The Present Status of Wire Transmission Theory and Some of its Outstanding Problems

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SYNOPSIS: The rapid development in the technique of wire transