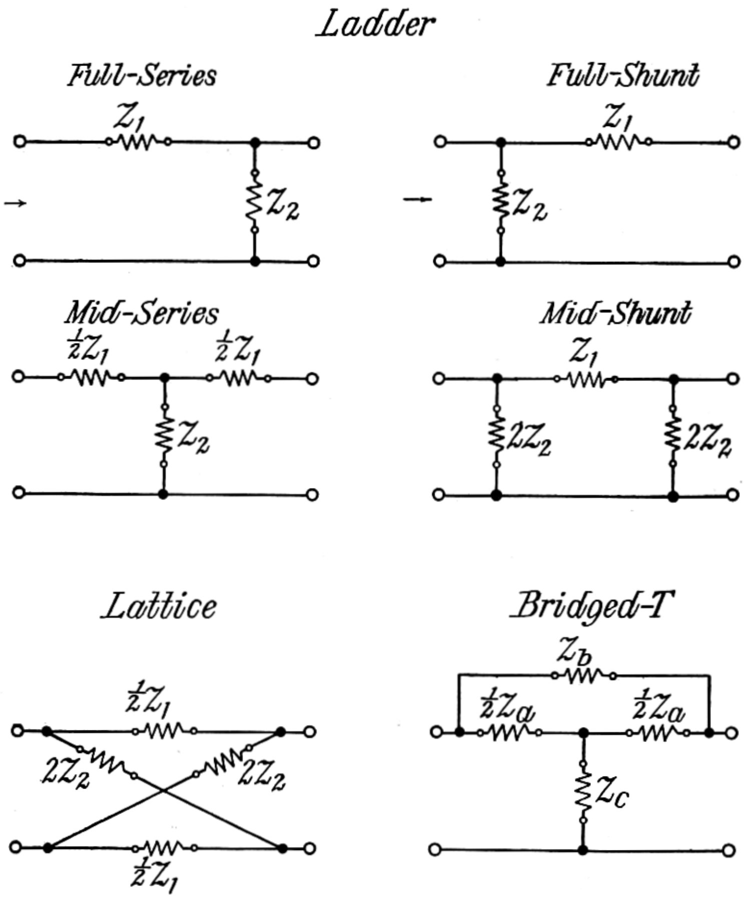


Fig. 2—Types of general recurrent network sections.

$$I_a,\ I_b;\ I_c,\ I_f;\ \text{and}\ I_d,\ I_e.$$

If one has the impedance R, the other must also, as is the case. The propagation constant, $\Gamma = A + iB$, of each of these *ladder type* sections is the same and is given by the simple relation

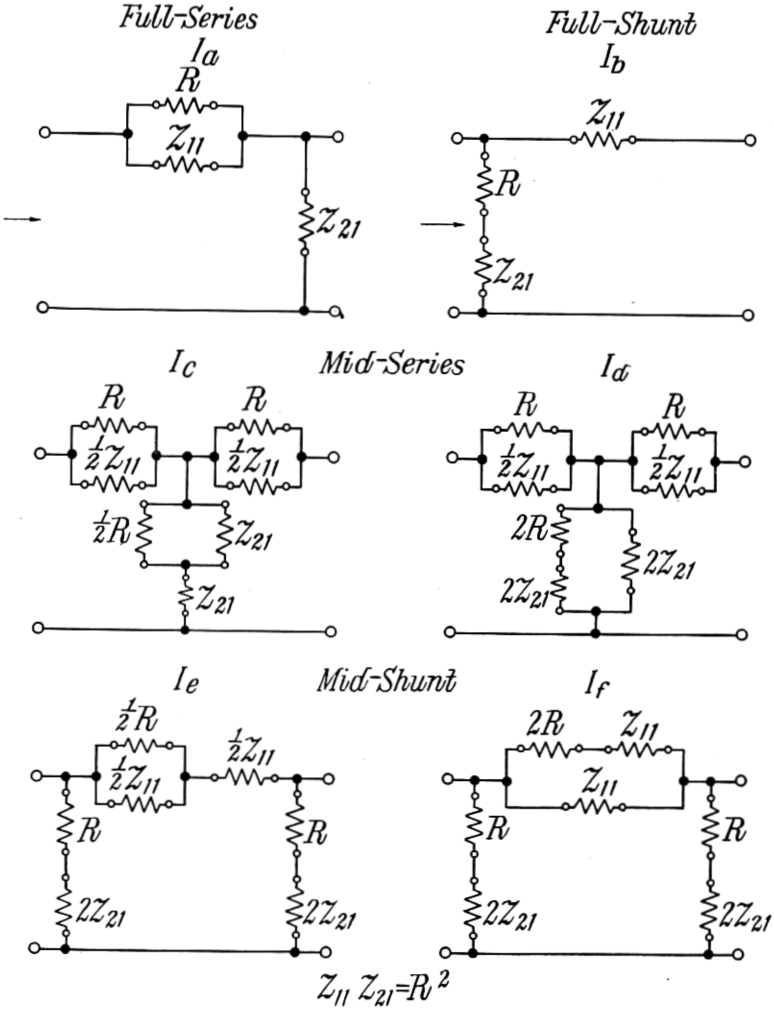


Fig. 3—Ladder type constant resistance sections.

$$e^{\Gamma} = 1 + z_{11}/R; \tag{12}$$

the particular iterative impedance is R. Here z_{11} is arbitrarily taken as the independent impedance determining the propagation constant with z_{21} dependent through the inverse network relation $z_{11}z_{21} = R^2$. This relation, besides ensuring a constant resistance iterative impedance, reduces the network parameters at least one half. *Since resistances occur explicitly in the structures, these sections will all be dissipative.*

The network of the *lattice type*, shown in Fig. 4, is symmetrical and has a propagation constant determined by

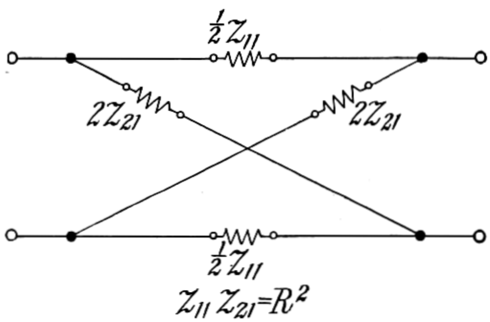


Fig. 4—Lattice type constant resistance section.

$$e^{\Gamma} = \frac{1 + z_{11}/2R}{1 - z_{11}/2R}, \tag{13}$$

where $z_{11}z_{21} = R^2$. *If z_{11} is a reactance, the network will introduce no attenuation, only phase difference.*

The networks of the *bridged-T type* are symmetrical and will be given in two groups, the members of each group having the same propagation constant. The two sections of the *first group* (I_a and I_b) in Fig. 5

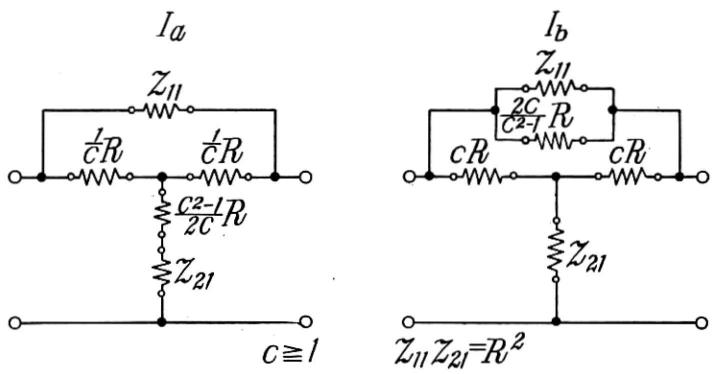


Fig. 5—Bridged-T(I) type constant resistance sections.

have a propagation constant formula

$$e^{\Gamma} = \frac{1 + (c + 1)z_{11}/2R}{1 + (c - 1)z_{11}/2R}, \tag{14}$$

where, besides the arbitrary impedance z_{11}, there is the arbitrary real $c \geqq 1$. *These sections will be dissipative owing to the ever present resistances.* Utilizing directly the rule given for inverse bridge networks, it can be seen that when closed by R these two structures are inverse networks of impedance product R^2.

The four bridged-T sections of the *second group* (II_a, II_b, II_c, and II_d) in Fig. 6 have the formula

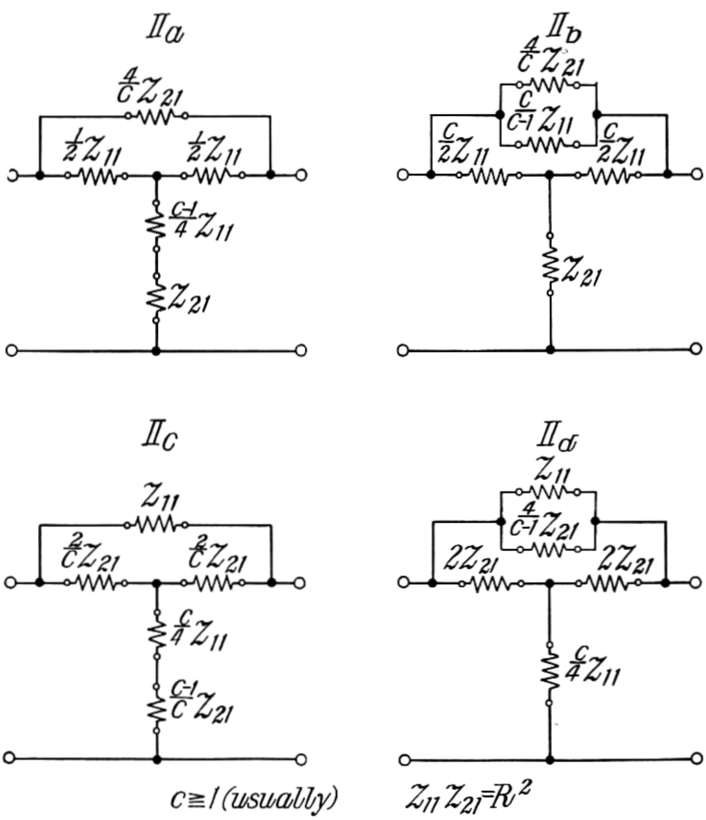


Fig. 6—Bridged-T(II) type constant resistance sections.

$$e^{\Gamma} = \frac{1 + z_{11}/2R + c(z_{11}/2R)^2}{1 - z_{11}/2R + c(z_{11}/2R)^2}, \tag{15}$$

where $c \geqq 1$, usually. In the very special cases of networks II_a and II_b, wherein z_{11} is an inductance, c may be less than unity and approach zero as a limit. For the latter values the negative inductance may be obtained physically as a negative mutual between the series coils. When $c = 1$, networks II_a and II_b become physically identical, as do also networks II_c and II_d. *If z_{11} is a reactance, there will be no attenuation.* Again, we shall find by applying the proper rule directly that when the four general sections are closed by resistances R there will result two pairs of inverse networks of impedance product R^2, respectively II_a, II_d and II_b, II_c.

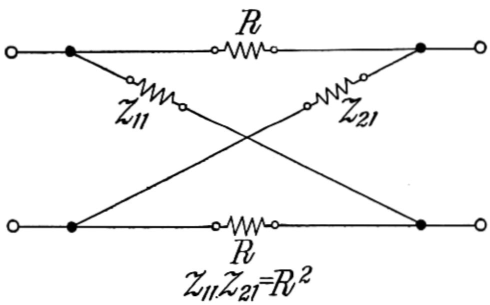


Fig. 7—Unbalanced lattice type constant resistance section.

A special network of the *unbalanced lattice type* may be mentioned briefly. This symmetrical structure as shown in Fig. 7 plays no direct part here as a distortion correcting network but is closely related to some of the other types and possesses interesting properties, among others that of conjugacy as in an ordinary balanced Wheatstone bridge. Its open-circuit impedance X and short-circuit impedance Y are both equal to R, hence its iterative impedance, $\sqrt{XY}$, is also R. Since $\tanh \Gamma = \sqrt{Y/X} = 1$, $\Gamma = \infty$, which means that no current would flow in a terminating resistance R due to an e.m.f. applied through a sending resistance R, these two impedance branches being conjugate. The network containing four resistances R, which is obtained by terminating this section at each end by a resistance R, may likewise be derived directly from the limiting case ($c = 1$) of the bridged-T (I) section which has similarly been terminated, merely by a rearrangement of form. It has these properties:

1. Opposite resistances are in conjugate branches.
2. Each of the four resistances is faced by a resistance R.

These properties can be seen as a result of the symmetry and also from a comparison with the full-series and full-shunt ladder type sections when terminated by resistances R. It is known that for one direction of propagation these two ladder sections have the same iterative impedance and propagation constant. In the full-series section terminated by R the junction point between R of the section and z_{21} is short-circuited with the point at the receiving side of z_{11}, while in the corresponding full-shunt network the structure is the same except that these two points are open-circuited. Because of the identity of propagation constants this can be possible only if the two points are at the same potential whence they can be connected by any impedance without altering propagation in the one direction. This being the case, a branch of resistance R, conjugate with the sending branch, can be connected across these points, and this results in giving the symmetrical bridged-T (I) type (where $c = 1$), or the equivalent network of Fig. 7 terminated by R. Thus the receiving-side series resistance R in the limiting case ($c = 1$) of the bridged-T (I) section plays no rôle and is superfluous for this direction of transmission, but it makes the section symmetrical and ensures similar propagation and impedance characteristics when transmitting in the opposite direction.[11]

If, in the network of Fig. 7, z_{11} is made resonant and anti-resonant at different frequencies, selective *maximum* energy transmission can be obtained at these frequencies between pairs of the four different resistance branches which might also be considered as different lines. The propagation constant between any pair of resistances can be determined from the relationships established above.

As an aid in obtaining an approximate value of the propagation constant for any of these types when its impedance elements are known, a simple chart may be drawn up if desired. This could be obtained in the following manner. The formulæ (12) to (15) are all of the form

$$e^{\Gamma} = e^{A+iB} = m + in;$$

whence

$$e^{A} = \sqrt{m^2 + n^2}, \tag{16}$$

and

$$\tan B = n/m.$$

Thus, it is evident that any locus of uniform attenuation constant, A, is represented in the m, n plane by a circle of radius, e^{A}, with center at the origin. Also, any locus of uniform phase constant, B, is a straight line of slope, $\tan B$, starting from the origin.

[11] Another method of deriving the section having directly the form given by putting $c = 1$ in the bridged-T (I) type was used by G. H. Stevenson, U. S. Patent No. 1,606,817, November 16, 1926.

2.5. *Relations for Equivalence of Propagation Constants*

All of the above networks have equivalent iterative impedances equal to R. It is sometimes useful to be able to transform readily from one type to another which has also an equivalent propagation constant, if that is physically possible. This may arise in an economic study of a final network design where account is taken of all practical factors, such as symmetry, line balance, number of the elements, their magnitudes, etc.

The structures which are important in this connection when dealing *with both attenuation and phase characteristics* comprise the ladder, lattice, and bridged-T (I) networks, whose propagation constant formulæ are given in (12), (13), and (14). For their propagation constants to be identical the impedance z_{11} in one type must bear a definite relation to that in another. In the following table, derived by equating these formulæ, a general impedance z is introduced. Each z_{11} may be expressed in terms of z and R. Here z is taken as the z_{11} for each type in succession. It then becomes a simple matter to transform from one type of structure to another having an equivalent propagation constant. The parameter c in a derived bridged-T (I) network would be taken such as to give the minimum number of elements.

TABLE II

Relations for Equivalence

Ladder z_{11}	Lattice z_{11}	Bridged-T (I) $z_{11},\ c \geqq 1$
z	$\dfrac{1}{\dfrac{1}{z}+\dfrac{1}{2R}}$	$\dfrac{1}{\dfrac{1}{z}+\dfrac{1}{-2R/(c-1)}}$
$\dfrac{1}{\dfrac{1}{z}+\dfrac{1}{-2R}}$	z	$\dfrac{1}{\dfrac{1}{z}+\dfrac{1}{-2R/c}}$
$\dfrac{1}{\dfrac{1}{z}+\dfrac{1}{2R/(c-1)}}$	$\dfrac{1}{\dfrac{1}{z}+\dfrac{1}{2R/c}}$	z

A transformation from the z_{11} of one type section to that of another equivalent one involves essentially only an alteration of the given impedance by a positive or negative resistance element in parallel with it. This will not always result in a physical network with positive elements. The following statements can be made, however:

1. The transformation of the ladder type to the equivalent bridged-T (I) type, and vice versa, is always possible.

2. The transformation of the ladder type, or the bridged-T (I) type, to the equivalent lattice type is always physically possible; the converse is not necessarily so.

Those structures which are potentially phase networks, and thus useful when requiring a non-attenuating network *with a phase characteristic only*, are the lattice type again and the bridged-T (II) type. Such networks are used to introduce various characteristics for the time-of-phase-transmission. It will be sufficient to give the relations for equivalence between these two types, obtained from (13) and (15), as

$$(z_{11})_{\text{lattice}} = \left(\frac{1}{\dfrac{1}{z_{11}} + \dfrac{1}{4z_{21}/c}}\right)_{\text{bridged-T (II)}}, \tag{17}$$

which is always physically possible if the bridged-T (II) network exists. On the other hand

$$(z_{11})_{\text{bridged-T (II)}} = \frac{2}{c}(z_{21} \pm \sqrt{z_{21}^2 - cR^2})_{\text{lattice}}, \tag{18}$$

where the c which belongs to the bridged-T (II) type must necessarily be taken so as to make the radical a perfect square, *if a physical equivalent is possible.* It is to be pointed out that in the propagation constant formula (15), considered as a general form, the range of values for the parameter c which will give a physical bridged-T (II) network is $c \geqq 1$, usually, while the range for a physical lattice network is $c \geqq 0$, as seen from (17). Thus, the lattice type can give a greater variety of propagation constants.

From all the comparisons made above this conclusion may be drawn. *The lattice type has a greater range for its propagation constant characteristic than has either a ladder or a bridged-T type. Hence, the lattice type might well be considered as the fundamental one, when designing such networks, from which other equivalent types may be obtained by transformations, if such physical structures are possible.*

2.6. *Propagation Constants Expressed as Frequency Functions*

In Section 2.4 the propagation constant of any of these networks was given as varying with frequency only implicitly, according to some function of the impedance ratio, $z_{11}/2R$. To express it more explicitly as a frequency function, I shall sketch briefly a satisfactory general method to be followed.

For an impedance z_{11} which is made up of lumped elements of resistance, inductance, and capacity we may express the impedance ratio $z_{11}/2R$ as the ratio of two frequency-polynomials in (if), where $i = \sqrt{-1}$ and f is frequency. Thus,

$$\frac{z_{11}}{2R} = \frac{a_0 + a_1(if) + a_2(if)^2 + \cdots}{b_0 + b_1(if) + b_2(if)^2 + \cdots} = s + iy. \tag{19}$$

The impedance coefficients a_0, b_0, etc., of which one is unity and some may be zero, are positive quantities and are algebraic combinations of the network elements. Their number is equal to, or greater than, the number of independent elements. For any given type of network the coefficients are fixed by the elements, and vice versa.

Putting this expression in any of the formulæ (12) to (15), there results *for the propagation constant* a form

$$e^{\Gamma} = \frac{g_0 + g_1(if) + g_2(if)^2 + \cdots}{h_0 + h_1(if) + h_2(if)^2 + \cdots}, \tag{20}$$

in which g_0, h_0, etc., are algebraic functions of a_0, b_0, etc., also of c if the network is a bridged-T type. From this the attenuation constant and phase constant can also be derived and expressed separately as functions of frequency.

For the attenuation constant, a form is obtained

$$F \equiv e^{2A} = 10^{TU/10} = \frac{P_0 + P_2 f^2 + \cdots}{Q_0 + Q_2 f^2 + \cdots}, \tag{21}$$

which is the ratio of two frequency-polynomials both in even powers of frequency. One of the attenuation coefficients is unity.

For the phase constant, a form

$$H \equiv \tan B = \frac{M_1 f + M_3 f^3 + \cdots}{N_0 + N_2 f^2 + \cdots}, \tag{22}$$

in which one of the phase coefficients is unity, is the ratio of two frequency-polynomials, odd powers of frequency in the numerator and even powers in the denominator. (It is sometimes convenient to use tan $(B/2)$.) In (21) and (22) the attenuation coefficients P_0, Q_0, etc., and the phase coefficients M_1, N_0, etc., are expressible in terms of the impedance coefficients a_0, b_0, etc.

It should be mentioned here that in deriving the above expressions certain assumptions have been made; namely, invariable elements and non-dissipative inductances and capacities. These restrictions are well justified from the fact that such departures are usually small and their effects in a network do not alter appreciably the general characteristics. However, to calculate accurate results for both the propagation constant and the iterative impedance of the final design of a physical network taking into account all factors, one should use the general formulæ given in Appendix III which have been simplified to give accurate results quite readily.

2.7. *Network Solutions from Their Propagation Characteristics*

It was assumed in the previous section that the recurrent network elements are invariable and that inductances and capacities are non-dissipative. On this basis general formulæ for the propagation characteristic were obtained in terms of these elements. The same assumptions are retained here but *reverse processes* will be carried through which derive the elements from the propagation characteristic of the recurrent network. Three methods will be outlined, necessarily in general terms.

Method 1. *Solutions from the Attenuation Constant*

Since attenuation is ordinarily of greatest importance, this method is the one most frequently used with networks having an attenuation characteristic and involves initially the determination of the attenuation coefficients P_0, Q_0, etc., from this characteristic. Using these coefficients, one derives from algebraic relations, first, the impedance coefficients a_0, b_0, etc., and finally the network elements in z_{11}. The elements of z_{21} follow from the inverse network relation (10).

The method is based upon the transformation of the attenuation formula (21) to a linear equation in P_0, Q_0, etc., whose number is equal to or greater than the number of independent network parameters. If we multiply equation (21) by the Q-polynomial, we obtain formally the *attenuation linear equation* which holds at all frequencies,

$$P_0 + f^2P_2 + \cdots - FQ_0 - f^2FQ_2 - \cdots = 0. \qquad (23)$$

Introducing in this the attenuation constant, and hence F, at a number of different frequencies equal to the number of independent network parameters, there results a system of independent simultaneous linear equations which can be solved for the coefficients. The simplest practical procedure is perhaps that of the *step-by-step elimination of the coefficients*.

When the number of coefficients and independent network parameters, hence equations, are the same, the solution of the latter offers no particular difficulty and results can readily be checked by substitution in the original equation (21).

When, as sometimes occurs, the number of coefficients is one greater than the number of independent network parameters, it means that one relation exists between the coefficients and hence any one of the latter may be assumed dependent. The dependent relation can be found from the formulæ for P_0, Q_0, etc., in terms of a_0, b_0, etc. However, in some such networks it is possible to use the attenuation constant at a particular frequency, say zero or infinite frequency, and thereby reduce the number of remaining coefficients and independent network parameters to equality, when the case is readily solvable. If this does not produce the desired reduction, it is usually best to first transfer the dependent coefficient to the right-hand member of (23) and after forming the set of linear equations solve them for the independent coefficients in terms of the dependent one. Substitution of these values in the dependent relation gives a polynomial in the dependent coefficient which can be solved by Horner's method. Its solution then determines the independent coefficients. This procedure might be extended similarly to cases where the number of coefficients is two or more greater than that of the linear equations, but obviously the process becomes quite involved.

The values of the attenuation coefficients P_0, Q_0, etc., are unique when determined from linear equations. The impedance coefficients a_0, b_0, etc., derived from them are also single-valued to give a physical solution in most types of networks, meaning that only one such physical network has the particular attenuation characteristic. However, in the *lattice type*, it has been found that there are usually possible *two or more physical solutions* for the impedance coefficients from the attenuation coefficients, which correspond to two or more similar appearing physical structures having identically the same attenuation characteristic but different phase constants.

Method 2. Solutions from the Phase Constant

This method is applicable particularly to phase networks which ideally have no attenuation and to other networks where the number of phase coefficients equals the number of independent network parameters. The procedure is the same as in the previous method where now we operate with the phase constant formula (22). Multiplying the latter by its N-polynomial, we obtain formally the *phase linear equation*, true at all frequencies,

$$fM_1 + f^3M_3 + \cdots - HN_0 - f^2HN_2 - \cdots = 0. \qquad (24)$$

Fixing the phase constant, and hence H, in this equation at frequencies equal in number to the phase coefficients gives us, if this number is equal to the number of independent network parameters, the desired set of linear equations to be solved by the usual methods. In a network where the number of phase coefficients is one less than the number of network parameters an additional relation will be needed to determine the network elements and this can be supplied from the attenuation characteristic. Here the attenuation characteristic can probably be lowered uniformly without altering the phase characteristic. (See Section 2.82.)

Method 3. *Solutions from the Propagation Constant*

Since it has been shown in Section 2.5 that any network of the type considered in this paper can always be represented physically by a lattice type having an equivalent propagation constant, we can simplify the discussion here by dealing entirely with the lattice network. From (13) the impedance ratio $z_{11}/2R$ for this type is derived in terms of its propagation constant as

$$\frac{z_{11}}{2R} = \frac{e^{\Gamma} - 1}{e^{\Gamma} + 1} = \tanh\ (\Gamma/2), \tag{25}$$

which holds at all frequencies. Thus, *a determination of the recurrent network from its propagation constant (attenuation and phase constants together) reduces to the solution of a two-terminal impedance network from its impedance characteristic.* The impedance ratio components s and y in (19) will become definite known functions of frequency determined through (25) by the propagation constant of the given lattice network.

A method of solving for the impedance coefficients a_0, b_0, etc., and hence the network elements from the components s and y, follows. Instead of attempting to separate the impedance ratio expression into its real and imaginary parts which can then separately be equated to s and y, which is the usual method, let us multiply (19) by the b-polynomial. Now equating separately the real and imaginary parts we obtain a pair of equations which are linear in the coefficients and hold at all frequencies. This pair of *impedance linear equations* are formally

$$\begin{aligned} &a_0 - f^2a_2 + \cdots - sb_0 + fyb_1 + f^2sb_2 + \cdots = 0, \\ \text{and} \quad & \\ &fa_1 - f^3a_3 + \cdots - yb_0 - fsb_1 + f^2yb_2 + \cdots = 0. \end{aligned} \tag{26}$$

By this means the formulæ are put in a form such as to require in all cases the solution of a set of equations linear in the coefficients, obtained from (26) at different frequencies. A procedure for their solution similar to that used in dealing with equation (23) can be applied and will not be repeated here. This process, apparently new, of obtaining linear equations for the impedance coefficients which contain powers of frequency and the impedance components, was applied by the writer to non-dissipative two-terminal networks in this *Journal*, January, 1923, p. 21, also in U. S. Patent No. 1,509,184, dated September 23, 1924; and to dissipative networks which simulate a smooth line impedance in U. S. Patent Application, Serial No. 134,515, filed September 9, 1926. It is merely outlined here.

2.8. *Useful Properties and Relations*

The following discussion covers a number of points concerning these networks which have been found quite useful. They can be verified readily from the fundamental formulæ and so need not be derived in detail.

2.81. *Analytical Simplifications*

Let it be desired to design a given network from its attenuation characteristic in a frequency range when the number of attenuation coefficients is one greater than the number of independent network elements. As previously stated, it is usually possible in such cases to choose as part of the attenuation data the attenuation constant at a particular frequency, such as zero or infinite frequency, and make the resulting number of attenuation coefficients and independent elements equal in number, with consequent ease of solution. Another method of simplifying the analysis might be to slightly alter the form of the given z_{11} by adding to it, or subtracting from it, a resistance element in series or in parallel. This may have the effect of making the resulting attenuation coefficients and independent elements equal in number without appreciably altering the general attenuation characteristic in the desired frequency range.

2.82. *Uniform Attenuation Change*

According to principles developed above, if the attenuation constant of a given network is changed uniformly over the entire frequency range without altering its phase constant, its distortion producing characteristics are not affected.

Let z_{11} correspond to a given *lattice type* network and z_{11}' to a derived one in which the attenuation only has been changed by a uniform amount A_0 at all frequencies. Then one form of structure for z_{11}' is

$$z_{11}' = \frac{1}{\dfrac{1}{m_1 z_{11} + m_2 R} + \dfrac{1}{m_3 R}}, \tag{27}$$

where $m_1 = \cosh^2 (A_0/2)$,

$m_2 = \sinh A_0$,

and $m_3 = 2 \coth (A_0/2)$,

m_1 being greater than unity, while m_2 and m_3 have the sign of A_0. This relation for z_{11}' stated approximately in words is as follows: *To raise the attenuation, magnify the given* z_{11} *and add series resistance, then add parallel resistance to the whole; to lower the attenuation, magnify* z_{11} *and add such negative resistances.* An example is given by Networks 1*a* and 3*a* of Appendix IV.

An impedance equivalent form of structure for z_{11}' is

$$z_{11}' = \frac{1}{\frac{1}{m_1' z_{11}} + \frac{1}{m_2' R}} + m_3' R, \tag{28}$$

where $m_1' = \text{sech}^2 (A_0/2)$,

$m_2' = 4 \text{ cosech } A_0$,

and $m_3' = 2 \tanh (A_0/2)$,

m_1' being positive and less than unity, while m_2' and m_3' have the sign of A_0. Hence with this form, *to raise the attenuation, reduce the given* z_{11} *and add parallel resistance, then add series resistance to the whole; to lower the attenuation, reduce* z_{11} *and add such negative resistances.* An example is given by Networks 1*b* and 3*b*, Appendix IV.

It will be seen from these relations derived from a physical z_{11} that when A_0 is positive a physical z_{11}' always results. When A_0 is negative, however, physical impedances would be obtained only under certain conditions, depending upon the given z_{11} and upon A_0.

One practical utility of the relations would occur in the following situation. Suppose that a design was being attempted from assumed attenuation values with a network having such a general characteristic and that z_{11} consists of some structure in series or in parallel with a resistance element. The latter resistance as determined from the linear equations may come out to be negative and give z_{11} an unphysical structure. In such a case we could apply the above relations and raise all the attenuation values uniformly such an amount A_0 that the resulting network z_{11}' would be physical.

Corresponding relations between two networks of the *ladder type* are

$$z_{11}' = e^{A_0} z_{11} + (e^{A_0} - 1)R; \tag{29}$$

and between two of the *bridged-T* (*I*) type are

$$z_{11}' = (c \sinh (A_0/2) + \cosh (A_0/2))^2 z_{11} + 2 \sinh (A_0/2)(c \sinh (A_0/2) + \cosh (A_0/2))R,$$

and

$$c' = \frac{c + \tanh (A_0/2)}{c \tanh (A_0/2) + 1}. \tag{30}$$

In the above process we would generally be increasing the number of network parameters without changing the number or magnitude of the phase coefficients.

2.83. *Phase Constant Comparisons of Certain Pairs of Lattice Type Networks*

It has already been stated that there are usually two physical networks of the same structural lattice form which have identical attenuation constants but different phase constants. They are derivable as two physical solutions from the same attenuation coefficients. In the case of a limited class of these networks, an interesting relation exists between the phase constants of such a pair which may be stated as follows.

Theorem.—The two lattice type networks of every pair having the same attenuation characteristic in each of which the series impedance (z_{11}) *consists of a resistance in parallel with any pure reactance network, of different proportions in each, have phase constants such that their sum or difference is identical with that of a non-dissipative lattice phase network whose series impedance* (z_{11}) *is a pure reactance network proportional to that in the series impedance of either of the pair.*

A *corollary* results from this.

One network of the pair is equivalent to the tandem combination of the other and the related phase network.

It should be pointed out here that results for the case in which z_{11} is a resistance in series with a reactance network are similar, except for a phase change of π, since then the lattice impedance z_{21}, the inverse network of z_{11}, is a resistance in parallel with a reactance network.

A procedure for proving the theorem will be sketched briefly. Assume as given one network in which z_{11} is made up of a resistance in parallel with a pure reactance network whose impedance is imy, where m is a positive constant and y is a function of frequency. This gives a form

$$F = e^{2A} = \frac{1 + P_2 y^2}{1 + Q_2 y^2}. \tag{31}$$

Reversing the process, we obtain from the same coefficients P_2 and Q_2 a second similarly constructed network besides the original one. The two physical networks differ in their phase constants but have the same attenuation constants. For one

$$\tan B' = \frac{(\sqrt{P_2} + \sqrt{Q_2})y}{-1 \quad \sqrt{P_2 Q_2} y^2}, \tag{32}$$

and for the other

$$\tan B'' = \frac{(\sqrt{P_2} - \sqrt{Q_2})y}{1 + \sqrt{P_2 Q_2} y^2}, \tag{33}$$

where B'' has a maximum or minimum depending upon whether y is positive or negative. As a result for the *sum*

$$\tan \left(\frac{B' + B''}{2} \right) = \sqrt{P_2} y, \tag{34}$$

and for the *difference*

$$\tan \left(\frac{B' - B''}{2} \right) = \sqrt{Q_2} y. \tag{35}$$

Now a non-dissipative lattice type network in which z_{11} is a reactance proportional to y has a formula

$$\tan (B/2) = M_1 y, \tag{36}$$

where M_1 is positive. Comparison of these latter formulæ indicates the proof of the theorem and its corollary.

A simple and useful relation exists between the maximum attenuation constant A_m occurring at $y = \infty$ and the maximum or minimum phase constant B_m'' of (33) occurring at $y = \pm 1/(P_2 Q_2)^{1/4}$. It is

$$\sinh (A_m/2) = \pm \tan B_m''. \tag{37}$$

An example is given by Networks 2*a*, Appendix IV, and a practical use of this relation will be made in Section 4.2.

2.84. *Composite Networks*

The tandem combination of two or more different sections of constant resistance networks can generally give propagation characteristics which are unattainable in a single section. For this reason it is sometimes advantageous to treat such a composite network of two or three simple sections as a single unit. When this is done it will be found that the composite network has attenuation coefficients, if any, which in number may be equal to, greater than, or even less than the sum for the individual networks when considered separately.

An example of a case in which the number of attenuation coefficients for the composite network equals the sum for the separate sections is furnished by two sections of Network 1*a* or of 2*a*, Appendix IV, both having four coefficients. On the other hand, a composite network of 1*a* and 2*a*, one of each, has five attenuation coefficients. Finally, a composite network of two sections of Network 3*a* has only five attenuation coefficients contrasted with a sum of six for the separate networks. In the latter case we can obtain only five linear equations from the attenuation characteristic which are not sufficient to determine the six series elements. This probably means that for the same attenuation characteristic the resistances in series with the two inductances can be given any ratio to each other from zero to infinity. A sixth relation can then be supplied by assuming the practical condition which makes the ratio of resistance to reactance the same in the inductance branches of both sections. This composite network can have an attenuation constant whose increase with frequency is approximately linear over a wide internal frequency range.

Composite phase networks of simple structure also lend themselves readily to such treatment as a single unit.

2.85. *Composite Lattice Networks Having Uniform Attenuation*

To a lattice type network of a certain class having a finite non-uniform attenuation characteristic there corresponds a single infinity of complementary ones, such that when any one of the latter is combined with it, the composite network has a uniform total attenuation constant and a zero total phase constant over the entire frequency range. *The separate attenuation constants are complementary while the phase constants are equal, but opposite in sign.* Such a conposite network we have seen would be *absolutely distortionless.* It is a relatively simple matter to obtain the necessary relations which such a complementary network must bear to the first if we impose these propagation conditions on the combination. Two sets of relations may be derived, each corresponding to a particular structure for the first network, with the following results.

If the given section (A, B) has *series impedances*

$$z_{11} = R_s + z_s, \tag{38}$$

where R_s is a resistance and z_s is any impedance, any equivalent transformation of which does not contain series resistance, and if a complementary network (A', B') is added such as to give a composite network (A_c, B_c) with the propagation constant

$$A_c = A + A' = \text{constant},$$

and

$$B_c = B + B' = 0, \tag{39}$$

then the complementary network is given by

$$z_{11}' = R_1 + \frac{1}{\dfrac{1}{z_2} + \dfrac{1}{R_3}}, \tag{40}$$

where $R_1 = 2 \coth (A_c/2)R$,

$z_2 = 4 \operatorname{cosech}^2 (A_c/2)R^2/z_s$,

and $R_3 = 4 \operatorname{cosech}^2 (A_c/2)R^2/(R_s - 2 \coth (A_c/2)R)$.

Here z_2 is the inverse network of z_s of impedance product 4cosech^2 $(A_c/2)R^2$. The network in (40) is R_1 in series with the parallel combination of z_2 and R_3. An equivalent form for z_{11}' is

$$z_{11}' = \frac{1}{\dfrac{1}{R_1' + z_2'} + \dfrac{1}{R_3'}}, \tag{41}$$

where $R_1' = \cosh^2 (A_c/2)(R_s - 2 \tanh (A_c/2)R)$,

$z_2' = \cosh^2 (A_c/2)(R_s - 2 \tanh (A_c/2)R)^2/z_s$,

and $R_3' = \dfrac{2R(\coth (A_c/2)R_s - 2R)}{(R_s - 2 \coth (A_c/2)R)}$.

It will be a *physical network* provided A_c satisfies the relation

$$1 < \coth (A_c/2) \leqq R_s/2R. \tag{42}$$

At the *minimum* A_c, $R_1 = R_1'$, $z_2 = z_2'$, and $R_3 = R_3' = \infty$.

If, on the other hand, the given section has *parallel impedances* (similar to the preceding network of (38) whose output terminals are reversed),

$$z_{11} = \frac{1}{\dfrac{1}{R_p} + \dfrac{1}{z_p}}, \tag{43}$$

where R_p is a resistance and z_p is any impedance, any equivalent transformation of which does not contain parallel resistance, then a corresponding complementary network has one form given by

$$z_{11}' = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{z_2 + R_3}}, \tag{44}$$

where $R_1 = 2 \tanh (A_c/2)R$,

$$z_2 = 4 \sinh^2 (A_c/2)R^2/z_p,$$

and $R_3 = 2 \sinh^2 (A_c/2)R(2R - \coth (A_c/2)R_p)/R_p$.

An equivalent form is

$$z_{11}' = \frac{1}{\dfrac{1}{R_1'} + \dfrac{1}{z_2'}} + R_3', \tag{45}$$

where $R_1' = 2 \operatorname{sech}^2 (A_c/2)RR_p/(2R - \tanh (A_c/2)R_p)$,

$$z_2' = 4 \operatorname{sech}^2 (A_c/2)R^2R_p^2/(2R - \tanh (A_c/2)R_p)^2 z_p,$$

and $R_3' = \dfrac{2R(2R - \coth (A_c/2)R_p)}{(2 \coth (A_c/2)R - R_p)}$.

There will be a *physical network* provided

$$1 < \coth (A_c/2) \leqq 2R/R_p. \tag{46}$$

At the *minimum* A_c, $R_1 = R_1'$, $z_2 = z_2'$, and $R_3 = R_3' = 0$.

It may be added that if (38) and (43) represent inverse networks of impedance product $4R^2$, then another such pair is given by (40) and (44), and still another by (41) and (45).

An extension of these results may now readily be made to give *two-section composite networks whose attenuation constants are uniform but whose phase constants are not zero.* It has been stated that to every lattice type network having finite attenuation there usually corresponds another one of the same structural form having the same attenuation but a different phase characteristic. Hence, in either case above where the two complementary sections giving a total uniform attenuation are known, we may derive by regular methods the alternative lattice sections, having, respectively, the same attenuation constants. Since we would then have two sections to give the one attenuation characteristic and two sections for the complementary characteristic, it would be possible to obtain *four composite networks* of similar structure, all of which give *the same uniform attenuation but four different phase characteristics.* One of these combinations would be the case in which the phase constant is zero. Four more phase characteristics, differing from the others by an amount π, can obviously be obtained by reversing the terminals of either section.

2.9. *Procedure for the Design of Distortion Correcting Networks*

It would be most gratifying to be able to obtain directly from a desired propagation characteristic the corresponding form of network.

This is generally a difficult problem and it becomes necessary to resort to simplifying methods somewhat similar to those employed in the design of electric wave-filters. One reason for this difficulty is that we are limited to physical resistance, inductance, and capacity elements, all of which must, in general, be positive. We would, therefore, begin with known forms of networks whose general propagation characteristics have been determined and choose from them one or more whose combination offers the possibility of giving a satisfactory desired result. A number of points which are applicable in the general case may be noted as follows:

1. First, determine the desired propagation characteristics of the distortion correcting network corresponding to formula (8).

2. If necessary, divide this propagation characteristic into several parts each of which has the approximate characteristic belonging to a known network structure.

3. Assume one of these networks physically capable of having such an alloted characteristic and attempt a design to approximately fit it according to one of the methods of Section 2.7. Where there is an attenuation characteristic, Method 1 is usually best, as attenuation is generally of more importance than phase and hence its simulation requires greater accuracy. The network will introduce a phase constant which will necessarily have to be taken into account. Of the two or more possible solutions for the lattice type network, the one with the most desirable phase constant would obviously be chosen and in some cases this may be close to requirements. Another reason for usually following this order of simulating the attenuation first and the resultant phase later is furnished as a consequence of Theorems I and II of Section 2.2. From them we see the physical possibility of introducing certain phase characteristics without attenuation (ideally), but not varying attenuation characteristics without phase. Method 3 imposes a rather severe requirement on a single network.

4. If the network design comes out to be unphysical with the particular characteristic values assumed, small variations from these values should be tried, since the natural varying curvatures in the propagation characteristic of the network must sometimes be allowed for. Otherwise, a different kind of network should be used, or a composite one, which has a similar characteristic.

5. In designing successive sections of the complete transducer, the effects of previous parts must be considered.

To facilitate the application of this method of distortion correction, general propagation characteristics together with formulæ have been derived for a representative number of lattice type structures. These are given in Appendix IV. Any pair of the networks, such as 1*a* and 1*b*, differ only by an interchange of series and lattice elements with a corresponding difference in their phase constants of an amount π. In order to simplify computations for some networks the formulæ were derived so as to require attenuation data at a limiting frequency, but other formulæ may also be obtained. By means of the relations in Section 2.5, transformations can readily be made to any of the other general types, if they lead to physical structures.

The type of network which a final design is to assume will be suggested by economic and practical considerations. However, an approximate statement can be made in this connection. If the sections are to be dissymmetrical as regards the two pairs of terminals and unbalanced as regards the two sides of the line, use the full-series or full-shunt ladder types; if symmetrical and unbalanced, use the bridged-T types; if symmetrical and balanced, use the bridged-T or lattice types.

Part 3. Arbitrary Impedance Recurrent Networks

In Part 2 consideration was given entirely to recurrent networks whose iterative impedances are a constant resistance at all frequencies and which depend upon the use of inverse networks; that is, $z_{11}z_{21} = R^2$. It is intended here merely to point out briefly that all the types in Section 2.4 can be generalized to have iterative impedances of arbitrary value K provided in them

$$R \text{ is generalized to } K,$$

and

$$z_{11}z_{21} = K^2; \tag{47}$$

that is, z_{11} and z_{21} are *inverse networks*[12] *of impedance product* K^2. The corresponding propagation constant formulæ hold also with these generalizations.

Where a recurrent network of arbitrary iterative impedance K is desirable, these structures would, theoretically at least, be applicable. Practically, however, considerable difficulties are usually encountered in physically realizing z_{11} and z_{21} to give a desired propagation constant, and perhaps even K when K is not a simple function of frequency. A few physical possibilities will be given here in which the structures for z_{11} and z_{21} are easily identified from the forms of the expressions. They may be used in the different types of networks, and, of course, z_{11} and z_{21} may be interchanged.

[12] The complete qualifying statement such as given is necessary here, not just simply "inverse networks."

1.

$$K = R + iL\omega;$$
$$z_{11} = iL_{11}\omega, \tag{48}$$

and

$$z_{21} = (2RL/L_{11}) + i(L^2/L_{11})\omega + 1/i(L_{11}/R^2)\omega.$$

The impedance z_{21} is series resistance, inductance and capacity.

2.

$$K = R + 1/iC\omega;$$
$$z_{11} = 1/iC_{11}\omega, \tag{49}$$

and

$$z_{21} = (2RC_{11}/C) + i(R^2C_{11})\omega + 1/i(C^2/C_{11})\omega.$$

Here z_{21} is the same type of structure as in (48).

3.

$$K = R + 1/iC\omega;$$
$$z_{11} = \cfrac{1}{\cfrac{1}{\cfrac{R}{m_1} + \cfrac{1}{im_1C\omega}} + \cfrac{1}{\cfrac{R}{m_2} + \cfrac{1}{im_2C_{11}\omega}}}, \tag{50}$$

and

$$z_{21} = \cfrac{1}{\cfrac{1}{R_{21}} + \cfrac{1}{iL_{22}\omega}} + R_{23} + \frac{1}{iC_{24}\omega},$$

where $R_{21} = m_2R(C - C_{11})^2/C^2$,

$$L_{22} = m_2R^2C_{11}(C - C_{11})^2/C^2,$$
$$R_{23} = R(m_1C^2 + m_2C_{11}(2C - C_{11}))/C^2,$$

and

$$C_{24} = C^2/(m_1C + m_2C_{11}).$$

The impedance z_{11} consists of two parallel branches each containing series resistance and capacity; z_{21} is made up of parallel resistance and inductance in series with both resistance and capacity. For certain values of the parameters m_1, m_2, and C_{11}, even though positive, the resistance R_{23} can become negative and hence unphysical as a passive element.

A lattice or equivalent network made up of such impedances, in addition to having the assumed iterative impedance which approximates that of an open-wire line at the upper frequencies, can have an attenuation constant decreasing with frequency which tends to equalize that of a length of such line; the attenuation formula has the form

$$F = e^{2A} = \frac{P_0 + P_2f^2}{Q_0 + f^2}. \tag{51}$$

4.

$$K = \sqrt{\frac{R' + iL'\omega}{G' + iC'\omega}}, \quad \text{(smooth line)};$$

$$z_{11} = mR' + imL'\omega, \tag{52}$$

and

$$z_{21} = \frac{1}{(mG' + imC'\omega)}.$$

Another possible simple pair is that in which z_{11} is a resistance and z_{21} is either series resistance and inductance in parallel with series resistance and capacity or parallel resistance and inductance in series with parallel resistance and capacity. These impedance elements may be used in the lattice or bridged-T (II) type structures where the impedance element K is not explicitly required. Extension to more complex structures can be made by the methods of Section 2.3. An application will be given in Section 4.7 which considers the simulation of a smooth line.

Owing to the much greater inherent difficulty of physically realizing inverse networks of impedance product K^2 when K is not R, the generalization does not add much practically for our purpose, but some structures in which K is not R may be of utility under particular conditions.

Part 4. Applications

4.1. *Complementary Distortion Correcting Networks*

The pair of networks in Fig. 8 illustrates in a very simple manner the general relations given in Section 2.85, as well as ideal distortion

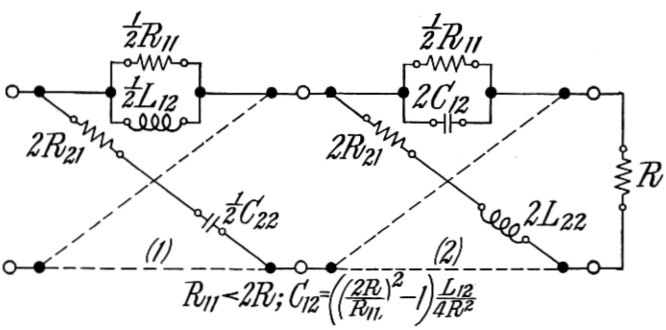


Fig. 8—Distortionless composite network.
(Broken lines indicate the other series and lattice branches, respectively identical).

correction over the entire frequency range. When placed in tandem they represent a composite network whose attenuation constant is uniform at all frequencies and whose phase constant is zero, which are characteristics for no distortion. Let us obtain the steady-state characteristics of each network and of the composite one; then consider transient conditions and obtain the indicial voltages of the corresponding networks to verify again by this illuminating example that the steady-state characteristics laid down for no distortion are quite sufficient when transient conditions exist.

The first section is Network 2*a*, Appendix IV, wherein z_{11} is parallel resistance R_{11} and inductance L_{12}, with R_{11} less than $2R$ and the characteristic 1. Let us put $m = R_{11}/2R$, and $n = L_{12}/2R$. Then

$$\frac{z_{11}}{2R} = \frac{imn\omega}{m + in\omega}, \tag{53}$$

and the propagation constant formula becomes from (13)

$$e^{\Gamma_1} = \frac{m + i(1 + m)n\omega}{m + i(1 - m)n\omega}. \tag{54}$$

To obtain a complementary second section let us assume that the total attenuation constant, A_c napiers, of the composite structure is to equal the maximum of the first section which occurs at infinite frequency. Then from the above

$$e^{A_c} = \frac{1 + m}{1 - m}$$

and $\tanh (A_c/2) = m$, giving as the correcting section by (44) one of Network 1*b*, Appendix IV, with characteristic 1 in which

$$R_{11} = 2mR, \text{ as in (53)},$$

and

$$C_{12} = \frac{(1 - m^2)n}{2m^2R} = \left[\left(\frac{2R}{R_{11}}\right)^2 - 1\right]\frac{L_{12}}{4R^2}. \tag{55}$$

For this second section then

$$\frac{z_{11}}{2R} = \frac{m^2}{m + i(1 - m^2)n\omega},$$

and

$$e^{\Gamma_2} = \left(\frac{1 + m}{1 - m}\right)\left(\frac{m + i(1 - m)n\omega}{m + i(1 + m)n\omega}\right). \tag{56}$$

Obviously, from (54) and (56),

$$e^{\Gamma_1 + \Gamma_2} = \frac{1 + m}{1 - m} = e^{A_c},$$

as was assumed.

31

The attenuation and phase constants of each of these two sections and the combined structure are shown in Fig. 9, as a function of $L_{12}\omega/2R$, where $R_{11} = R$. It will be seen that A_1 and A_2 are complementary while B_1 and B_2 are equal but opposite. For the composite network A_c = constant and $B_c = 0$; thus the latter phase constant

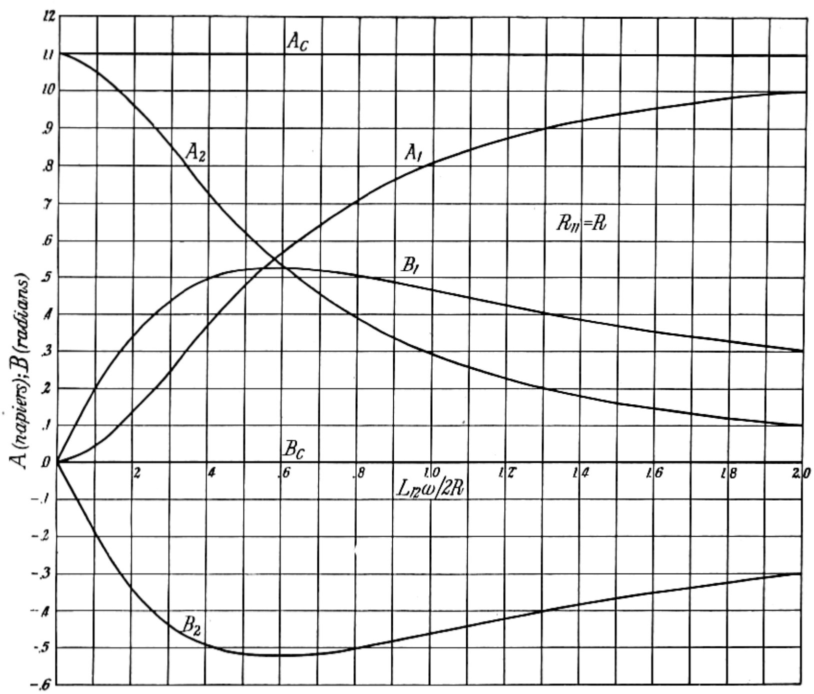


Fig. 9—Propagation constants in distortionless network.

has a zero slope with frequency. Whatever steady periodic voltage exists at one end would appear across the terminating resistance R in the same phase but attenuated by an amount A_c napiers. Since these conditions hold for the composite network at all frequencies, we should expect to obtain for it an indicial voltage and time-of-transmission, respectively,

$$g_c(t) = e^{-A_c},$$

and

$$\tau = \frac{B_c}{\omega} = \frac{dB_c}{d\omega} = 0. \tag{57}$$

Let us next determine the indicial voltages of the individual sections when each is closed by a resistance R. Substitute the operator p for $i\omega$ and obtain symbolically from (54) and (56)

$$e^{-\Gamma_1} = 1 - 2mn\left(\frac{p}{m+(1+m)np}\right), \tag{58}$$

and

$$e^{-\Gamma_2} = \frac{1-m}{1+m} + \frac{2mn(1-m)}{1+m}\left(\frac{p}{m+(1-m)np}\right). \tag{59}$$

Introducing these expressions in the general relation, where the network is terminated by R,

$$\frac{e^{-\Gamma}}{p} = \int_0^\infty e^{-pt}g(t)dt, \tag{60}$$

there results for the indicial voltage of the first section, since

$$\frac{1}{u+vp} = \int_0^\infty e^{-pt}\left(\frac{e^{-(ut/v)}}{v}\right)dt, \tag{61}$$

$$g_1(t) = 1 - \frac{2m}{1+m}e^{-[mt/(1+m)n]}; \tag{62}$$

and for the second section

$$g_2(t) = \frac{1-m}{1+m} + \frac{2m}{1+m}e^{-[mt/(1-m)n]}. \tag{63}$$

These functions are given in Fig. 10.

It will now be shown that, whereas the indicial voltage of each section alone is a varying function of time, that of the composite network is a constant, which represents the transient condition for no distortion with zero time-of-transmission.

For the composite network terminated by R the indicial voltage $g_c(t)$ may be derived from the usual formula for such a combination, equivalent to (5),

$$g_c(t) = g_2(0)g_1(t) + \int_0^t g_1(t-y)g_2'(y)dy. \tag{64}$$

Upon carrying through the integration we get

$$g_c(t) = \frac{1-m}{1+m} = e^{-A_c} = \text{constant}, \tag{65}$$

which agrees with the prediction from the steady state and is so shown in Fig. 10.

Obviously the two sections can be interchanged.

The composite network appears at first hand to behave in a rather remarkable manner. For if a periodic voltage is suddenly impressed at one end, the steady state will not be established within the network until after some lapse of time, whereas it occurs at the terminating resistance instantaneously. This property is, of course, to be expected from its steady-state characteristics.

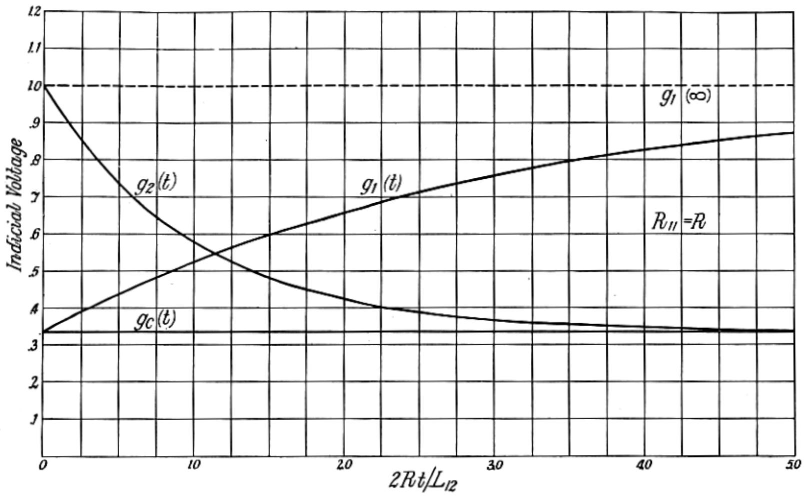


Fig. 10—Indicial voltages in distortionless network.

It may be added that such networks would still give complementary results if separated for any purpose by a symmetrical line in a circuit which is terminated at each end by a resistance R and which has an e.m.f. applied through one of the resistances. The separation of the two complementary networks under these conditions would result in the same current being received by the terminating resistance as when both networks are together at one end, where it is known the networks would produce no distortion. This follows immediately from the reciprocal theorem. For by it we readily see that the same current would be transmitted to the input terminals of the complementary receiving network whether the first network was at one end or the other. (These two cases are equivalent from the standpoint of received current to turning the combined transmission line and first network end for end.)

4.2. *Distortion Correction in Submarine Cable Circuit*

The following illustration shows the improvement which can be made in the shape of the arrival voltage at the end of a long submarine cable circuit by distortion correction at the very low frequencies only. Such an improvement would increase the speed of building up of d-c. telegraph signals and hence allow a greater speed of signaling.

The circuit assumed is a submarine cable whose length, l, is 1700 miles and whose parameters are to have the constant values per mile

$$R' = 2.74 \text{ ohms}; \quad L' = .001 \text{ h.};$$
$$G' = 0 \quad ; \quad C' = .296 \text{ mf.}$$

It is terminated at the receiving end only by a resistance $R = \sqrt{L'/C'} = 58.12$ ohms. The transfer exponent, $a + ib$, of this circuit at the terminal resistance is computed from the formula, easily derived,

$$e^{a+ib} = (k/R) \sinh \gamma l + \cosh \gamma l, \tag{66}$$

where

$$\gamma = \sqrt{(R' + iL'\omega)iC'\omega},$$

and

$$k = \sqrt{(R' + iL'\omega)/iC'\omega}.$$

These results are shown in Fig. 12.

It is desired to obtain distortion correction in this circuit from 0 to 25 cycles per second by introducing a terminal constant resistance transducer which will approximately equalize the attenuation over this range and make the resultant phase linear with frequency. Since in practice there is interference between different cables at higher frequencies, the correcting network should introduce increased attenuation above this range. Calculations gave

$$\text{at } f = 0,\ a = 4.40 \text{ napiers};$$

and

$$\text{at } f = 25\sim,\ a = 14.10 \text{ napiers}.$$

Assuming arbitrarily that the network will have at $f = 25\sim$ an attenuation of only .30 napier, the ideal total attenuation for the frequency range is

$$a' = 14.10 + .30 = 14.40 \text{ napiers}. \tag{67}$$

The attenuation of the network should decrease from a maximum value of $(14.40 - 4.40) = 10.00$ napiers at $f = 0$ to a value of .30 napier at $f = 25\sim$ and then increase with frequency. If a linear relation for the resultant phase is assumed so as to cross the b curve at about $f = 25\sim$, the phase which the network should give is negative in the range with a minimum of about -2.75 radians, and is zero at $f = 0$ and $f = 25\sim$.

A network having this desired general type of propagation constant is Network 8, Appendix IV, with the characteristic 1, but a single section will not be sufficient since its minimum phase is between 0 and $-\pi/2$ radians. The best number of sections to use is determined by the total minimum phase required and can be found here quite readily, as follows. Because of the comparatively small amount of attenuation assumed for the total correcting network at $f = 25\sim$, this type of network is one in which z_{11} consists of a resistance in parallel with an approximate reactance so that we may apply for the present purpose the relation (37) between maximum attenuation and minimum phase of such a section. For a total maximum attenuation of 10.00 napiers this relation gives for two sections a total minimum phase of -2.81 radians, which is close to the required value -2.75 radians. Three sections give -3.59 radians, showing the best number to be two. (If the result with two identical sections had been a negative phase considerably greater than the required value, it would have been possible to proportion the total maximum attenuation at zero frequency between two such different sections so as to give approximately the desired total minimum phase. In such a case each section could be designed from its corresponding proportion of the total attenuations at the other frequencies.)

Each of two such identical sections was designed by the formulæ given in Appendix IV, using attenuation data fixed by the values of $(a' - a)/2$. Allowances had to be made at $f_1 = 5\sim$ and $f_2 = 15\sim$ for necessary curvature in the attenuation characteristic so as to obtain a physical result. It was assumed that the phase constant would turn out to be satisfactory since it had already been given some consideration when determining the number of sections. The frequencies and corresponding attenuations used were

$$f_0 = 0, \qquad A_0 = 5.00 \text{ napiers};$$
$$f_1 = 5\sim, \qquad A_1 = 3.25 \text{ napiers};$$
$$f_2 = 15\sim, \qquad A_2 = 1.78 \text{ napiers};$$
$$f_3 = 25\sim, \qquad A_3 = .15 \text{ napier}.$$

The solution of the attenuation linear equations gave

$$P_2 = -68.737; \qquad Q_2 = 1.1929; \qquad Q_4 = 2.5537 \cdot 10^{-6}.$$

Whence

$$a_0 = .98661 \quad; \qquad a_1 = 1.1854 \cdot 10^{-3};$$
$$b_1 = 15.829 \cdot 10^{-3}; \qquad b_2 = 1.5980 \cdot 10^{-3}.$$

Also,

$$R_{11} = 9.42 \text{ ohms}; \qquad L_{12} = 1.994 \text{ h.};$$
$$C_{13} = 20.30 \text{ mf.}; \qquad R_{14} = 114.68 \text{ ohms};$$

where $R = 58.12$ ohms.

These results were transformed to give a ladder type network according to Section 2.5 and then incorporated in two of the dis-symmetrical unbalanced full-shunt sections, as shown in Fig. 11.

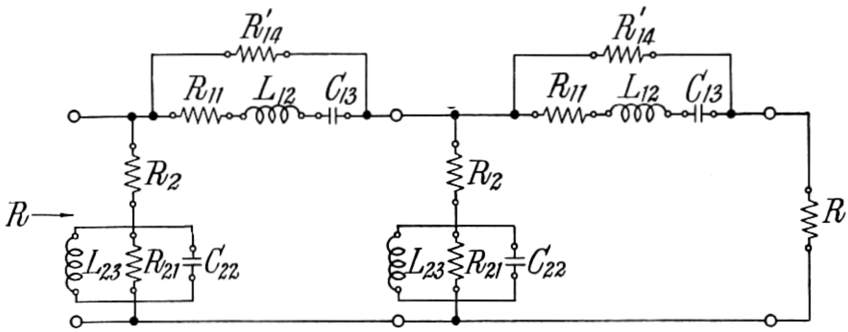


Fig. 11—Distortion correcting network for submarine cable circuit.

This transformation gives a different parallel resistance in the series branch, namely,

$$R_{14}' = 2a_0R/(1 - a_0). \tag{68}$$

Here $R_{14}' = 8565$ ohms. The elements of z_{21} in the shunt branch of the ladder type were determined from the inverse network relations

$$R_{11}R_{21} = L_{12}/C_{22} = L_{23}/C_{13} = R_{14}'R_{24}' = R^2.$$

Finally combining two resistances which are in series, $R_2 = R + R_{24}'$, we have

$$R_{21} = 359 \text{ ohms}; \qquad C_{22} = 590.3 \text{ mf.};$$

$$L_{23} = .0686 \text{ h.}; \qquad R_{24}' = .39 \text{ ohm};$$

and $$R_2 = 58.51 \text{ ohms}.$$

In Fig. 12 are shown the steady-state propagation characteristics of the uncorrected circuit, the correcting network, and the corrected circuit; the latter indicates approximately ideal conditions up to 25 cycles per second.

The improvement in shape of the arrival voltage due to this distortion correction can be seen from Fig. 13 which gives the ratio of indicial to final voltage for both the uncorrected and corrected circuit, a constant e.m.f. being impressed at the sending end at time $t = 0$. (These were computed from the steady-state characteristics of the respective circuits, using formulæ based upon those given by J. R. Carson in *B. S. T. J.*, 1924, p. 563.) The building-up speed has been increased, perhaps fourfold. The arrival voltage for the corrected circuit is within 3 per cent of its final value when that for the uncorrected circuit has reached but half value. The initial maximum in the former is similar to that in the case of a low-pass wave-filter [13] and may be

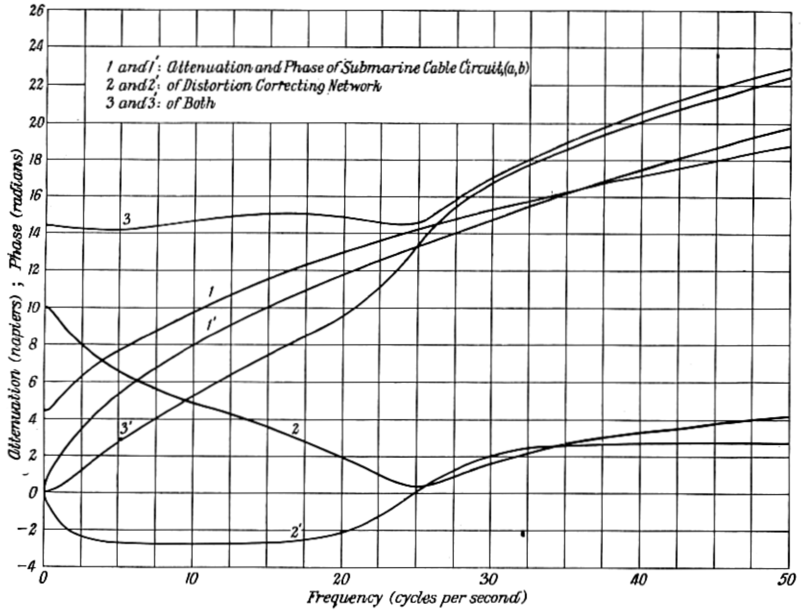


Fig. 12—Transmission characteristics of submarine cable circuit and distortion correcting network.

due to the increasing attenuation beyond the equalized range. It is probable that had but partial equalization been obtained without a

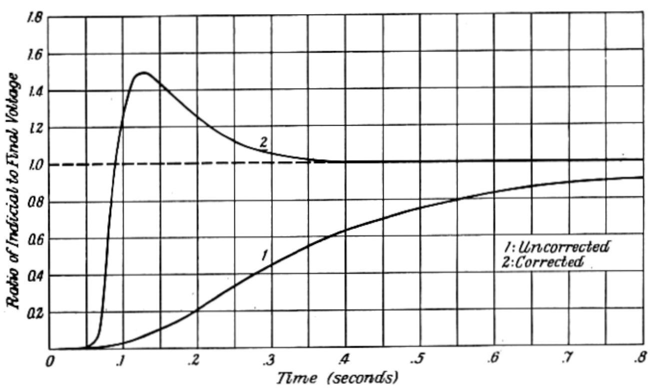


Fig. 13—Ratio of indicial to final voltage for (1) uncorrected and (2) corrected submarine cable circuit.

[13] "Transient Oscillations in Electric Wave-Filters," J. R. Carson and O. J. Zobel, *B. S. T. J.*, July, 1923.

sharp change in the attenuation, such a maximum would not have been produced. However, it is desirable to sharply attenuate the higher frequencies as has been done here, for the reason stated above. It is of interest to point out that the time-of-transmission which might be expected for the corrected circuit from the low-frequency slope with angular frequency of the steady-state phase, approximately $\tau = .076$ second, is actually the time at which the indicial voltage increases most rapidly and has reached about .4 its final value, a quite satisfactory agreement.

4.3. *Distortion Correction in Loaded-Cable Program Transmission Circuits*

Circuits which transmit programs originating at distant points to a radio broadcasting station need to be of considerably better quality over a wider frequency range than those used for ordinary telephone transmission and must be reliable under various weather conditions. Such circuits can be obtained economically with lightly loaded cable pairs which have been corrected by terminal networks for each repeater section.

The design of distortion correcting networks applicable to a 50-mile repeater section of 16-gauge H-44 cable follows. The section is terminated at *each end* by a resistance $R = 600$ ohms, the generator which impresses the voltage E having an internal impedance R. Since the received voltage would be only $.5E$ with the cable removed, in this case we are interested in the ratio

$$\frac{v}{.5E} = e^{-a-ib},$$

where a then represents the *insertion loss* in napiers.

If Γ and K are the propagation constant and iterative impedance (here used at mid-section) of the loaded cable[14] of length, l, it can be shown that the transfer exponent is

$$a + ib = \Gamma l + M_1 + M_2, \tag{69}$$

where Γl = propagation length,

$$e^{M_1} = \frac{1}{2}\left[1 + \frac{1}{2}\left(\frac{K}{R} + \frac{R}{K}\right)\right],$$

and

$$e^{M_2} = 1 - \left(\frac{K-R}{K+R}e^{-\Gamma l}\right)^2.$$

The above, of course, includes the effects of circuit terminations.

[14] Accurate computations for the propagation constant of the loaded cable were made readily by means of an improved formula for $\cosh^{-1}(x + iy)$, given in Appendix III.

It was desired to equalize the attenuation over a frequency range from zero to 4500 cycles per second and improve the time-of-phase-transmission at the lower frequencies. Computations for this 50-mile cable circuit gave values of attenuation (a in T.U.) and time-of-phase-transmission ($b/2\pi f$) as shown in Fig. 15. These circuit characteristics suggested the use of two different networks in tandem shown separately in Fig. 14, one equalizing principally at the lower frequencies, the other at the higher frequencies of the required range.

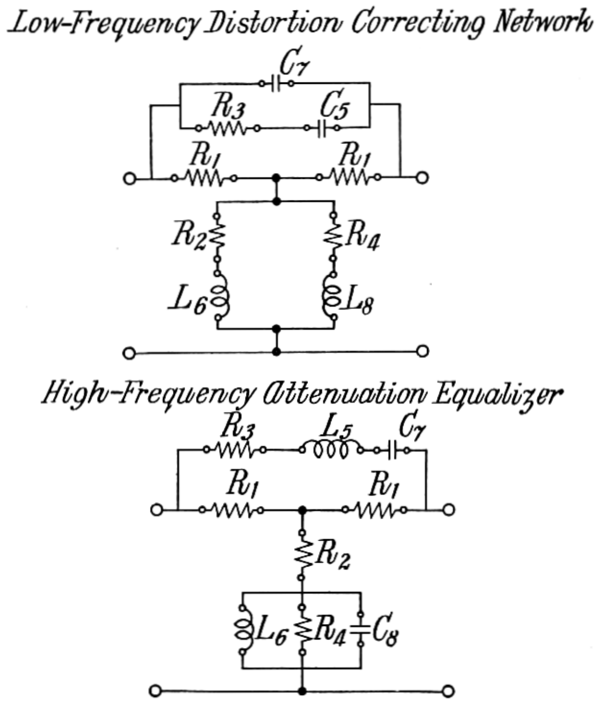


Fig. 14—Distortion correcting networks for program transmission circuit.

The low-frequency correcting network, shown as the upper section in Fig. 14, is of the symmetrical unbalanced bridged-T (Ia) type and was transformed from Network 7, Appendix IV. In the design of the latter the attenuation data corresponding to (8) were

$$f_1 = 40\sim, \quad A_1 = .536 \text{ napier};$$
$$f_2 = 200\sim, \quad A_2 = .291 \text{ napier};$$
$$f_3 = 800\sim, \quad A_3 = .176 \text{ napier};$$
$$f_4 = 2000\sim, \quad A_4 = .100 \text{ napier}.$$

Solution of the resulting four attenuation linear equations gave

$$P_0 = 102.007\cdot 10^9; \qquad P_2 = 5.06037\cdot 10^6;$$
$$Q_0 = 32.200\cdot 10^9; \qquad Q_2 = 3.43087\cdot 10^6;$$

from which

$$a_0 = .28054; \qquad a_1 = .88319\cdot 10^{-3};$$
$$b_1 = 8.6884\cdot 10^{-3}; \qquad b_2 = 4.0094\cdot 10^{-6}.$$

Then, where $R = 600$ ohms, the series elements in the lattice structures are

$$R_{11} = 248.40 \text{ ohms}; \qquad C_{12} = 2.0171 \text{ mf.};$$
$$C_{13} = .6021 \text{ mf.}; \qquad R_{14} = 336.65 \text{ ohms}.$$

Transforming from this lattice type to the equivalent bridged-T (Ia) type, we eliminate a parallel resistance in the bridged series branch (corresponding to R_{14}) by letting

$$c = 1/a_0. \tag{70}$$

Then in Fig. 14, where $c = 3.5645$,

$$R_1 = 168.3 \text{ ohms}; \qquad R_3 = 248.4 \text{ ohms};$$
$$C_5 = 2.0171 \text{ mf.}; \qquad C_7 = .6021 \text{ mf.};$$

and in the shunt branch

$$R_2 = 3037.4 \text{ ohms}; \qquad R_4 = 1458.1 \text{ ohms};$$
$$L_6 = .243 \text{ h.}; \qquad L_8 = 2.010 \text{ h.}$$

This latter useful form in which resistances are in series with inductances was obtained from the regular bridged-T (Ia) shunt elements by means of Transformation *C*, *B. S. T. J.*, January, 1923, p. 45.

The high-frequency network, shown as the lower section in Fig. 14, is well suited to extend the range of attenuation equalization above that so far considered and was derived from Network 8, Appendix IV. Allowing for both cable and low-frequency network attenuations, and arbitrarily assuming this network to have an attenuation of .300 napier at 4500 cycles per second, the data became (as from (8))

$$f_0 = 0, \qquad A_0 = .796 \text{ napier};$$
$$f_1 = 3000\sim, \qquad A_1 = .747 \text{ napier};$$
$$f_2 = 4000\sim, \qquad A_2 = .530 \text{ napier};$$
$$f_3 = 4500\sim, \qquad A_3 = .300 \text{ napier}.$$

The solution is

$$P_2 = -46.207\cdot 10^{-8}; \qquad Q_2 = -9.0092\cdot 10^{-8}; \qquad Q_4 = 23.198\cdot 10^{-16}.$$

Whence

$$a_0 = .37824; \qquad a_1 = 8.4245\cdot 10^{-6};$$

$$b_1 = 57.522\cdot 10^{-6}; \qquad b_2 = 4.8164\cdot 10^{-8}.$$

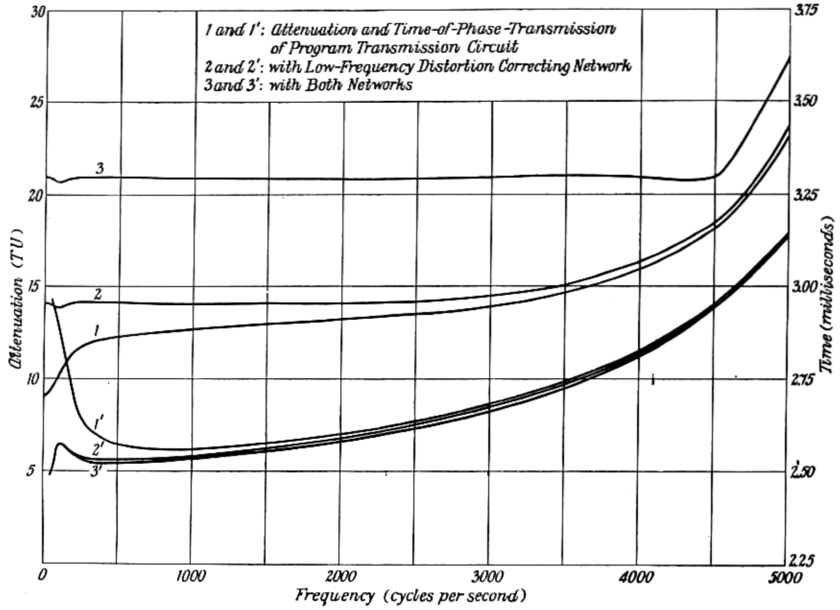


Fig. 15—Transmission characteristics of program transmission circuit with and without distortion correcting networks.

The series elements of the lattice structure are

$$R_{11} = 286.8 \text{ ohms}; \qquad L_{12} = .0987 \text{ h.};$$

$$C_{13} = .01236 \text{ mf.}; \qquad R_{14} = 453.9 \text{ ohms}.$$

Transforming to the equivalent bridged-T (Ia) type, we take c similarly as in (70); thus $c = 2.6438$. The series elements in Fig. 14 then become

$$R_1 = 226.9 \text{ ohms}; \qquad R_3 = 286.8 \text{ ohms};$$

$$L_5 = .0987 \text{ h.}; \qquad C_7 = .01236 \text{ mf.};$$

and the shunt elements

$$R_2 = 679.7 \text{ ohms}; \qquad R_4 = 1255.0 \text{ ohms};$$

$$L_6 = .00445 \text{ h.}; \qquad C_8 = .2741 \text{ mf.}$$

The effect of adding these two sections successively to the cable circuit is shown in Fig. 15. It will be seen that the first section, besides equalizing the attenuation up to about 2000 cycles per second, produces as well approximately ideal results on the time-of-phase-transmission at the lower frequencies. The complete circuit attenuation departs less than .2 T.U. from a constant value everywhere over the assumed frequency range. If desired, the time-of-phase-transmission could be improved also at the upper frequencies by the addition of proper phase networks. Such a type of correction will be made in the following application.

4.4. *Distortion Correction in Open-Wire Television Circuit*

The networks to be described here were designed by the writer especially for the particular open-wire circuit which was used for the television demonstrations from Washington, D. C., to New York City on April 7, 1927. They were designed entirely from calculated data, some of which had previously been derived from measurements on other similar lines, as the complete circuit was not available for measurements until later.

The circuit had a total length of about 285 miles, being made up principally of 276.4 miles of 165-mil open-wire pair together with 8.43 miles of necessary entrance, submarine and underground 13-gauge carrier-loaded cable (C-4.1). The iterative impedances of these two types of lines are very nearly the same in the frequency range considered and were so assumed in what follows. Hence, the propagation length of the circuit was taken as the sum of the propagation lengths of the parts. In order that such a circuit be suitable for television transmission it must be made to have extremely high quality over a very wide frequency range by means of distortion correcting networks. The requirements which the design of the present networks aimed to meet follow.

Design Requirements

1. An impedance of 600 ohms is to terminate the line at each end.
2. The attenuation, or insertion loss, of the corrected circuit is to be constant within $\pm$ 1 T.U. over the entire frequency range from 10 to 20,000 cycles per second.
3. The time-of-phase-transmission of the corrected circuit is to be constant within $\pm$ 500 microseconds (10^{-6}) from 10 to 400 cycles per second, and to be the same constant within $\pm$ 10 microseconds from 400 to 20,000 cycles per second.
4. Provision is to be made for distortion correction under various

weather conditions of the open-wire line. Details of the process of arriving at some of these requirements, also measurements and performance of the complete television circuit, have been given in a previous number of this *Journal*.[15]

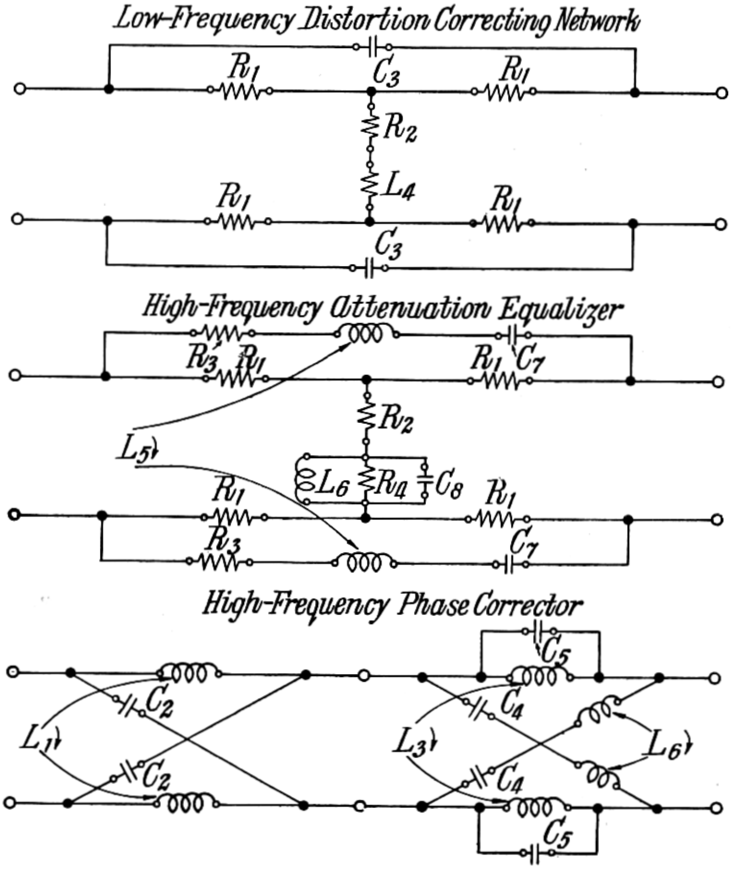


Fig. 16—Distortion correcting networks for television circuit. (Dry weather.)

Since the general circuit arrangement is here the same as in the previous problem, formula (69) is directly applicable. The transfer exponents, $a + ib$, were calculated from it for two weather conditions of the open-wire line, called dry weather and average-wet weather.

[15] "Wire Transmission System for Television," D. K. Gannett and E. I. Green, *B. S. T. J.*, October, 1927, pp. 616–632. See also "The Production and Utilization of Television Signals," Frank Gray, J. W. Horton, and R. C. Mathes, pp. 560–603. Three other papers on Television by H. E. Ives, by H. M. Stoller and E. R. Morton, and by E. L. Nelson are given in the same number of the *Journal*.

From a study of these characteristics and those of certain distortion correcting networks it appeared possible to obtain a base network design for the dry weather condition and supplementary networks for various degrees of wet weather.

The dry weather network consists of three parts in tandem, a low-frequency distortion correcting network, a high-frequency attenuation equalizer, and a high-frequency phase corrector, which were designed in the order given. The final structures are shown in Fig. 16, the first two being put in the form of balanced bridged-T (Ia) types. The low-frequency distortion correcting section corresponds to Network 1b, Appendix IV, and, while designed to approximately equalize the attenuation at low frequencies, it gave at the same time sufficient phase correction in that frequency range. The attenuation data used were

$$f_1 = 50\sim, \qquad A_1 = .409 \text{ napier};$$
$$f_2 = 500\sim, \qquad A_2 = .060 \text{ napier};$$

from which, where $R = 600$ ohms,

$$P_0 = 60{,}307; \qquad Q_0 = 25{,}217;$$
$$a_0 = .2145; \qquad b_1 = 4.945\cdot 10^{-3};$$
$$R_{11} = 257.4 \text{ ohms}; \qquad C_{12} = 3.056 \text{ mf.}$$

Transformation in the usual manner to the bridged-T (Ia) type, letting $c = 1/a_0 = 4.662$, gave the balanced structure of Fig. 16 in which

$$R_1 = 64.35 \text{ ohms}; \qquad R_2 = 1334 \text{ ohms};$$
$$C_3 = 6.112 \text{ mf.}; \qquad L_4 = 1.100 \text{ h.}$$

The high-frequency attenuation equalizer was derived from Network 8, Appendix IV, with this data, which followed formula (8) and allowed for the attenuation of the preceding network. The amount of attenuation at the highest frequency was arbitrarily assumed to be .400 napier.

$$f_0 = 0, \qquad A_0 = 2.551 \text{ napiers};$$
$$f_1 = 5{,}000\sim, \qquad A_1 = 2.100 \text{ napiers};$$
$$f_2 = 10{,}000\sim, \qquad A_2 = 1.476 \text{ napiers};$$
$$f_3 = 20{,}000\sim, \qquad A_3 = .400 \text{ napier}.$$

Solution of the linear equations gave

$$P_2 = -53.683\cdot 10^{-8}; \qquad Q_2 = 5.0669\cdot 10^{-8}; \qquad Q_4 = 3.7662\cdot 10^{-18};$$

whence

$$a_0 = .85529; \qquad a_1 = 6.1045 \cdot 10^{-6};$$
$$b_1 = 39.906 \cdot 10^{-6}; \qquad b_2 = 1.9407 \cdot 10^{-9}.$$

Then in the lattice structure

$$R_{11} = 223.8 \text{ ohms}; \qquad L_{12} = 9.668 \text{ mh.};$$
$$C_{13} = .005081 \text{ mf.}; \qquad R_{14} = 1026.3 \text{ ohms}.$$

Transformation to the bridged-T (Ia) type, using as in (70) $c = 1/a_0 = 1.1692$, gave as the elements of the balanced structure of Fig. 16

$$R_1 = 256.6 \text{ ohms}; \qquad R_2 = 94.20 \text{ ohms};$$
$$R_3 = 111.9 \text{ ohms}; \qquad R_4 = 1609 \text{ ohms};$$
$$L_5 = 9.668 \text{ mh.}; \qquad L_6 = 1.829 \text{ mh.};$$
$$C_7 = .010162 \text{ mf.}; \qquad C_8 = .02686 \text{ mf.}$$

Having equalized the dry weather attenuation over the desired frequency range from 10 to 20,000 cycles per second and improved phase conditions at low frequencies, there remained the problem of total phase correction at the higher frequencies. It was found that the high-frequency attenuation equalizer introduced phase distortion at the higher frequencies which was of the same nature but more than twice as great as that due to the original circuit itself. Letting D be the departure from linearity to the value at 20,000 cycles per second of the total phase due to the circuit and the two networks above, the departures at three important frequencies were

$$f_1 = 5{,}000\sim, \qquad D_1 = -\ .686 \text{ radian};$$
$$f_2 = 10{,}000\sim, \qquad D_2 = -\ 1.053 \text{ radians};$$
$$f_3 = 20{,}000\sim, \qquad D_3 = 0.$$

A phase characteristic which when combined with these departures can give an approximate linear resultant phase in that frequency range is that of the composite phase Network 16, Appendix IV, containing three parameters. Its phase constant B was therefore taken to satisfy at these three frequencies the relation $B + D = Cf$, or explicitly

$$B = Cf - D. \tag{71}$$

The constant C was arbitrarily chosen so that the network became physical and satisfactory results were given at intermediate frequencies also. After a number of trials the final value taken was $C = .370 \cdot 10^{-3}$.

This then gave

$$M_1 = 2.8015 \cdot 10^{-4}; \qquad M_3 = -1.3929 \cdot 10^{-12}; \qquad N_2 = -2.4671 \cdot 10^{-8}.$$

Whence

$$a_1 = 1.8846 \cdot 10^{-4}; \qquad a_1' = .9169 \cdot 10^{-4}; \qquad b_2' = .7391 \cdot 10^{-8}.$$

These gave as the elements of the high-frequency phase corrector of Fig. 16

$$L_1 = 35.99 \text{ mh.}; \qquad C_2 = .04999 \text{ mf.};$$

$$L_3 = 17.51 \text{ mh.}; \qquad C_4 = .02432 \text{ mf.};$$

$$C_5 = .02138 \text{ mf.}; \qquad L_6 = 15.40 \text{ mh.}$$

The small attenuation effects of coil dissipation were not included in this design and they were later found to be negligible. An equivalent network, namely, Network 15, Appendix IV, might have been used instead of Network 16 for this phase corrector. But since it gives less uniform or practical magnitudes for the inductances and capacities, it would not be the simplest network to construct.

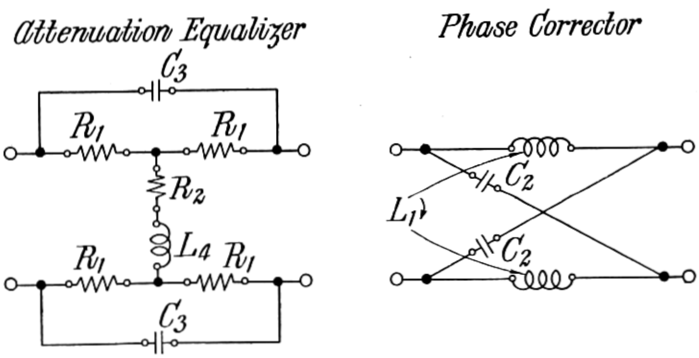


Fig. 17—Weather change distortion correcting networks for television circuit. (One step.)

To provide for wet weather effects on the open-wire part of the circuit, three identical weather change networks were designed each of which was capable of correcting one half the increase in circuit distortion caused by a change from dry weather to average-wet weather. With 0, 1, 2, or 3 of these supplementary networks added in tandem to the dry weather network, provision was thus made for a total of four assumed weather conditions which for convenience I have designated dry, semi-wet, wet, and extra-wet weather. These conditions differed by small equal steps and their number was later found to be sufficient to cover the weather range ordinarily experienced. The increase, $u + iv$, in the transfer exponent due to a change of one step, such as from dry to semi-wet, was taken as one half the difference of these exponents computed for dry and average-wet weather conditions under which the circuit constants were known. Thus

$$u + iv = \tfrac{1}{2}[(a + ib)_{\text{av.-wet}} - (a + ib)_{\text{dry}}]. \tag{72}$$

The weather change network which corrected this consists of two parts shown in Fig. 17, an attenuation equalizer and a phase corrector, the latter being required primarily because of the phase constant necessarily introduced by the former.

This attenuation equalizer has the same form as the low-frequency network for dry weather and was designed similarly from the data (according to (8))

$$f_1 = 0, \qquad A_1 = .466 \text{ napier};$$

$$f_2 = 20{,}000\sim, \qquad A_2 = .150 \text{ napier}.$$

The assumption for the network of .150 napier at 20,000 cycles per second was found to result in a satisfactory attenuation characteristic over the entire frequency range. Then

$$P_0 = 29{,}872 \cdot 10^4; \qquad Q_0 = 11{,}763 \cdot 10^4;$$

$$a_0 = .22887; \qquad b_1 = .71099 \cdot 10^{-4};$$

$$R_{11} = 274.62 \text{ ohms}; \qquad C_{12} = .04120 \text{ mf}.$$

Transforming to the bridged-T (Ia) structure, $c = 1/a_0 = 4.369$ and the elements of Fig. 17 become

$$R_1 = 68.65 \text{ ohms}; \qquad R_2 = 1242 \text{ ohms};$$

$$C_3 = .08240 \text{ mf.}; \qquad L_4 = 14.83 \text{ mh}.$$

The phase corrector was Network 13, Appendix IV, designed in a manner somewhat different from that usually employed. If D again represents the phase departure of the uncorrected phase from linearity to the value at 20,000 cycles per second, it was found that

$$\text{at } f_1 = 10{,}000\sim, \qquad D_1 = -\ .111 \text{ radian};$$

$$\text{at } f_2 = 20{,}000\sim, \qquad D_2 = 0.$$

To give a satisfactory resultant phase which is linear through f_1 and f_2 irrespective of its slope, the phase corrector only needed to have a phase constant, B_1 at f_1 and B_2 at f_2, such that

$$D_1 + B_1 = \tfrac{1}{2}B_2, \tag{73}$$

since $f_1 = \frac{1}{2}f_2$. Imposing this condition on the phase relation

$$H = \tan (B/2) = a_1 f, \tag{74}$$

there resulted

$$2 \tan^{-1} (H_2/2) - \tan^{-1} H_2 = - D_1,$$

which can be reduced to

$$\frac{H_2^3}{4 + 3H_2^2} = - \tan D_1,$$

and the cubic in H_2,

$$H_2^3 + 3 \tan D_1 H_2^2 + 4 \tan D_1 = 0. \tag{75}$$

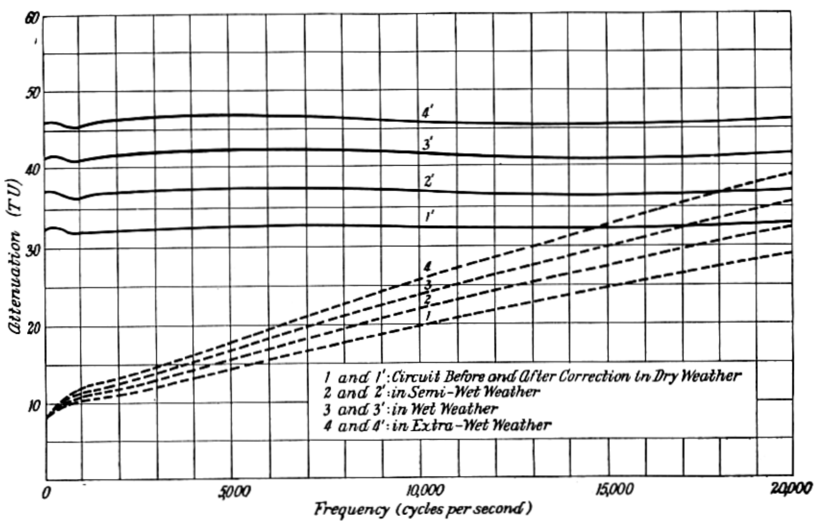


Fig. 18—Attenuation characteristics of television circuit before and after distortion correction.

The solution of the latter with $D_1 = - .111$ radian gave here

$$H_2 = .89363, \qquad \text{whence} \qquad a_1 = H_2/f_2 = .44681 \cdot 10^{-4},$$

and in Fig. 17

$$L_1 = 8.533 \text{ mh.}; \qquad C_2 = .01185 \text{ mf.}$$

For the purpose of showing the amount and precision of distortion correction produced by the addition of these various networks to the open-wire circuit under different weather conditions, attenuation and time-of-phase-transmission characteristics are given in Figs. 18 and 19, respectively. The final results indicate that the design requirements were fulfilled. (For measurements and performance of the complete line circuit see footnote 15.)

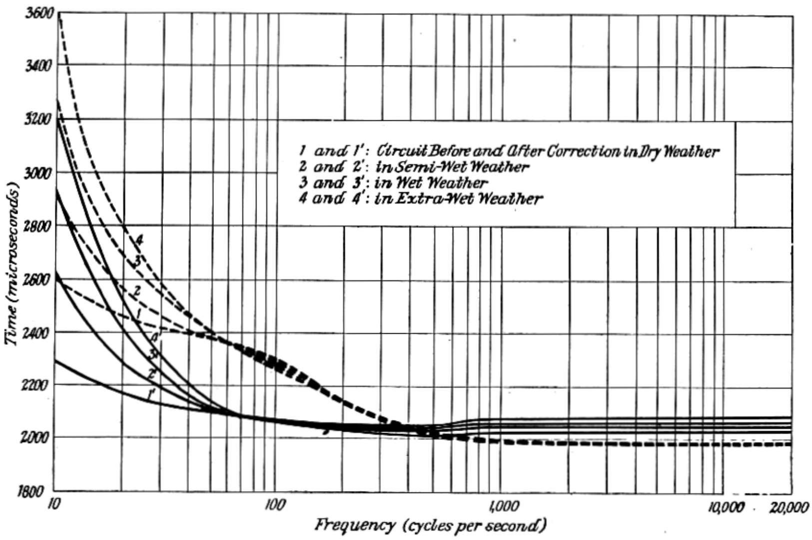


Fig. 19—Time-of-phase-transmission characteristics of television circuit before and after distortion correction.

4.5. *Equalization of Variable Attenuation in Carrier Telephone Circuits*

An open-wire circuit, such as used in a carrier system, is exposed to various weather conditions along the line and consequently experiences considerable changes in its transmission characteristics, primarily its attenuation. For satisfactory operation of carrier circuits the total circuit attenuation must ordinarily be kept reasonably constant.

One practical and advantageous method of maintaining a constant circuit attenuation which takes into account weather changes as well as length differences in the successive repeater sections is the following. Each repeater section is built out and equalized with terminal networks such that at all times the total attenuation has the same uniform value in the desired frequency range. This is done by means of two kinds of networks, *a variable artificial line* and *a base attenuation equalizer.* The variable artificial line builds out any given section to correspond to what is effectively under wet weather conditions the maximum line section used, and the base attenuation equalizer makes this total attenuation of the section uniform in the frequency range under consideration. Then the total attenuation of any line section, artificial line, and attenuation equalizer has the same constant value over the frequency range and will thus be in proper adjustment with a repeater having a fixed gain.

Such an artificial line is made up of a number of different sections whose various tandem combinations can build up by small steps a considerable length of repeater section. A mechanism for switching the various sections of artificial line in and out of a repeater section might be operated by means of regulating apparatus which is automatically controlled from circuit conditions existing on a single-frequency pilot channel or channels. In Fig. 20 is shown a type of network suitable for a section of such artificial line. It is equivalent to Network *3a*, Appendix IV, from which it can be transformed. The

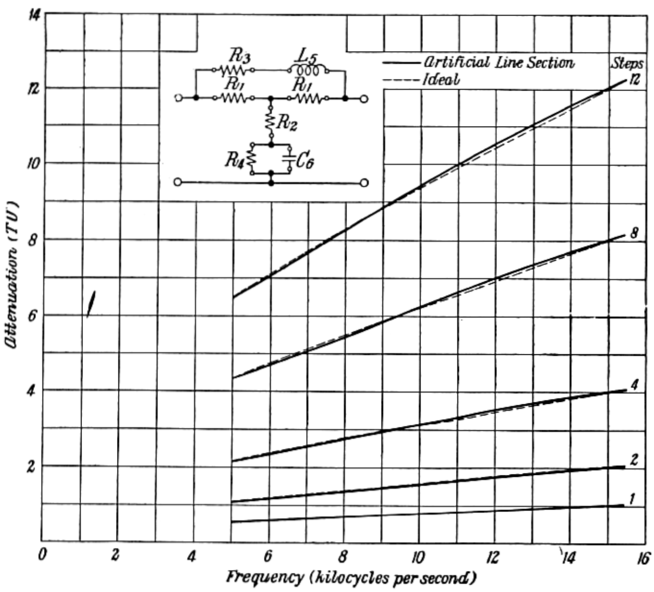


Fig. 20—Sections of variable artificial line and their attenuation characteristics for carrier telephone circuits.

following table gives the network elements for a group of such sections. I need not discuss any of the design details here but shall merely state that these sections were designed according to formulæ in Appendix IV from attenuation data which represent average requirements on the open-wire pairs used for carrier systems. The frequency range for these networks, 5.0–15.4 kilocycles per second, includes a lower group of adjacent carrier channels each having a band width of about 2500 cycles per second.

The attenuation characteristics of these individual sections are also given in Fig. 20. By properly combining them the desired maximum amount of artificial line can be obtained in equal steps, each step corresponding to approximately 1 T.U. at the highest frequency of the range.

TABLE III

ARTIFICIAL LINE CONSTANTS (Fig. 20)

(5.0–15.4 kilocycles per second)

	Steps				
	1	2	4	8	12
R_1 (ohms)......	54.2	108.2	212.9	401.4	605.0
R_2............	3348.	1637.	753.2	255.3	0
R_3............	40.2	80.5	160.9	318.4	445.3
R_4............	9108.	4544.	2275.	1150.	822.1
L_5 (mh.).......	1.62	3.24	6.57	13.83	23.13
C_6 (mf.)........	.004426	.008863	.01794	.03779	.06318

Iterative Impedance R = 605 ohms.

A structure suitable for a base attenuation equalizer is that of Fig. 21, transformed from Network 11, Appendix IV. In designing it to simulate the required attenuation characteristic shown, the procedure

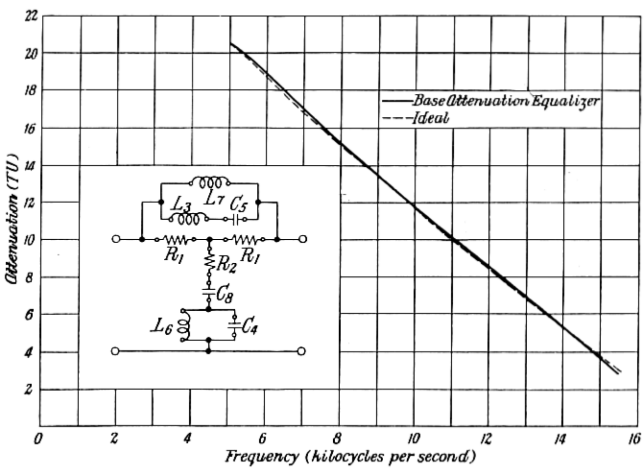


Fig. 21—Base attenuation equalizer and its attenuation characteristic for carrier telephone circuits.

was first to choose arbitrarily a plausible maximum attenuation for the network and then to use in the attenuation linear equations the three desired attenuation values at the mean and the extreme frequencies of the frequency range. At the highest frequency the attenuation was lowered slightly to allow for coil dissipation. Several such computations were made with different values of this maximum until a network was derived which gave a satisfactory result at all frequencies within the range. The magnitudes of the elements corresponding to the partial attenuation characteristic shown in Fig. 21, where $R = 605$ ohms, are

$$R_1 = 508.4 \text{ ohms}; \qquad R_2 = 105.8 \text{ ohms};$$
$$L_3 = 12.69 \text{ mh.}; \qquad C_4 = .03469 \text{ mf.};$$
$$C_5 = .005852 \text{ mf.}; \qquad L_6 = 2.14 \text{ mh.};$$
$$L_7 = 238.8 \text{ mh.}; \qquad C_8 = .6525 \text{ mf.}$$

The departures of the attenuation from the desired values are less than .2 T.U. At the highest frequencies small coil dissipation tends to improve this result.

4.6. *Phase Correction in Transatlantic Telephone System*

At the receiving stations of the transatlantic telephone system it is necessary to use phase correctors in connection with the antenna arrays. These networks serve in two capacities, either (*a*) as artificial lines or delay networks which build out the phase characteristics of short transmission lines until they are equivalent to certain longer lines used elsewhere in the system, or (*b*) as phase correctors which secure adjustable and arbitrary phase characteristics when combining the outputs of the antennæ which form the array. For satisfactory operation the phase correctors had to meet these design requirements.

1. A constant iterative impedance of $R = 600$ ohms.
2. A continuously variable phase change which is proportional to frequency over the frequency range from 50 to 65 kilocycles per second, the total phase change being from 0 to 250 degrees at 50 kilocycles per second.
3. Over any frequency band of 5 kilocycles per second in the range the variations should be less than .100 degree for the phase and less than .025 T.U. for the attenuation.
4. A balanced structure.

In making the design it was found that the continuously variable phase change to the desired maximum could be provided by means of one variable section having a small phase constant and five fixed sections of Networks 13, 14, and 16, Appendix IV. Designated in terms of their phase constants at 50 kilocycles per second as in Fig. 22, the variable section has a range of 0–15 degrees, while the fixed sections have phase constants of 10, 20, 40, 80, and 160 degrees, respectively. The variable section is normally required to give a maximum of only 10 degrees but an extension of its range to 15 degrees is provided so as to ensure phase overlapping at any transition point where a section is put in or taken out of the circuit. By properly combining these sections it is seen that a continuous range from 0 to 325 degrees is obtainable.

The sections were designed from the formulæ of Appendix IV so as to give the desired individual linear phase characteristics shown in Fig. 22. It need only be stated that the data taken from the phase

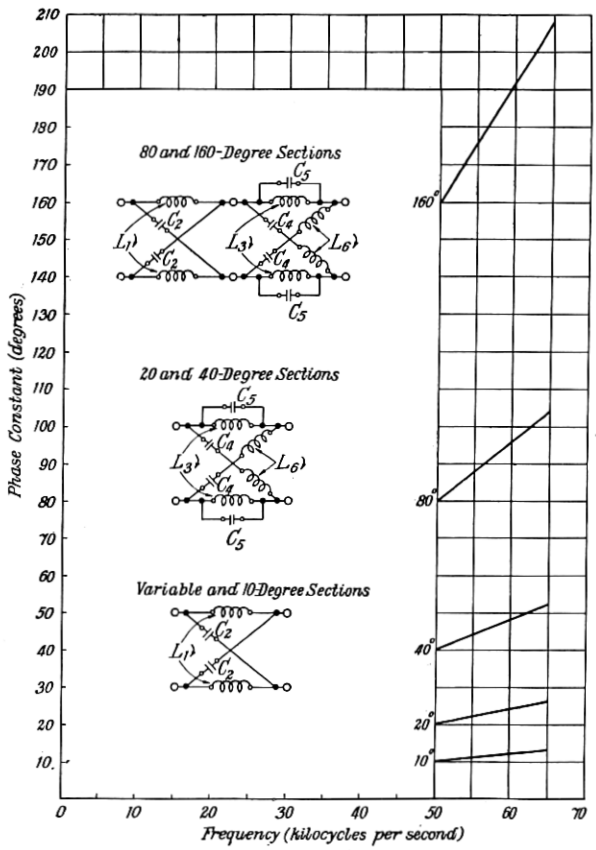


Fig. 22—Sections of variable phase corrector and their phase characteristics used in the transatlantic telephone system.

characteristics in the one-parameter sections were those at 50, in the two-parameter sections those at 50 and 65, and in the three-parameter composite sections those at 50, 57.5, and 65 kilocycles per second. The elements for the *variable section* in Fig. 22 are continuously variable and have their magnitudes given in terms of the variable phase constant B at 50 kilocycles per second as

$$L_1 = \frac{\tan (B/2)R}{50\pi} \text{mh.}; \qquad C_2 = \frac{10 \tan (B/2)}{\pi R} \text{mf.}$$

The results for the fixed sections follow in Table IV.

TABLE IV

PHASE CORRECTOR CONSTANTS

(Fig. 22)

	Fixed Sections (Degrees at 50 kilocycles per second)				
	10	20	40	80	160
L_1 (mh.)	.334			1.211	2.941
C_2 (10^{-3} mf.)	.464			1.682	4.084
L_3 (mh.)		.667	1.333	1.456	2.466
C_4 (10^{-3} mf.)		.926	1.851	2.023	3.425
C_5 (10^{-3} mf.)		.309	.631	1.051	2.408
L_6 (mh.)		.223	.454	.756	1.734

In any one of these sections the computed departures of the phase constant from ideal proportionality to frequency in the frequency range 50 to 65 kilocycles per second was usually much less than .02 degree. The practical construction of the networks gave similar high precision, and by using coils of small dissipation constant, d = (resistance/reactance), the attenuation requirements were likewise satisfied. The frequency band now in use is from 58.5 to 61.5 kilocycles per second.

It may be added that these designs can readily be altered so as to apply to other frequency ranges. In order to translate the phase constants from the 50-kilocycle designation to any other frequency range with a minimum frequency, f_0, designation, multiply all inductances and capacities by the translation factor $(50{,}000/f_0)$.[16]

4.7. *Simulation of Smooth Line*

This application is based upon and illustrates the general results of Part 3 which discusses recurrent networks having arbitrary iterative impedances. A network design will be given which has the following characteristics.

1. A propagation constant which simulates a moderate propagation length of any smooth line, or its equivalent.

[16] For a discussion of other applications of constant resistance networks see footnote 10; also "Transmission Circuits for Telephonic Communication," K. S. Johnson.

2. An iterative impedance which equals that of the smooth line at all frequencies.

Such a network could have a number of uses. For example, it could serve as a substitute for a small length of smooth line where approximately exact simulation is required as in certain laboratory tests, or as part of an artificial line in a balancing network. Leakance changes can be provided for by means of particular adjustable resistances. The design can represent the special case of a distortionless line at the lower frequencies and, if non-dissipative, give a phase network having a constant time-of-phase-transmission in this frequency range.

The method of solution differs considerably from those previously used for the other networks and so will be given here. To begin with let

$$\left.\begin{aligned} z_a &= \text{series} \\ z_b &= \text{shunt} \end{aligned}\right\} \begin{array}{l}\text{impedance of any section of smooth line, or its equivalent,} \\ \text{of propagation constant } \gamma, \text{ iterative impedance } k, \text{ and} \\ \text{length } l.\end{array}$$

Also let

$$\left.\begin{aligned} X &= \text{open-circuit} \\ Y &= \text{short-circuit} \end{aligned}\right\} \text{impedance of the smooth line section.}$$

Then

$$\gamma l = \sqrt{z_a/z_b} = \tanh^{-1} \sqrt{Y/X},$$

and

$$k = \sqrt{z_a z_b} = \sqrt{XY}. \tag{76}$$

(*B. S. T. J.*, October, 1924, p. 617.)

From these

$$z_a = k\gamma l = \sqrt{XY} \tanh^{-1} \sqrt{Y/X},$$

and

$$z_b = k/\gamma l = \sqrt{XY}/\tanh^{-1} \sqrt{Y/X}; \tag{77}$$

thus z_a and z_b are inverse networks of impedance product k^2. In a physical smooth line z_a is simulated by series resistance and inductance and z_b by parallel resistance and capacity (assuming the line constants to be independent of frequency), both represented by simple physical networks. In other cases they may be realized in desired frequency ranges, more or less approximately, by physical networks. It will be assumed in what follows that z_a and z_b are given by the above formulæ.

The structure which is to simulate the smooth line is shown in its general form as Network 18, Appendix IV, wherein z_a and z_b are considered as two types of physical elements whose combinations in different proportions make up the network. It consists of a composite lattice network of two sections having four real, positive parameters, m_1, m_2, m_1', and m_2', two in each section.

In the first section put for the series impedance

$$z_{11} = \frac{1}{\dfrac{1}{2m_1 z_a} + \dfrac{1}{2z_b/m_2}}. \tag{78}$$

To satisfy the condition for the desired iterative impedance at all frequencies,

$$K = \sqrt{z_{11}z_{21}} = k = \sqrt{z_a z_b}, \tag{79}$$

it follows that the lattice impedance must be

$$z_{21} = \frac{m_2 z_a}{2} + \frac{z_b}{2m_1}. \tag{80}$$

That is, z_{11} and z_{21} are also inverse networks of impedance product k^2. The propagation constant, by generalized (13) (that is, R replaced by K), is

$$e^{\Gamma} = \frac{1 + m_1 y + m_1 m_2 y^2}{1 - m_1 y + m_1 m_2 y^2}, \tag{81}$$

where for convenience $y = \sqrt{z_a/z_b} = \gamma l =$ propagation length.

In the second section, similarly,

$$z_{11}' = \frac{1}{\dfrac{1}{2m_1' z_a} + \dfrac{1}{2z_b/m_2'}},$$

$$z_{21}' = \frac{m_2' z_a}{2} + \frac{z_b}{2m_1'}, \tag{82}$$

and

$$e^{\Gamma'} = \frac{1 + m_1' y + m_1' m_2' y^2}{1 - m_1' y + m_1' m_2' y^2}.$$

For the composite structure made up of these two sections in tandem, the iterative impedance condition is already fulfilled independently of the values of the coefficients, since (79) holds for each section. Its propagation constant is given from (81) and (82) by

$$\begin{aligned} e^{\Gamma_c} &= e^{\Gamma + \Gamma'} \\ &= \frac{\begin{aligned}1 + (m_1 + m_1')y + (m_1 m_2 + m_1 m_1' + m_1' m_2')y^2 \\ + (m_1 m_2 m_1' + m_1 m_1' m_2')y^3 + m_1 m_2 m_1' m_2' y^4\end{aligned}}{\begin{aligned}1 - (m_1 + m_1')y + (m_1 m_2 + m_1 m_1' + m_1' m_2')y^2 \\ - (m_1 m_2 m_1' + m_1 m_1' m_2')y^3 + m_1 m_2 m_1' m_2' y^4\end{aligned}}. \end{aligned} \tag{83}$$

It remains to choose the coefficients m_1, m_2, m_1', and m_2' so that for moderate propagation lengths, $y = \gamma l$, the composite network will give

$$\Gamma_c \text{ approximately } = y = \gamma l. \tag{84}$$

At this point let us introduce an important simplification by writing the function

$$e^y = \frac{e^{y/2}}{e^{-y/2}} = \frac{1 + \frac{1}{2}y + \frac{1}{8}y^2 + \frac{1}{48}y^3 + \frac{1}{384}y^4 + \frac{1}{3840}y^5 + \cdots}{1 - \frac{1}{2}y + \frac{1}{8}y^2 - \frac{1}{48}y^3 + \frac{1}{384}y^4 - \frac{1}{3840}y^5 + \cdots}. \tag{85}$$

Then upon comparing (83) and (85) we see that fortunately for small values of y we can satisfy (84) providing we identify the coefficients of powers of y in (83) as

$$\begin{aligned} m_1 + m_1' &= \frac{1}{2}, \\ m_1 m_2 + m_1 m_1' + m_1' m_2' &= \frac{1}{8}, \\ m_1 m_2 m_1' + m_1 m_1' m_2' &= \frac{1}{48}, \end{aligned} \tag{86}$$

and

$$m_1 m_2 m_1' m_2' = \frac{1}{384}.$$

The solution of the equations gives a sixth degree equation for m_1, namely,

$$m_1^6 - \frac{3}{2}m_1^5 + m_1^4 - \frac{3}{8}m_1^3 + \frac{5}{64}m_1^2 - \frac{1}{128}m_1 + \frac{1}{4608} = 0;$$

and for the others

$$m_2 = \frac{6m_1 - 48m_1^2 m_1' - 1}{48m_1(m_1 - m_1')},$$

$$m_1' = .5 - m_1,$$

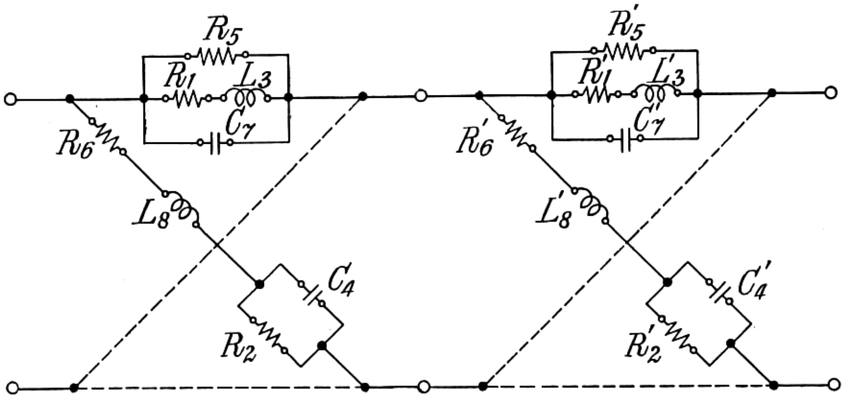
and

$$m_2' = \frac{1 + 48m_1(m_1')^2 - 6m_1'}{48m_1'(m_1 - m_1')}.$$

From these we get this set of real positive coefficients, determined once for all, namely,

$$\begin{aligned} m_1 &= .45737; & m_2 &= .14456; \\ m_1' &= .04263; & m_2' &= .92403. \end{aligned} \tag{87}$$

With the above fixed values of the coefficients and formulæ (77), (78), (80), and (82), the network can be constructed which is to simulate any smooth line having physically realizable z_a and z_b. This simula-



(Broken lines indicate the other series and lattice branches, respectively identical.)

$R_1 = m_1R'l,$	$R_2 = 1/m_1G'l,$	$R_1' = m_1'R'l,$	$R_2' = 1/m_1'G'l,$
$L_3 = m_1L'l,$	$C_4 = m_1C'l,$	$L_3' = m_1'L'l,$	$C_4' = m_1'C'l,$
$R_5 = 1/m_2G'l,$	$R_6 = m_2R'l$	$R_5' = 1/m_2'G'l,$	$R_6' = m_2'R'l,$
$C_7 = m_2C'l,$	$L_8 = m_2L'l,$	$C_7' = m_2'C'l,$	$L_8' = m_2'L'l,$
$m_1 = .45737,$	$m_2 = .14456,$	$m_1' = .04263,$	$m_2' = .92403.$

Fig. 23—Artificial smooth line which simulates a moderate length, l, of line having the primary constants R', L', G', and C' per unit length. (If $R' = G' = 0$, it becomes a non-dissipative phase network whose time-of-phase-transmission at the lower frequencies has the constant value, $\tau_p = \sqrt{L'C'}l$.)

tion is very accurate for small values of y. As y increases, the departure of the network propagation characteristic from the smooth line values also increases, but it amounts to less than 1.4 per cent even at $|y| = 3.0$, as may be derived from a comparison of (83) and (85).

As an illustration of this type of design, these results were analytically applied to the case of a 104-mil open-wire smooth line having the constants per loop mile (for wet weather, and assumed independent of frequency),

$$R' = 10.12 \text{ ohms}; \qquad L' = 3.66 \text{ mh.};$$
$$G' = 3.20 \text{ micromhos}; \qquad C' = .00837 \text{ mf.}$$

The corresponding simulating network for a length l is shown structurally in Fig. 23, where

$$z_a = (R' + iL'\omega)l,$$

and (88)

$$z_b = 1/(G' + iC'\omega)l.$$

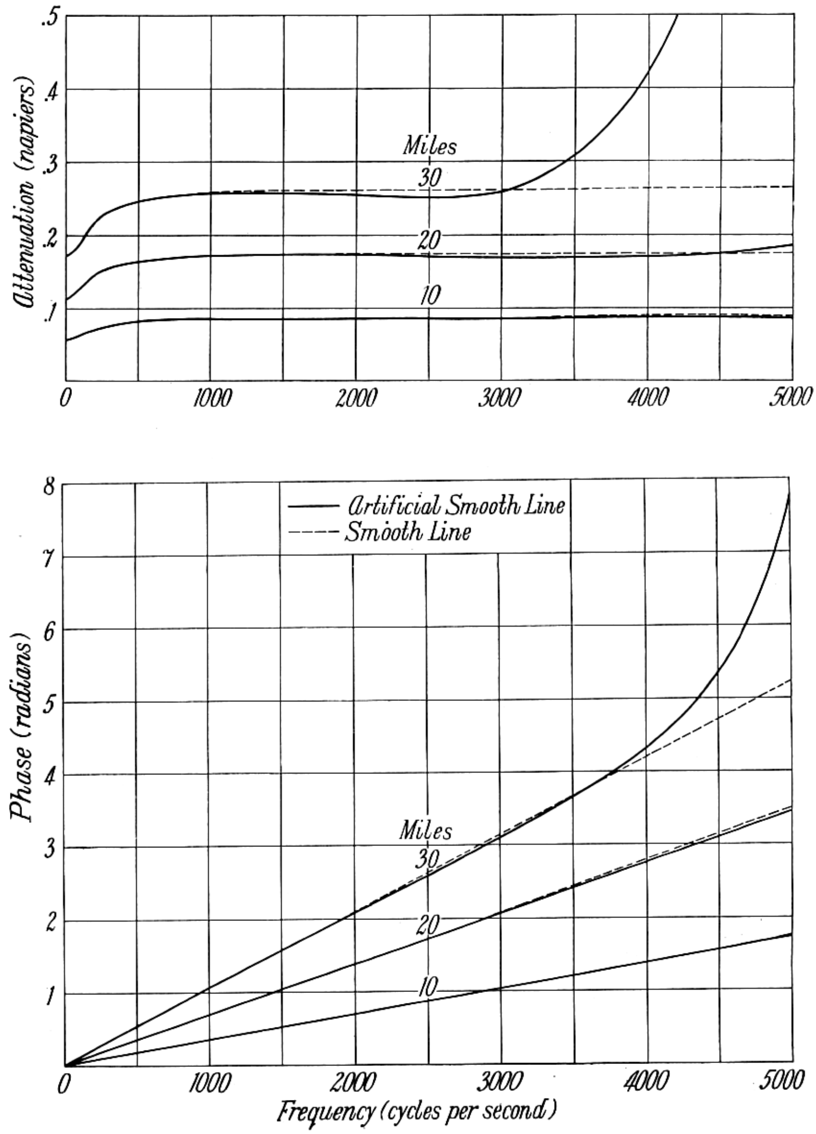


Fig. 24—Propagation characteristics of 10-, 20- and 30-mile lengths of 104-mil open-wire smooth line and of the simulating artificial smooth lines. (R' = 10.12 ohms, L' = 3.66 mh., G' = 3.20 micromhos (wet weather), and C' = .00837 mf. per loop mile.)

A comparison of the propagation characteristic of a line section and that of its simulating network is shown in Fig. 24 for three different line lengths, l = 10, 20, and 30 miles. Even in the longest section the simulation is good up to 3000 cycles per second. The iterative impedances are, of course, identical as in Fig. 25.

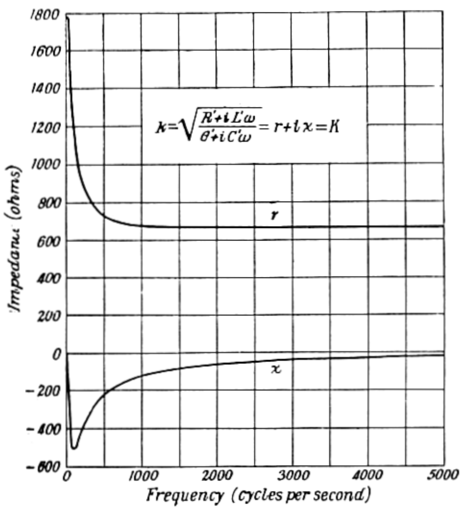


Fig. 25—Iterative impedance of 104-mil open-wire smooth line and of the simulating artificial smooth lines.

While the above general design considered four parameters, a similar procedure can be followed with other networks having a smaller or greater number of parameters. The structure can be obtained by building the series impedance of any section out of various combinations of the impedance elements z_a and z_b. However, several of the above four-parameter composite sections can perhaps meet most design requirements.

Appendix I

Discussion of Linear Phase Intercept

Let us first consider steady-state transmission over a circuit where the impressed e.m.f., consisting of simple sinusoids of any two angular frequencies ω_1 and ω_2, is given by

$$\begin{aligned} E(t) &= \sin \omega_1 t + \sin \omega_2 t, \\ &= 2 \cos \tfrac{1}{2}(\omega_1 - \omega_2)t \sin \tfrac{1}{2}(\omega_1 + \omega_2)t. \end{aligned} \tag{89}$$

Assume that the circuit has at these frequencies the transfer exponents $a_1 + ib_1$ and $a_2 + ib_2$ such that $a_1 = a_2 = a'$. A straight line drawn through b_1 and b_2 in the ω, b plane will have a slope τ, say, and at $\omega = 0$ a linear phase intercept b_0 which may have any value. Hence, the transfer exponent may be expressed as a function of frequency at these two frequencies by the relations

$$a = a' = \text{constant},$$

and

$$b = \tau\omega + b_0. \tag{90}$$

The received voltage across R will then be a periodic function which is attenuated by an amount a' napiers and is

$$\begin{aligned} v(t) &= e^{-a'}\Big[\sin\big(\omega_1(t-\tau) - b_0\big) + \sin\big(\omega_2(t-\tau) - b_0\big)\Big], \\ &= 2e^{-a'}\cos\tfrac{1}{2}(\omega_2 - \omega_1)(t-\tau)\sin\big(\tfrac{1}{2}(\omega_1+\omega_2)(t-\tau) - b_0\big). \end{aligned} \tag{91}$$

How the transmitting property of this circuit for the two frequencies depends upon the phase intercept can be seen from a comparison of (91) with (89). In order that the received voltage may be a time-function of identically the same shape as the impressed voltage, but with a time-of-transmission over the circuit of τ seconds, it is necessary that $b_0 = 2n\pi$ radians, where n is any positive or negative integer. This would mean no distortion of the impressed steady-state signal made up of the two frequency components. If $b_0 = (2n \pm 1)\pi$, there would be an apparent distortion only of a reversal in sign. However, if $b_0 = (2n \pm \frac{1}{2})\pi$, there would be maximum distortion in the transmitted voltage. These conclusions may be tabulated briefly as follows:

If $b_0 = 2n\pi$, no distortion;

If $b_0 = (2n \pm 1)\pi$, apparent distortion of sign reversal;

If $b_0 = (2n \pm \frac{1}{2})\pi$, maximum distortion.

The above discussion considered the case of any two frequencies. If now we assume that the circuit has the characteristics (90) for several or a range of frequencies, then the conclusions above obviously apply as well to the steady-state transmission of an impressed e.m.f. which is made up of any of those frequencies. Thus, *for distortionless steady-state transmission (without change of signal shape), the transfer exponent must have for the frequency components impressed not only a uniform attenuation and a linear phase relation, but also a proper linear phase intercept* $b_0 = 2n\pi$. If, in a physical system, (90) is satisfied over a frequency range which includes zero frequency, then τ would necessarily be positive and $b_0 = 0$ or a multiple of 2π.

Proceeding next to the transmission of an e.m.f. impressed suddenly at time $t = 0$, we note that since the e.m.f. can be expressed in terms of a Fourier integral representation from $t = -\infty$ to $t = +\infty$ we may regard it as made up of a distribution of steady-state components. For example, let the e.m.f. of (89) be impressed on a circuit at time $t = 0$. Then

$$\begin{aligned} E(t) &= \left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin ty}{y}dy\right)(\sin \omega_1 t + \sin \omega_2 t), \\ &= \tfrac{1}{2}(\sin \omega_1 t + \sin \omega_2 t) + \frac{1}{\pi}\int_0^\infty \left(\frac{\omega_1}{\omega_1^2 - \omega^2} + \frac{\omega_2}{\omega_2^2 - \omega^2}\right)\cos t\omega d\omega. \end{aligned} \tag{92}$$

This represents the impressed voltage for negative as well as positive values of t since in the first equation the factor of the sinusoids represents a function which is zero for all negative and unity for all positive values of t. We may then interpret the last equation as giving for all values of time the frequency distribution of steady-state components of all frequencies which give the same result as the sinusoids of (89) impressed suddenly at $t = 0$. This distribution extends over the entire frequency range and has the largest amplitudes about $\omega = \omega_1$ and $\omega = \omega_2$.

Hence, if the initial part of the impressed e.m.f., as well as the final steady state, is to be transmitted without distortion, the circuit transfer voltage must have a characteristic which is distortionless not only with respect to ω_1 and ω_2 but also to all angular frequencies about them as obtained from the analysis. That is, since the steady state is only the limiting case of the transient state, an ideal circuit characteristic for its distortionless transmission is only a part of and is included in that for the transient state. Or, vice versa, ideal circuit characteristics for the steady state are at least the same as for the transient state.

These results are useful in studying a circuit whose attenuation is constant and whose phase characteristic is approximately linear over an internal frequency band. An extrapolation of this linear phase characteristic to zero frequency may give a phase intercept which is not ideal for preservation of wave-shape even in the steady state of frequencies within the band, as we have seen. Increasing the frequency range over which an ideal phase relation holds obviously improves the transmission of transient voltages. Practically, good results are obtained in a circuit wherein the attenuation is approximately constant and the phase is approximately proportional to frequency over the required internal band of frequencies; then the phase intercept, b_0, is zero and the time-of-phase-transmission, $\tau_p = b/\omega$, is approximately constant and represents the time-of-transmission of the circuit for those frequencies.

Appendix II

Proofs of Linear Transducer Theorems

Theorem I: Any passive network whose attenuation constant is zero at all frequencies is a limiting case of a physical wave-filter wherein the transmitting band extends over the entire frequency range. The proof that the phase constant increases with frequency in the transmitting band of any wave-filter has already been given by the writer in the paper, "Theory and Design of Uniform and Composite Electric Wave-Filters," *B. S. T. J.*, January, 1923, pages 37–38. In the present case, therefore, the phase constant increases throughout the frequency range.

The proof relating to the iterative impedance will be given in two steps which comprise essentially the proofs of two impedance theorems. From the first of these it will follow immediately that the transducer under consideration has everywhere a real iterative impedance because of symmetry and a transmitting band extending over the entire frequency range; from the second, this real iterative impedance is a constant resistance throughout the frequency range.

Wave-Filter Impedance Theorem: In all transmitting bands the iterative impedances of a recurrent section of any electric wave-filter are conjugate impedances. If the section is symmetrical, they are equal and real without a reactance component.

From the general formulæ on page 617 of *B. S. T. J.*, October, 1924, we may write the iterative impedances as:

$$\left.\begin{matrix} K_a \\ K_b \end{matrix}\right\} = \tfrac{1}{2}((X_a + X_b) \tanh \Gamma \pm (X_a - X_b)), \qquad (93)$$

where X_a and X_b are the open-circuit driving-point impedances at the ends a and b of the transducer. In a wave-filter recurrent section which is made up of non-dissipative reactance elements the impedances X_a and X_b have only reactance components. Also, in a transmitting band the attenuation constant is zero, so that here $\Gamma = iB$ and $\tanh \Gamma = i \tan B$. From this, it follows readily that in any transmitting band the first term of the right-hand member of (93) represents a positive resistance component and the second term a reactance component. Hence, the resistance components of K_a and K_b are identical while their reactance components differ only in sign; that is, K_a and K_b are conjugate impedances in all transmitting bands.

As results of the above we may state parenthetically:

Corollary I: The absolute values of the iterative impedances of a wave-filter recurrent section are equal at any frequency in all transmitting bands; and

Corollary II: The iterative impedances of a wave-filter recurrent section are such as to give maximum energy transfer from section to section in all transmitting bands.

When the section is symmetrical, $X_a = X_b$, and therefore $K_a = K_b = r$, a resistance in those frequency ranges.

Non-Reactive Impedance Theorem: The impedance of any two-terminal network whose reactance component is zero at all frequencies must have a resistance component which is constant, independent of frequency. To prove the theorem, let the impedance of any two-terminal network whose reactance component is zero at all frequencies be represented as:

$$Z = r, \tag{94}$$

where r is a real function of frequency.

The general relations between the components of the steady-state admittance, $\alpha(\omega) + i\beta(\omega)$, of a network and the corresponding indicial admittance, $h(t)$, are known from electric circuit theory to be:

$$\alpha(\omega) = h(o) + \int_0^\infty \cos \omega y h'(y) dy$$

and

$$\beta(\omega) = -\int_0^\infty \sin \omega y h'(y) dy; \tag{95}$$

also

$$h(t) = \alpha(o) + \frac{2}{\pi}\int_0^\infty \frac{\beta(\omega)}{\omega} \cos t\omega d\omega, \quad t > 0.$$

(See pages 18 and 180 of the reference in footnote 5.)

In the passive network under discussion here, the admittance components at all frequencies from (94) are

$$\alpha(\omega) = 1/r,$$

and

$$\beta(\omega) = 0. \tag{96}$$

Upon substituting them in (95) it is found that

$$h(t) = \alpha(o) = \text{a constant}, \quad t > 0,$$

$$h'(t) = 0 \tag{97}$$

and

$$\alpha(\omega) = 1/r = h(o) = \text{a constant}.$$

This relation demands that the resistance component r be constant, independent of frequency, as stated in the theorem.

The converse of the above theorem does not follow, that is, if the resistance component of a two-terminal network impedance is constant, independent of frequency, it is not necessary that the reactance component be zero throughout the frequency range. This may be seen from the relations above. A simple example is series resistance and inductance.

Theorem II: If the iterative impedance of a network is real at all frequencies, it must be constant according to the latter impedance theorem above.

For the second part of Theorem II we have as assumptions regarding the propagation constant, $\Gamma = A + iB$, and iterative impedance, K, effectively

$$B = \tau\omega$$

and (98)

$$K = \text{a constant} = R,$$

where τ is some positive constant. The transfer admittance components with respect to a resistance R which terminates the transducer are then

$$\alpha(\omega) = \frac{e^{-A}}{R} \cos \tau\omega$$

and (99)

$$\beta(\omega) = -\frac{e^{-A}}{R} \sin \tau\omega.$$

By means of these and (95) we shall prove that A is uniform at all frequencies.

To satisfy (95) with (99) at all frequencies the transducer must be such as to give the relations

$$h(o) = o,$$

$$h'(t) = o, \quad t \neq \tau,$$

and (100)

$$\int_{\tau-}^{\tau+} h'(y)dy = \frac{e^{-A}}{R}.$$

Since the left-hand member of the last relation is independent of frequency, it follows necessarily that the attenuation constant, A, must be uniform. That uniform attenuation together with (98) is also sufficient to satisfy the other relations of (100) can be seen if the parameter characteristics at all frequencies are

$$A = \text{a constant},$$

$$B = \tau\omega \qquad (101)$$

and

$$K = \text{a constant} = R.$$

From electric circuit theory, the fundamental integral equation for the indicial admittance $h(t)$ becomes

$$\frac{1}{pZ(p)} = \frac{e^{-A-\tau p}}{pR} = \int_0^\infty e^{-py}h(y)dy, \tag{102}$$

where p replaces $i\omega$. Its solution is

$$h(t) = o, \quad t < \tau$$

and

$$h(t) = \frac{e^{-A}}{R} = \text{a constant}, \quad t > \tau; \tag{103}$$

whence also $h'(t) = o$ for $t \neq \tau$, thus satisfying (100). These results hold as well for the limiting case of $B = o$, meaning $\tau = o$.

It may be pointed out here also that *the converse of the latter theorem does not follow.* That is, if the transducer has a uniform attenuation constant and a constant resistance iterative impedance, it is not necessary that the phase constant be proportional to frequency throughout the range. This is seen from the general equations or from the fact that we can alter the phase characteristic non-linearly by means of phase networks having zero attenuation and a constant resistance iterative impedance.

Theorem III: A symmetrical transducer made up entirely of resistances would have the characteristics

$$A = \text{a constant},$$

$$B = o \tag{104}$$

and

$$K = \text{a constant} = R.$$

Many other more complicated networks satisfying (104) are known to exist, as in Section 4.1. We need not, therefore, seek further to prove the possible existence of such a combination of parameters.

For networks in which B is not zero, but

$$A = \text{a constant}$$

and

$$K = \text{a constant} = R, \tag{105}$$

the transfer admittance components with respect to a terminating resistance R are given as

$$\alpha(\omega) = \frac{e^{-A}}{R}\cos B$$

and

$$\beta(\omega) = -\frac{e^{-A}}{R}\sin B. \tag{106}$$

Using these and the general relations (95), we can obtain

$$\frac{dB}{d\omega} = \frac{\int_0^\infty y \sin \omega y h'(y) dy}{\int_0^\infty \sin \omega y h'(y) dy}, \tag{107}$$

which is independent of A.

Since (if B is not everywhere zero) $dB/d\omega$ is positive when $A = o$ according to Theorem I, and since by (107) it is independent of A (a constant), it will be positive whatever the value of A. Hence, B increases with frequency in such transducers.

Appendix III

Propagation Constant and Iterative Impedance Formulæ for General Ladder, Lattice and Bridged-T Types

These formulæ apply to the general types of structures shown in Fig. 2 and should be used whenever it is desired to take into account accurately the actual physical impedances. Network designs which follow the methods given in this paper are made under the assumption of invariable lumped elements. In constructing physical networks according to such designs, however, certain departures from this assumption unavoidably make their appearance and must be taken into consideration whenever extreme accuracy is required. The departures include dissipation in coils and condensers, distributed capacity in coils, as well as inaccuracies due to manufacture.

Some of these formulæ have been given in previous papers but all can be derived readily either by the method given in *B. S. T. J.*, January, 1923, p. 34, or by that in *B. S. T. J.*, October, 1924, p. 617.

Ladder Type:

$$\cosh \Gamma = 1 + \tfrac{1}{2}\frac{z_1}{z_2}\cdot \tag{108}$$

The iterative impedances at different terminations are:

$$\begin{aligned} \text{At full-series } &= K_1 + \tfrac{1}{2}z_1, \\ \text{At full-shunt } &= K_1 - \tfrac{1}{2}z_1, \\ \text{At mid-series } &= K_1 = \sqrt{z_1 z_2 + \tfrac{1}{4}z_1^2}, \\ \text{At mid-shunt } &= K_2 = z_1 z_2 / K_1. \end{aligned} \tag{109}$$

Lattice Type:

$$\cosh \Gamma = 1 + \frac{2z_1}{4z_2 - z_1} \tag{110}$$

and

$$K = \sqrt{z_1 z_2}. \tag{111}$$

Bridged-T Type:

$$\cosh \Gamma = 1 + \frac{2z_a z_b}{z_a(z_a + 4z_c) + 4z_b z_c} \tag{112}$$

and

$$K = \sqrt{\frac{z_a z_b (z_a + 4z_c)}{4(z_a + z_b)}}. \tag{113}$$

As an aid in obtaining the propagation constant, $\Gamma = A + iB$, from any of the three hyperbolic cosine formulæ it will be found convenient to use the following formulæ.

Computation Formulæ for the Complex Anti-Hyperbolic Cosine

It is known that many formulæ have already been derived for such evaluations but those below appear to give accurate results more readily.

Let it be desired to obtain A and B from the formula

$$\cosh (A + iB) = x + iy, \tag{114}$$

wherein x and y are known. A transformation of the x and y variables is first made so as to use the form of substitution and formulæ given in *B. S. T. J.*, October, 1924, pages 577 and 578. A further substitution and the application of hyperbolic formulæ give the following results where

$$U = \tfrac{1}{2}(x - 1),$$

$$V = \tfrac{1}{2}y,$$

$$P = 4(U + U^2 + V^2), \tag{115}$$

and

$$Q = \tfrac{1}{2} \sinh^{-1} \left| \frac{V}{U + U^2 + V^2} \right|.$$

When P is Positive:

$$A = \sinh^{-1} (\sqrt{P} \cosh Q)$$

and (116)

$$B = \pm \sin^{-1} (\sqrt{P} \sinh Q).$$

When P is Negative:

$$A = \sinh^{-1} (\sqrt{-P} \sinh Q)$$

and (117)

$$B = \pm \sin^{-1} (\sqrt{-P} \cosh Q).$$

When P is Zero, a Special Case:

$$A = \sinh^{-1} \sqrt{2|V|} = \tfrac{1}{2} \cosh^{-1} (1 + 4|V|)$$

and (118)

$$B = \pm \sin^{-1} \sqrt{2|V|} = \pm \tfrac{1}{2} \cos^{-1} (1 - 4|V|).$$

In All Cases:

$$B = \cos^{-1}\left(\frac{1 + 2U}{\cosh A}\right) = \sin^{-1}\left(\frac{2V}{\sinh A}\right). \tag{119}$$

The latter anti-cosine formula is particularly useful when B is in the neighborhood of $(2n + 1)\pi/2$, and both formulæ of (119) when considered together determine the sign of B.

The above formulæ give the solution of (114) which has a positive value for A (as in the propagation constant of a passive network). The other solution, since $\cosh(-\Gamma) = \cosh \Gamma$, would have values for both A and B which are the negative of those in the first solution (as may be possible in an active network).

It has been found that, when x and y are given to five or six decimals, it is possible to derive A and B to about this same degree of accuracy from these formulæ and the Smithsonian Mathematical Tables of Hyperbolic Functions. The formulæ may be used to advantage in accurately obtaining the propagation constant of a loaded line where x and y are calculated from the known circuit constants. (See footnote 2.)

Appendix IV

Propagation Characteristics and Formulæ for Various Lattice Type Networks

Networks of the lattice type only are specifically considered here since they have more general propagation characteristics than ladder or bridged-T types. However, transformations of any lattice type design obtained can be made to equivalent networks of these other types, if physical, by means of the simple relations given in Table II and the corresponding Section 2.5.

The network drawings show only half of the elements so as to avoid confusion; it is to be understood that the broken lines indicate the other series and lattice branches, respectively identical. The double subscript notation adopted for the elements is to be interpreted as follows: the first subscript on any element denotes the general position of the element in the network, 1 for the series branch and 2 for the lattice branch; the second subscript denotes the serial number of the element in either branch. Elements in the two branches which have the same serial numbers for their second subscripts correspond to each other according to the inverse network relations.

This group of networks, while not exhaustive, includes the simpler and perhaps most useful structures, but it could readily be extended. The propagation characteristics shown for each structure and derived from computed results are representative and serve to give an idea of the possibilities of the network for design purposes. All networks except the last have a constant resistance iterative impedance R. Networks 1*a*–12 have attenuation so that they will usually be designed from their attenuation characteristics in terms of which the formulæ are given. There is usually more than one physical solution from the same attenuation characteristic, and in Networks 9 and 10 as many as four have been found possible. These multiple solutions all have different phase constants. A possible practical advantage of one solution over another may lie either in its phase constant or the magnitudes of its elements. It is of interest to point out that if these networks were designed from the phase characteristic some of them might have multiple solutions with different attenuation characteristics. For example, Networks 3*a* and 3*b* corresponding to the phase characteristics 1′ and 2′ each can have two such solutions.

The Networks 1*b*, 2*b*, etc., with their output terminals interchanged are, respectively, identical with Networks 1*a*, 2*a*, etc. Hence, any pair of these networks have the same attenuation constants but phase constants differing by π radians. An extension of this list to include Networks 6*b*, 7*b*, etc., was not thought to be necessary.

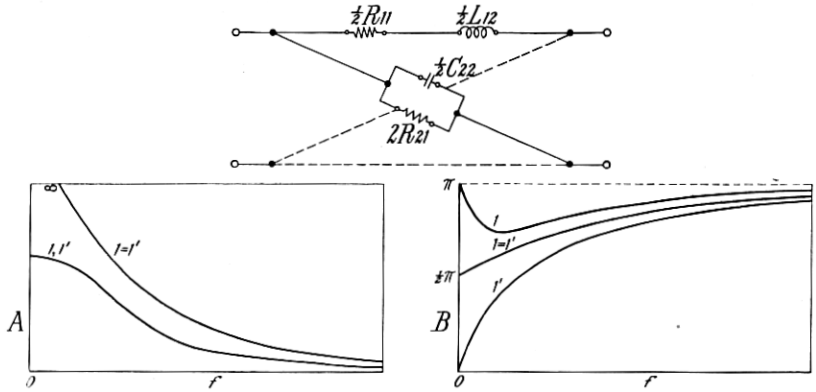
Several networks may have the same form of frequency function for F or H. Some values of the attenuation or phase coefficients will give a physical structure to one network but not to another. Whether a network having a definite A- or B-characteristic is physical or not can be determined most readily by a direct substitution of the coefficients in the formulæ for the elements. In certain cases these latter formulæ show easily that one network may give a physical result where another cannot. For example, Networks 6 and 10 both have the same F formula, but when one network is physical the other is not; similarly with Networks 7 and 9. These particular results would be expected from the fact that those pairs of networks cannot have the same attenuation characteristics, as seen from their structures.

Networks 13–17 have no attenuation and are designed from their phase characteristics. Network 18 represents a somewhat general form of artificial line and has other types of formulæ.

Examples of networks which are potentially complementary are Networks 1*a* and 2*b*; 1*b* and 2*a*; 3*a* and 3*b*; 11 and 12.

Transformations of impedance branches to equivalent ones can be made in some of the networks by means of the general transformation formulæ given in *B. S. T. J.*, January, 1923, pages 45 and 46.

Network 1*a*



$$R_{11} = 2a_0R; \qquad L_{12} = \frac{a_1R}{\pi}.$$

$$R_{11}R_{21} = L_{12}/C_{22} = R^2.$$

$$F = e^{2A} = 10^{\frac{\mathrm{TU}}{10}} = \frac{P_0 + f^2}{Q_0 + f^2}.$$

Attenuation Linear Equation:

$$P_0 - FQ_0 = f^2(F - 1).$$

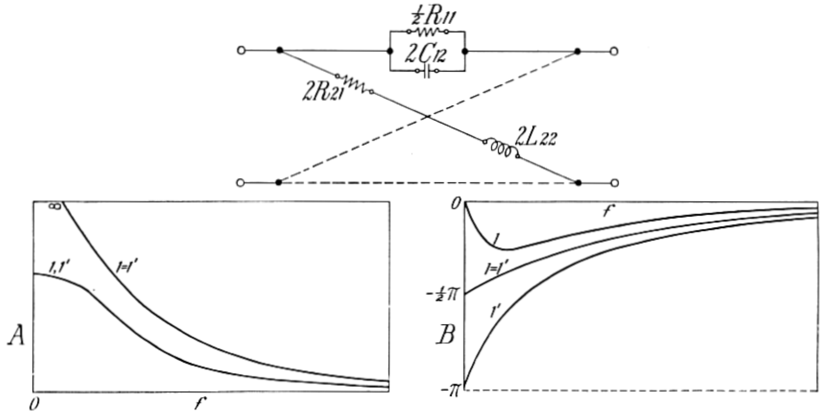
In physical solutions $O \leqq Q_0 \leqq P_0$.

$$1. \quad a_0 = \frac{\sqrt{P_0} + \sqrt{Q_0}}{\sqrt{P_0} - \sqrt{Q_0}}; \qquad a_1 = \frac{2}{\sqrt{P_0} - \sqrt{Q_0}}.$$

$$1'. \quad a_0 = \frac{\sqrt{P_0} - \sqrt{Q_0}}{\sqrt{P_0} + \sqrt{Q_0}}; \qquad a_1 = \frac{2}{\sqrt{P_0} + \sqrt{Q_0}}.$$

$$H = \tan B = \frac{2a_1f}{(1 - a_0^2) - a_1^2f^2}.$$

NETWORK 1b



$$R_{11} = 2a_0R; \qquad C_{12} = \frac{b_1}{4\pi a_0 R}.$$

$$R_{11}R_{21} = L_{22}/C_{12} = R^2.$$

$$F = e^{2A} = 10^{\frac{\text{TU}}{10}} = \frac{P_0 + f^2}{Q_0 + f^2}.$$

Attenuation Linear Equation:

$$P_0 - FQ_0 = f^2(F - 1).$$

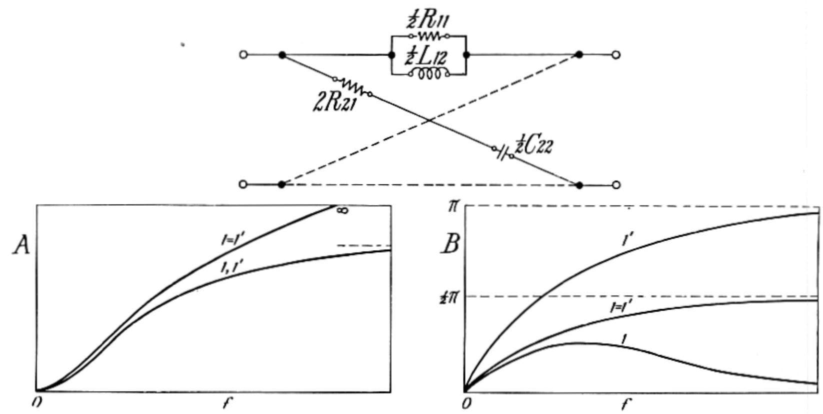
In physical solutions $O \leqq Q_0 \leqq P_0$.

$$1.\ \ a_0 = \frac{\sqrt{P_0} - \sqrt{Q_0}}{\sqrt{P_0} + \sqrt{Q_0}}; \qquad b_1 = \frac{2}{\sqrt{P_0} + \sqrt{Q_0}}.$$

$$1'.\ \ a_0 = \frac{\sqrt{P_0} + \sqrt{Q_0}}{\sqrt{P_0} - \sqrt{Q_0}}; \qquad b_1 = \frac{2}{\sqrt{P_0} - \sqrt{Q_0}}.$$

$$H = \tan B = \frac{-\ 2a_0b_1f}{(1 - a_0^2) + b_1^2f^2}.$$

NETWORK 2a



$$R_{11} = \frac{2a_1R}{b_1}; \qquad L_{12} = \frac{a_1R}{\pi}.$$

$$R_{11}R_{21} = L_{12}/C_{22} = R^2.$$

$$F = e^{2A} = 10^{\frac{\text{TU}}{10}} = \frac{1 + P_2f^2}{1 + Q_2f^2}.$$

Attenuation Linear Equation:

$$P_2 - FQ_2 = (F - 1)/f^2.$$

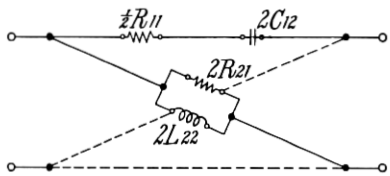
In physical solutions $O \leqq Q_2 \leqq P_2$.

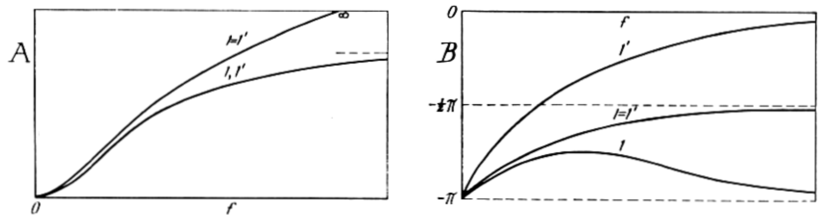
$$1. \quad \left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \tfrac{1}{2}(\sqrt{P_2} \mp \sqrt{Q_2}).$$

$$1'. \quad \left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \tfrac{1}{2}(\sqrt{P_2} \pm \sqrt{Q_2}).$$

$$H = \tan B = \frac{2a_1f}{1 - (a_1^2 - b_1^2)f^2}.$$

NETWORK 2*b*





$$R_{11} = \frac{2a_1R}{b_1}; \qquad C_{12} = \frac{b_1}{4\pi R}.$$

$$R_{11}R_{21} = L_{22}/C_{12} = R^2.$$

$$F = e^{2A} = 10^{\frac{\text{TU}}{10}} = \frac{1 + P_2f^2}{1 + Q_2f^2}.$$

Attenuation Linear Equation:

$$P_2 - FQ_2 = (F - 1)/f^2.$$

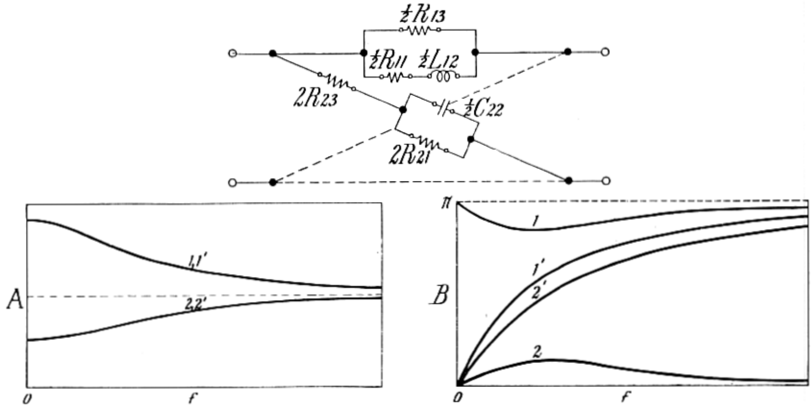
In physical solutions $O \leqq Q_2 \leqq P_2$.

$$1. \quad \left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \tfrac{1}{2}(\sqrt{P_2} \pm \sqrt{Q_2}).$$

$$1'. \quad \left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \tfrac{1}{2}(\sqrt{P_2} \mp \sqrt{Q_2}).$$

$$H = \tan B = \frac{2b_1f}{1 + (a_1^2 - b_1^2)f^2}.$$

NETWORK 3*a*



$$R_{11} = \frac{2a_0 a_1 R}{a_1 b_0 - a_0}; \qquad L_{12} = \frac{a_1^2 R}{\pi(a_1 b_0 - a_0)}; \qquad R_{13} = 2a_1 R.$$

$$R_{11}R_{21} = L_{12}/C_{22} = R_{13}R_{23} = R^2.$$

$$F = e^{2A} = 10^{\frac{\text{TU}}{10}} = \frac{P_0 + f^2}{Q_0 + Q_2 f^2}.$$

Attenuation Linear Equation:

$$-P_0 + FQ_0 + f^2 FQ_2 = f^2.$$

In physical solutions $O \leqq Q_0 \leqq P_0$; $O \leqq Q_2 \leqq 1$.

If $Q_0 < P_0 Q_2$ (A decreases with frequency):

$$1.\ \begin{rcases} a_0 \\ b_0 \end{rcases} = \frac{\sqrt{P_0} \pm \sqrt{Q_0}}{1 - \sqrt{Q_2}}; \qquad a_1 = \frac{1 + \sqrt{Q_2}}{1 - \sqrt{Q_2}}.$$

$$1'.\ \begin{rcases} a_0 \\ b_0 \end{rcases} = \frac{\sqrt{P_0} \mp \sqrt{Q_0}}{1 - \sqrt{Q_2}}; \qquad a_1 = \frac{1 + \sqrt{Q_2}}{1 - \sqrt{Q_2}}.$$

If $Q_0 > P_0 Q_2$ (A increases with frequency):

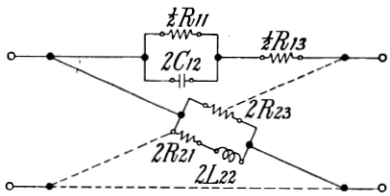
$$2.\ \begin{rcases} a_0 \\ b_0 \end{rcases} = \frac{\sqrt{P_0} \mp \sqrt{Q_0}}{1 + \sqrt{Q_2}}; \qquad a_1 = \frac{1 - \sqrt{Q_2}}{1 + \sqrt{Q_2}}.$$

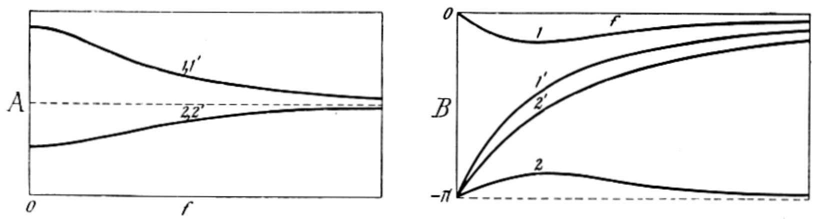
2′. Same formulæ as in 1′.

If $Q_0 = P_0 Q_2$, $F = 1/Q_2$ (A is constant).

$$H = \tan B = \frac{2(a_1 b_0 - a_0)f}{(b_0^2 - a_0^2) + (1 - a_1^2)f^2}.$$

NETWORK 3b





$$R_{11} = \frac{2(a_0 b_1 - a_1)R}{b_1}; \qquad C_{12} = \frac{b_1^2}{4\pi(a_0 b_1 - a_1)R}; \qquad R_{13} = \frac{2a_1 R}{b_1}.$$

$$R_{11}R_{21} = L_{22}/C_{12} = R_{13}R_{23} = R^2.$$

$$F = e^{2A} = 10^{\frac{\text{TU}}{10}} = \frac{P_0 + f^2}{Q_0 + Q_2 f^2}.$$

Attenuation Linear Equation:

$$-P_0 + FQ_0 + f^2 FQ_2 = f^2.$$

In physical solutions $O \leqq Q_0 \leqq P_0$; $O \leqq Q_2 \leqq 1$.

If $Q_0 < P_0 Q_2$ (A decreases with frequency):

$$1. \quad a_0 = \frac{\sqrt{P_0} - \sqrt{Q_0}}{\sqrt{P_0} + \sqrt{Q_0}}; \qquad \left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \frac{1 \mp \sqrt{Q_2}}{\sqrt{P_0} + \sqrt{Q_0}}.$$

$$1'. \quad a_0 = \frac{\sqrt{P_0} + \sqrt{Q_0}}{\sqrt{P_0} - \sqrt{Q_0}}; \qquad \left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \frac{1 \mp \sqrt{Q_2}}{\sqrt{P_0} - \sqrt{Q_0}}.$$

If $Q_0 > P_0 Q_2$ (A increases with frequency):

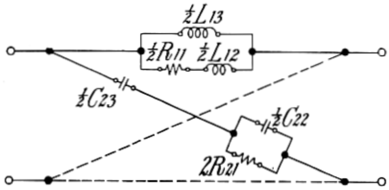
$$2. \quad a_0 = \frac{\sqrt{P_0} + \sqrt{Q_0}}{\sqrt{P_0} - \sqrt{Q_0}}; \qquad \left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \frac{1 \pm \sqrt{Q_2}}{\sqrt{P_0} - \sqrt{Q_0}}.$$

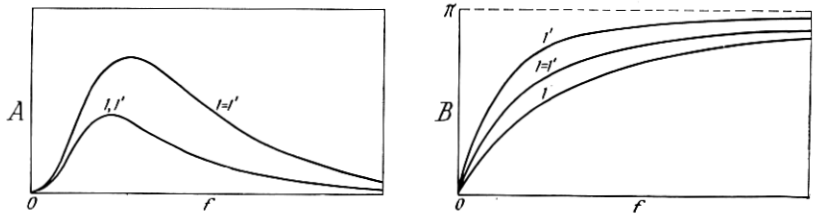
2'. Same formulæ as in 1'.

If $Q_0 = P_0 Q_2$, $F = 1/Q_2$ (A is constant).

$$H = \tan B = \frac{-2(a_0 b_1 - a_1)f}{(1 - a_0^2) + (b_1^2 - a_1^2)f^2}.$$

NETWORK 4a





$$R_{11} = \frac{2a_1{}^2R}{a_1b_1 - a_2}; \qquad L_{12} = \frac{a_1a_2R}{\pi(a_1b_1 - a_2)}; \qquad L_{13} = \frac{a_1R}{\pi}.$$

$$R_{11}R_{21} = L_{12}/C_{22} = L_{13}/C_{23} = R^2.$$

$$F = e^{2A} = 10^{\frac{\mathrm{TU}}{10}} = \frac{1 + P_2f^2 + P_4f^4}{1 + Q_2f^2 + P_4f^4}.$$

Attenuation Linear Equation:

$$P_2 - f^2(F - 1)P_4 - FQ_2 = (F - 1)/f^2.$$

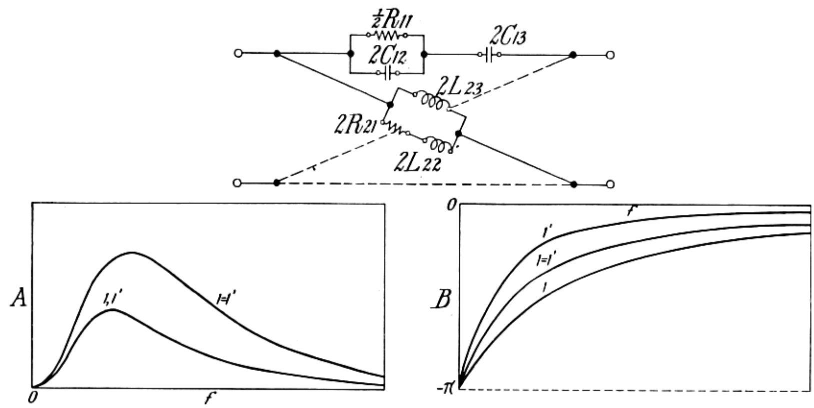
In physical solutions $O \leqq 2\sqrt{P_4} \leqq Q_2 \leqq P_2$.

1. $\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \frac{1}{2}(\sqrt{P_2 + 2\sqrt{P_4}} \mp \sqrt{Q_2 - 2\sqrt{P_4}}); \qquad a_2 = \sqrt{P_4}.$

1′. Same formulæ as in 1, but with a_1 and b_1 interchanged.

$$H = \tan B = \frac{2a_1f + 2a_2b_1f^3}{1 - (a_1{}^2 - b_1{}^2)f^2 - a_2{}^2f^4}.$$

NETWORK 4b



$$R_{11} = \frac{2(a_1b_1 - b_2)R}{b_1^2}; \qquad C_{12} = \frac{b_1b_2}{4\pi(a_1b_1 - b_2)R}; \qquad C_{13} = \frac{b_1}{4\pi R}.$$

$$R_{11}R_{21} = L_{22}/C_{12} = L_{23}/C_{13} = R^2.$$

$$F = e^{2A} = 10^{\frac{\text{TU}}{10}} = \frac{1 + P_2f^2 + P_4f^4}{1 + Q_2f^2 + P_4f^4}.$$

Attenuation Linear Equation:

$$P_2 - f^2(F - 1)P_4 - FQ_2 = (F - 1)/f^2.$$

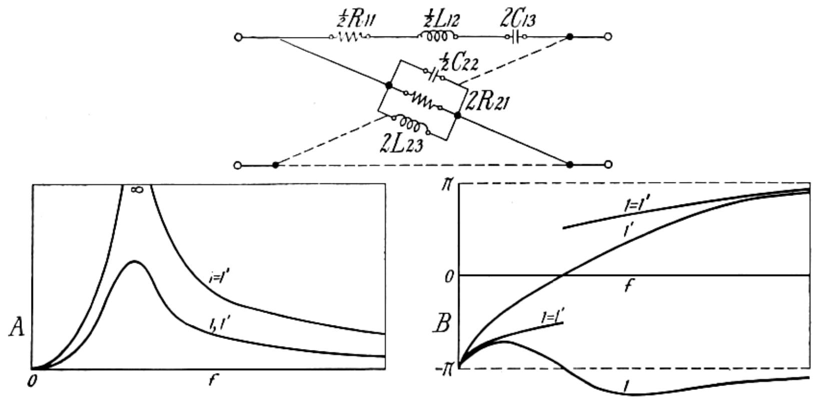
In physical solutions $O \leqq 2\sqrt{P_4} \leqq Q_2 \leqq P_2$.

1. $\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \frac{1}{2}(\sqrt{P_2 + 2\sqrt{P_4}} \pm \sqrt{Q_2 - 2\sqrt{P_4}}); \qquad b_2 = \sqrt{P_4}.$

1′. Same formulæ as in 1, but with a_1 and b_1 interchanged.

$$H = \tan B = \frac{2b_1f + 2a_1b_2f^3}{1 + (a_1^2 - b_1^2)f^2 - b_2^2f^4}.$$

Network 5a



$$R_{11} = \frac{2a_1 R}{b_1}; \qquad L_{12} = \frac{a_2 R}{\pi b_1}; \qquad C_{13} = \frac{b_1}{4\pi R}.$$

$$R_{11}R_{21} = L_{12}/C_{22} = L_{23}/C_{13} = R^2.$$

$$F = e^{2A} = 10^{\frac{\text{TU}}{10}} = \frac{1 + P_2 f^2 + P_4 f^4}{1 + Q_2 f^2 + P_4 f^4}.$$

Attenuation Linear Equation:

$$P_2 - FQ_2 - f^2(F-1)P_4 = (F-1)/f^2.$$

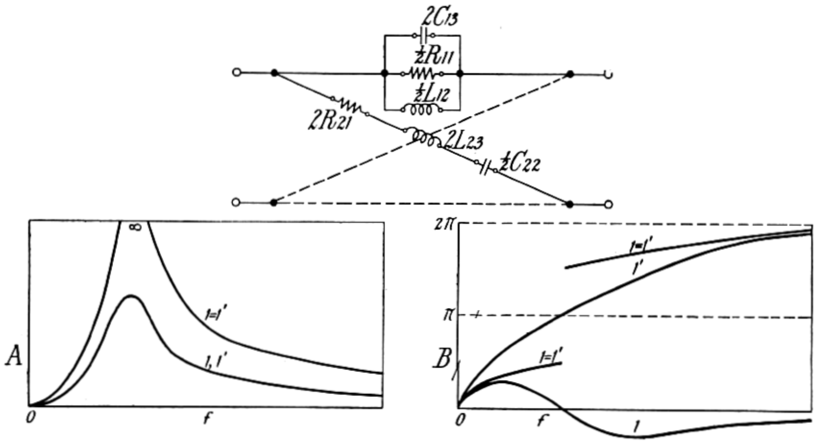
In physical solutions $-2\sqrt{P_4} \leqq Q_2 \leqq P_2$.

$$1.\ \left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \tfrac{1}{2}\left(\sqrt{P_2 + 2\sqrt{P_4}} \pm \sqrt{Q_2 + 2\sqrt{P_4}}\right); \qquad a_2 = \sqrt{P_4}.$$

1′. Same formulæ as in 1, but with a_1 and b_1 interchanged.

$$H = \tan B = \frac{2b_1 f - 2a_2 b_1 f^3}{1 + (a_1^2 - 2a_2 - b_1^2)f^2 + a_2^2 f^4}.$$

NETWORK 5*b*



$$R_{11} = \frac{2a_1R}{b_1}; \qquad L_{12} = \frac{a_1R}{\pi}; \qquad C_{13} = \frac{b_2}{4\pi a_1 R}.$$

$$R_{11}R_{21} = L_{12}/C_{22} = L_{23}/C_{13} = R^2.$$

$$F = e^{2A} = 10^{\frac{\mathrm{TU}}{10}} = \frac{1 + P_2f^2 + P_4f^4}{1 + Q_2f^2 + P_4f^4}.$$

Attenuation Linear Equation:

$$P_2 - FQ_2 - f^2(F-1)P_4 = (F-1)/f^2.$$

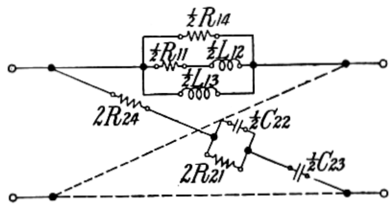
In physical solutions $-2\sqrt{P_4} \leqq Q_2 \leqq P_2$.

1. $\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \frac{1}{2}\left(\sqrt{P_2 + 2\sqrt{P_4}} \mp \sqrt{Q_2 + 2\sqrt{P_4}}\right); \qquad b_2 = \sqrt{P_4}.$

1′. Same formulæ as in 1, but with a_1 and b_1 interchanged.

$$H = \tan B = \frac{2a_1f - 2a_1b_2f^3}{1 - (a_1^2 - b_1^2 + 2b_2)f^2 + b_2^2f^4}.$$

NETWORK 6



A- and B-characteristics are similar to those of Networks $2a$ and $4a$.

$$R_{11} = \frac{2a_1^2 a_2 R}{a_1 a_2 b_1 - a_1^2 b_2 - a_2^2}; \qquad L_{12} = \frac{a_1 a_2^2 R}{\pi(a_1 a_2 b_1 - a_1^2 b_2 - a_2^2)};$$

$$L_{13} = \frac{a_1 R}{\pi}; \qquad R_{14} = \frac{2a_2 R}{b_2}.$$

$$R_{11}R_{21} = L_{12}/C_{22} = L_{13}/C_{23} = R_{14}R_{24} = R^2.$$

$$F = e^{2A} = 10^{\frac{\text{TU}}{10}} = \frac{1 + P_2 f^2 + P_4 f^4}{1 + Q_2 f^2 + Q_4 f^4}.$$

Attenuation Linear Equation:

$$P_2 + f^2 P_4 - FQ_2 - f^2 FQ_4 = (F - 1)/f^2.$$

In unrestricted solutions, where $O \leqq Q_4 \leqq P_4$:

$$\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} \text{ or } \left.\begin{matrix} b_1 \\ a_1 \end{matrix}\right\} = \tfrac{1}{2}\left(\sqrt{P_2 + 2\sqrt{P_4}} \pm \sqrt{Q_2 - 2\sqrt{Q_4}}\right);$$

$$\left.\begin{matrix} a_2 \\ b_2 \end{matrix}\right\} = \tfrac{1}{2}(\sqrt{P_4} \pm \sqrt{Q_4}).$$

Also

$$\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} \text{ or } \left.\begin{matrix} b_1 \\ a_1 \end{matrix}\right\} = \tfrac{1}{2}\left(\sqrt{P_2 + 2\sqrt{P_4}} \pm \sqrt{Q_2 + 2\sqrt{Q_4}}\right);$$

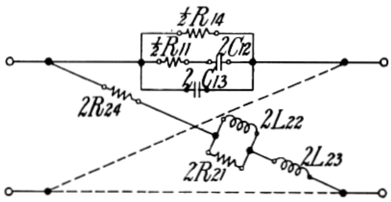
$$\left.\begin{matrix} a_2 \\ b_2 \end{matrix}\right\} = \tfrac{1}{2}(\sqrt{P_4} \mp \sqrt{Q_4}).$$

In physical solutions a_1, a_2, b_1, b_2 are positive;

$$a_1^2 b_2 + a_2^2 \leqq a_1 a_2 b_1.$$

$$H = \tan B = \frac{2a_1 f - 2(a_1 b_2 - a_2 b_1) f^3}{1 - (a_1^2 - b_1^2 + 2b_2) f^2 - (a_2^2 - b_2^2) f^4}.$$

NETWORK 7



A- and B-characteristics are similar to those of Networks 1b and 4b.

$$R_{11} = \frac{2a_0a_1^2R}{a_0a_1b_1 - a_1^2 - a_0^2b_2}; \qquad C_{12} = \frac{a_0a_1b_1 - a_1^2 - a_0^2b_2}{4\pi a_0^2a_1R};$$

$$C_{13} = \frac{b_2}{4\pi a_1R}; \qquad R_{14} = 2a_0R.$$

$$R_{11}R_{21} = L_{22}/C_{12} = L_{23}/C_{13} = R_{14}R_{24} = R^2.$$

$$F = e^{2A} = 10^{\frac{TU}{10}} = \frac{P_0 + P_2f^2 + f^4}{Q_0 + Q_2f^2 + f^4}.$$

Attenuation Linear Equation:

$$P_0 + f^2P_2 - FQ_0 - f^2FQ_2 = f^4(F - 1).$$

In unrestricted solutions, where $O \leqq Q_0 \leqq P_0$:

$$a_0 = \frac{\sqrt{P_0} - \sqrt{Q_0}}{\sqrt{P_0} + \sqrt{Q_0}}; \qquad b_2 = \frac{2}{\sqrt{P_0} + \sqrt{Q_0}};$$

$$\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} \text{or} \left.\begin{matrix} b_1 \\ a_1 \end{matrix}\right\} = \frac{\sqrt{P_2 + 2\sqrt{P_0}} \pm \sqrt{Q_2 + 2\sqrt{Q_0}}}{\sqrt{P_0} + \sqrt{Q_0}}.$$

Also

$$a_0 = \frac{\sqrt{P_0} + \sqrt{Q_0}}{\sqrt{P_0} - \sqrt{Q_0}}; \qquad b_2 = \frac{2}{\sqrt{P_0} - \sqrt{Q_0}};$$

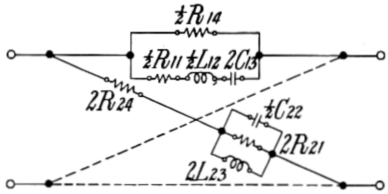
$$\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} \text{or} \left.\begin{matrix} b_1 \\ a_1 \end{matrix}\right\} = \frac{\sqrt{P_2 + 2\sqrt{P_0}} \pm \sqrt{Q_2 - 2\sqrt{Q_0}}}{\sqrt{P_0} - \sqrt{Q_0}}.$$

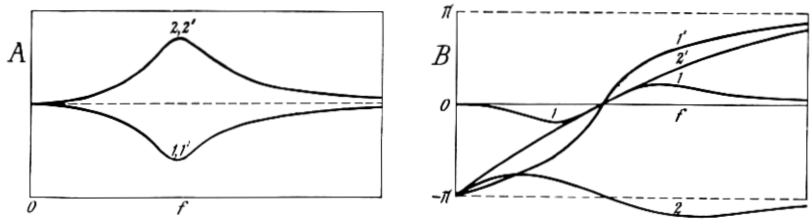
In physical solutions a_0, a_1, b_1, b_2 are positive;

$$a_1^2 + a_0^2b_2 \leqq a_0a_1b_1.$$

$$H = \tan B = \frac{2(a_1 - a_0b_1)f - 2a_1b_2f^3}{(1 - a_0^2) - (a_1^2 - b_1^2 + 2b_2)f^2 + b_2^2f^4}.$$

NETWORK 8





$$R_{11} = \frac{2a_0a_1R}{a_0b_1 - a_1}; \qquad L_{12} = \frac{a_0^2b_2R}{\pi(a_0b_1 - a_1)};$$

$$C_{13} = \frac{a_0b_1 - a_1}{4\pi a_0^2R}; \qquad R_{14} = 2a_0R.$$

$$R_{11}R_{21} = L_{12}/C_{22} = L_{23}/C_{13} = R_{14}R_{24} = R^2.$$

$$F = e^{2A} = 10^{\frac{\text{TU}}{10}} = \frac{F_0 + P_2f^2 + F_0Q_4f^4}{1 + Q_2f^2 + Q_4f^4}.$$

$$F_0(f = o) = F_\infty(f = \infty); \qquad A_0 = A_\infty.$$

Attenuation Linear Equation:

$$-P_2 + FQ_2 - f^2(F_0 - F)Q_4 = (F_0 - F)/f^2.$$

In physical solutions $O \leqq Q_2 + 2\sqrt{Q_4} \equiv n \leqq P_2 + 2F_0\sqrt{Q_4} \equiv m.$
If $P_2 < F_0Q_2$ (A has a minimum):

1. $a_0 = \tanh(A_0/2); \qquad b_2 = \sqrt{Q_4};$

$$\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \tfrac{1}{2}(1 - \tanh(A_0/2))(\sqrt{m} \mp \sqrt{n}).$$

1′. $a_0 = \coth(A_0/2); \qquad b_2 = \sqrt{Q_4};$

$$\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \tfrac{1}{2}(\coth(A_0/2) - 1)(\sqrt{m} \mp \sqrt{n}).$$

If $P_2 > F_0Q_2$ (A has a maximum):

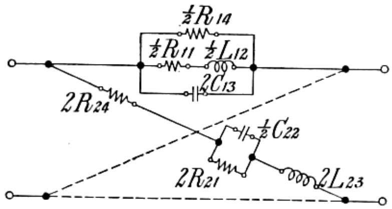
2. $a_0 = \coth(A_0/2); \qquad b_2 = \sqrt{Q_4};$

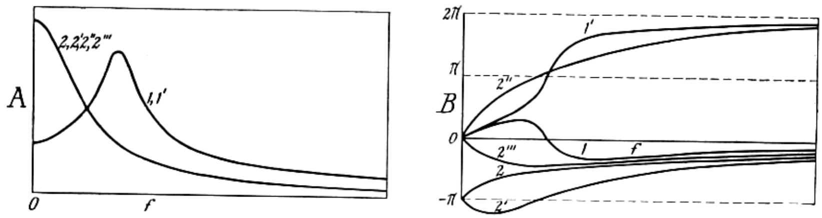
$$\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} = \tfrac{1}{2}(\coth(A_0/2) - 1)(\sqrt{m} \pm \sqrt{n}).$$

2′. The same formulæ as in 1′.

$$H = \tan B = \frac{2(a_0b_1 - a_1)(-f + b_2f^3)}{(1 - a_0^2) - (a_1^2 - b_1^2 + 2(1 - a_0^2)b_2)f^2 + (1 - a_0^2)b_2^2f^4}.$$

Network 9





$$R_{11} = \frac{2a_0a_1^2R}{a_0^2b_2 + a_1^2 - a_0a_1b_1}\,; \qquad L_{12} = \frac{a_1^3R}{\pi(a_0^2b_2 + a_1^2 - a_0a_1b_1)}\,;$$

$$C_{13} = \frac{b_2}{4\pi a_1 R}\,; \qquad R_{14} = \frac{2a_1^2R}{a_1b_1 - a_0b_2}\,.$$

$$R_{11}R_{21} = L_{12}/C_{22} = L_{23}/C_{13} = R_{14}R_{24} = R^2.$$

$$F = e^{2A} = 10^{\frac{\text{TU}}{10}} = \frac{P_0 + P_2f^2 + f^4}{Q_0 + Q_2f^2 + f^4}\,.$$

Attenuation Linear Equation:

$$P_0 + f^2P_2 - FQ_0 - f^2FQ_2 = f^4(F - 1).$$

In unrestricted solutions, where $O \leqq Q_0 \leqq P_0$:

$$a_0 = \frac{\sqrt{P_0} - \sqrt{Q_0}}{\sqrt{P_0} + \sqrt{Q_0}}\,; \qquad b_2 = \frac{2}{\sqrt{P_0} + \sqrt{Q_0}}\,;$$

$$\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} \text{or} \left.\begin{matrix} b_1 \\ a_1 \end{matrix}\right\} = \frac{\sqrt{P_2 + 2\sqrt{P_0}} \pm \sqrt{Q_2 + 2\sqrt{Q_0}}}{\sqrt{P_0} + \sqrt{Q_0}}\,.$$

Also

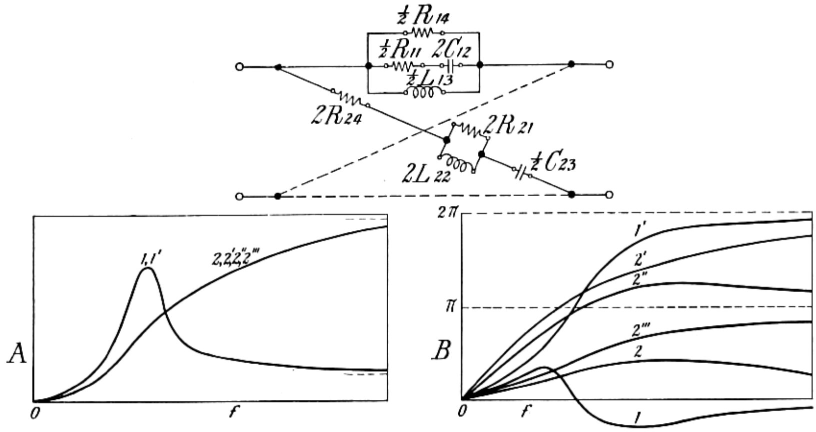
$$a_0 = \frac{\sqrt{P_0} + \sqrt{Q_0}}{\sqrt{P_0} - \sqrt{Q_0}}\,; \qquad b_2 = \frac{2}{\sqrt{P_0} - \sqrt{Q_0}}\,;$$

$$\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} \text{or} \left.\begin{matrix} b_1 \\ a_1 \end{matrix}\right\} = \frac{\sqrt{P_2 + 2\sqrt{P_0}} \pm \sqrt{Q_2 - 2\sqrt{Q_0}}}{\sqrt{P_0} - \sqrt{Q_0}}\,.$$

In physical solutions $Q_2 \leqq P_2$; $a_0a_1b_1 \leqq a_0^2b_2 + a_1^2$.

$$H = \tan B = \frac{2(a_1 - a_0b_1)f - 2a_1b_2f^3}{(1 - a_0^2) - (a_1^2 - b_1^2 + 2b_2)f^2 + b_2^2f^4}\,.$$

Network 10



$$R_{11} = \frac{2a_1^2a_2R}{a_1^2b_2 + a_2^2 - a_1a_2b_1}; \qquad C_{12} = \frac{a_1^2b_2 + a_2^2 - a_1a_2b_1}{4\pi a_1^3R};$$

$$L_{13} = \frac{a_1R}{\pi}; \qquad R_{14} = \frac{2a_1^2R}{a_1b_1 - a_2}.$$

$$R_{11}R_{21} = L_{22}/C_{12} = L_{13}/C_{23} = R_{14}R_{24} = R^2.$$

$$F = e^{2A} = 10^{\frac{\mathrm{TU}}{10}} = \frac{1 + P_2f^2 + P_4f^4}{1 + Q_2f^2 + Q_4f^4}.$$

Attenuation Linear Equation:

$$P_2 + f^2P_4 - FQ_2 - f^2FQ_4 = (F - 1)/f^2.$$

In unrestricted solutions, where $O \leqq Q_4 \leqq P_4$:

$$\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} \text{ or } \left.\begin{matrix} b_1 \\ a_1 \end{matrix}\right\} = \tfrac{1}{2}\left(\sqrt{P_2 + 2\sqrt{P_4}} \pm \sqrt{Q_2 - 2\sqrt{Q_4}}\right);$$

$$\left.\begin{matrix} a_2 \\ b_2 \end{matrix}\right\} = \tfrac{1}{2}(\sqrt{P_4} \pm \sqrt{Q_4}).$$

Also

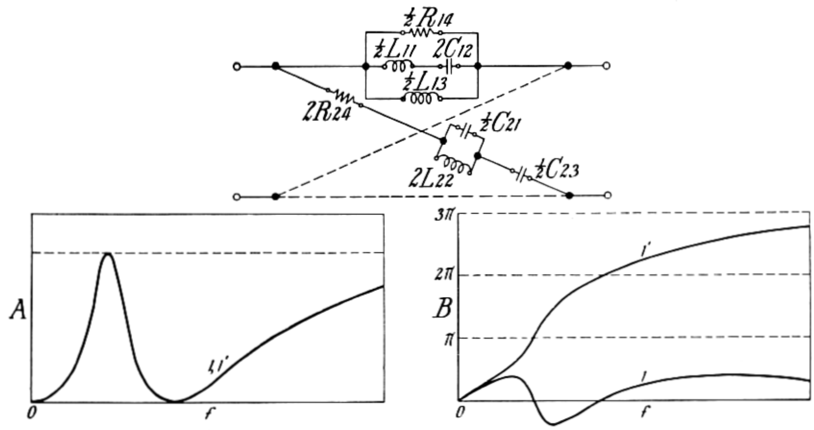
$$\left.\begin{matrix} a_1 \\ b_1 \end{matrix}\right\} \text{ or } \left.\begin{matrix} b_1 \\ a_1 \end{matrix}\right\} = \tfrac{1}{2}\left(\sqrt{P_2 + 2\sqrt{P_4}} \pm \sqrt{Q_2 + 2\sqrt{Q_4}}\right);$$

$$\left.\begin{matrix} a_2 \\ b_2 \end{matrix}\right\} = \tfrac{1}{2}(\sqrt{P_4} \mp \sqrt{Q_4}).$$

In physical solutions $Q_2 \leqq P_2$; $a_1a_2b_1 \leqq a_1^2b_2 + a_2^2$.

$$H = \tan B = \frac{2a_1f - 2(a_1b_2 - a_2b_1)f^3}{1 - (a_1^2 - b_1^2 + 2b_2)f^2 - (a_2^2 - b_2^2)f^4}$$

NETWORK 11



$$L_{11} = \frac{a_1 a_3 R}{\pi(a_1 b_2 - a_3)}; \qquad C_{12} = \frac{a_1 b_2 - a_3}{4\pi a_1{}^2 R};$$

$$L_{13} = \frac{a_1 R}{\pi}; \qquad R_{14} = \frac{2R}{m}.$$

$$L_{11}/C_{21} = L_{22}/C_{12} = L_{13}/C_{23} = R_{14}R_{24} = R^2.$$

$$F = e^{2A} = 10^{\frac{\text{TU}}{10}} = \frac{1 + (1+m)^2 y^2}{1 + (1-m)^2 y^2},$$

where

$$y = \frac{a_1 f - a_3 f^3}{1 - b_2 f^2}$$

is the total parallel reactance in z_{11} divided by $2R$.

$$1.\quad m = \coth \tfrac{1}{2}A_\infty;$$
$$1'.\quad m = \tanh \tfrac{1}{2}A_\infty;$$

where A_∞ is the maximum attenuation at $f = \infty$ and at the internal frequency $f = 1/\sqrt{b_2}$.

Attenuation Linear Equation:

$$a_1 - f^2 a_3 + f y b_2 = y/f,$$

where

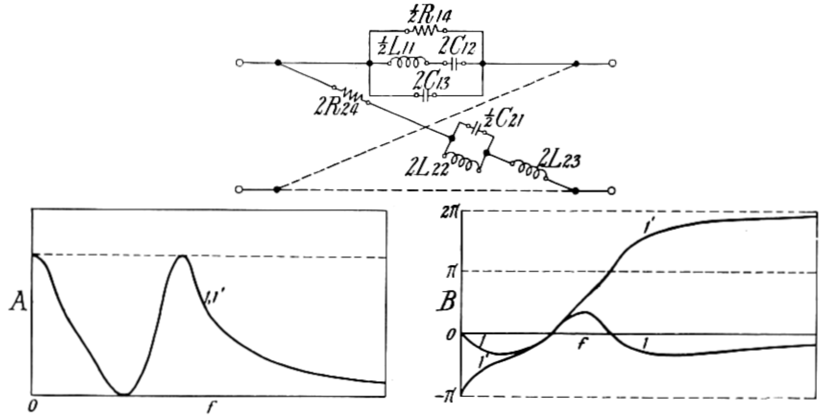
$$y = \pm\sqrt{\frac{F - 1}{(1+m)^2 - (1-m)^2 F}}$$

and the signs to be taken for y correspond to the particular reactance branches involved, whose signs in order on the frequency scale are $+$, $-$, and $+$.

In physical solutions $a_3 \leqq a_1 b_2$.

$$H = \tan B = \pm\frac{2y}{1 - (1 - m^2)y^2}.$$

NETWORK 12



$$L_{11} = \frac{a_2{}^2 R}{\pi(a_2 b_1 - b_3)}; \qquad C_{12} = \frac{a_2 b_1 - b_3}{4\pi a_2 R};$$

$$C_{13} = \frac{b_3}{4\pi a_2 R}; \qquad R_{14} = \frac{2R}{m}.$$

$$L_{11}/C_{21} = L_{22}/C_{12} = L_{23}/C_{13} = R_{14}R_{24} = R^2.$$

$$F = e^{2A} = 10^{\frac{\text{TU}}{10}} = \frac{1 + (1+m)^2 y^2}{1 + (1-m)^2 y^2},$$

where

$$y = \frac{-1 + a_2 f^2}{b_1 f - b_3 f^3}$$

is the total parallel reactance in z_{11} divided by $2R$.

1. $m = \coth \frac{1}{2}A_0;$

1′. $m = \tanh \frac{1}{2}A_0;$

where A_0 is the maximum attenuation at $f = 0$ and at the internal frequency $f = \sqrt{b_1/b_3}$.

Attenuation Linear Equation:

$$a_2 - (y/f)b_1 + fyb_3 = 1/f^2,$$

where

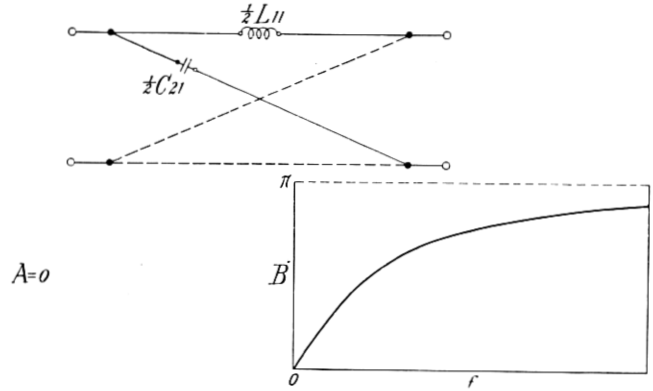
$$y = \pm\sqrt{\frac{F-1}{(1+m)^2 - (1-m)^2 F}}$$

and the signs to be taken for y correspond to the particular reactance branches involved, whose signs in order on the frequency scale are $-$, $+$, and $-$.

In physical solutions $b_3 \leqq a_2 b_1$.

$$H = \tan B = \frac{2y}{1 - (1 - m^2)y^2}.$$

NETWORK 13



$$L_{11} = \frac{a_1 R}{\pi}; \qquad L_{11}/C_{21} = R^2.$$

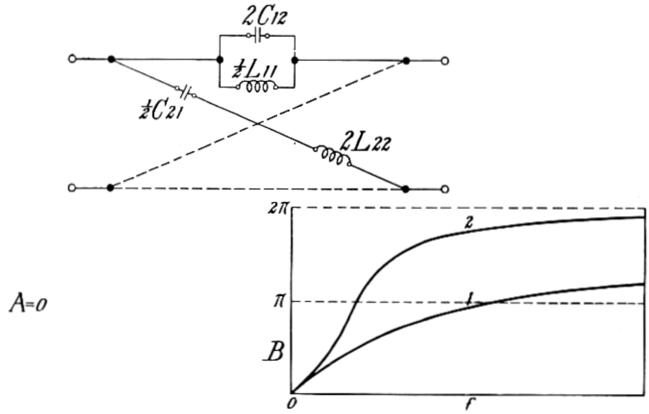
$$H = \tan \tfrac{1}{2}B = a_1 f.$$

Phase Linear Equation:

$$a_1 = H/f.$$

(See also formula (75).)

NETWORK 14



$$L_{11} = \frac{a_1 R}{\pi}; \qquad C_{12} = \frac{b_2}{4\pi a_1 R}\,.$$

$$L_{11}/C_{21} = L_{22}/C_{12} = R^2.$$

$$H = \tan \tfrac{1}{2}B = \frac{a_1 f}{1 - b_2 f^2}\,.$$

Phase Linear Equation:

$$a_1 + fHb_2 = H/f.$$

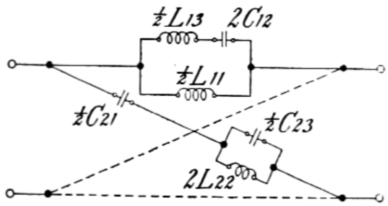
$$1.\ b_2 < \tfrac{1}{3}a_1^2. \qquad 2.\ b_2 > \tfrac{1}{3}a_1^2.$$

Equivalent Network, if $b_2 \leqq \frac{1}{4}a_1^2$:

Two sections (a_1' and a_1'') of Network 13:

$$\left.\begin{matrix} a_1' \\ a_1'' \end{matrix}\right\} = \tfrac{1}{2}(a_1 \pm \sqrt{a_1^2 - 4b_2}).$$

NETWORK 15



$A = 0.$

B-characteristic is the sum of those for Networks 13 and 14.

$$L_{11} = \frac{a_1 R}{\pi}; \qquad C_{12} = \frac{a_1 b_2 - a_3}{4\pi a_1{}^2 R}; \qquad L_{13} = \frac{a_1 a_3 R}{\pi(a_1 b_2 - a_3)}.$$

$$L_{11}/C_{21} = L_{22}/C_{12} = L_{13}/C_{23} = R^2.$$

$$H = \tan \tfrac{1}{2}B = \frac{a_1 f - a_3 f^3}{1 - b_2 f^2}.$$

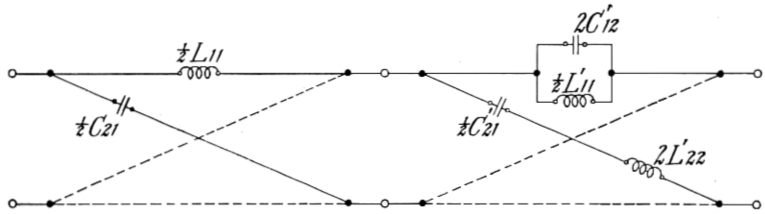
Phase Linear Equation:

$$a_1 - f^2 a_3 + fHb_2 = H/f.$$

In physical solutions $a_3 \leqq a_1 b_2$.

Equivalent to Network 16.

NETWORK 16



$A = 0.$

B-characteristic is the sum of those for Networks 13 and 14.

$$L_{11} = \frac{a_1 R}{\pi}; \qquad L_{11}' = \frac{a_1' R}{\pi}; \qquad C_{12}' = \frac{b_2'}{4\pi a_1' R}.$$

$$L_{11}/C_{21} = L_{11}'/C_{21}' = L_{22}'/C_{12}' = R^2.$$

$$H = \tan \tfrac{1}{2}B = \frac{M_1 f + M_3 f^3}{1 + N_2 f^2}.$$

Phase Linear Equation:

$$M_1 + f^2 M_3 - fHN_2 = H/f.$$

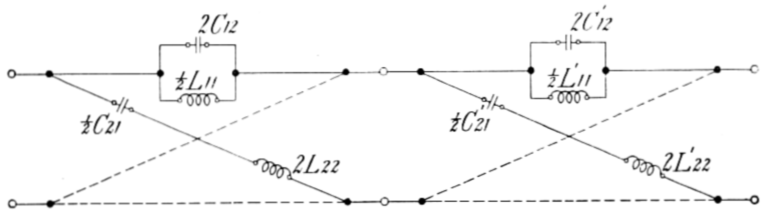
$$a_1^3 - M_1 a_1^2 - N_2 a_1 + M_3 = 0;$$

$$a_1' = M_1 - a_1;$$

$$b_2' = -M_3/a_1.$$

Equivalent to Network 15.

NETWORK 17



$A = 0$.

B-characteristic is the sum of those for two sections of Network 14.

$$L_{11} = \frac{a_1 R}{\pi}; \qquad C_{12} = \frac{b_2}{4\pi a_1 R}; \qquad L_{11}' = \frac{a_1' R}{\pi}; \qquad C_{12}' = \frac{b_2'}{4\pi a_1' R}.$$

$$L_{11}/C_{21} = L_{22}/C_{12} = L_{11}'/C_{21}' = L_{22}'/C_{12}' = R^2.$$

$$H = \tan \tfrac{1}{2}B = \frac{M_1 f + M_3 f^3}{1 + N_2 f^2 + N_4 f^4}.$$

Phase Linear Equation:

$$M_1 + f^2 M_3 - fHN_2 - f^3 HN_4 = H/f.$$

In physical solutions M_1 and N_4 are positive, as are also a_1, a_1', b_2, and b_2'. M_3 and N_2 are negative.

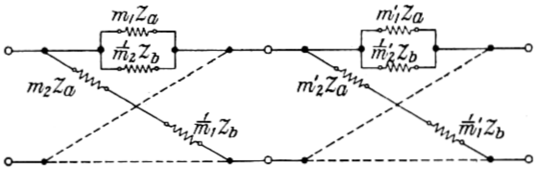
$$q^3 + 2N_2 q^2 + (-M_1 M_3 + N_2^2 - 4N_4)q + (M_1^2 N_4 - M_1 M_3 N_2 + M_3^2) = 0;$$

$$\left.\begin{matrix} a_1 \\ a_1' \end{matrix}\right\} = \tfrac{1}{2}(M_1 \pm \sqrt{M_1^2 - 4q});$$

$$\left.\begin{matrix} b_2 \\ b_2' \end{matrix}\right\} \text{ or } \left.\begin{matrix} b_2' \\ b_2 \end{matrix}\right\} = \tfrac{1}{2}(-(N_2 + q) \pm \sqrt{(N_2 + q)^2 - 4N_4}),$$

the determining condition being that $a_1 b_2' + a_1' b_2 = -M_3$.

Network 18



(To simulate a short symmetrical line or circuit)

Symmetrical Section of Line or Circuit:

$$X = \text{open-circuit impedance;}$$

$$Y = \text{short-circuit impedance;}$$

$$\tanh^{-1}\sqrt{Y/X} = \text{propagation length;}$$

$$\sqrt{XY} = \text{iterative impedance.}$$

Simulating Network:

$$z_a = \sqrt{XY}\tanh^{-1}\sqrt{Y/X};$$

$$z_b = \sqrt{XY}/\tanh^{-1}\sqrt{Y/X};$$

$$m_1 = .45737; \qquad m_2 = .14456; \qquad m_1' = .04263; \qquad m_2' = .92403.$$

The impedances z_a and z_b are to be realized in desired frequency ranges, more or less approximately, by comparatively simple physical networks.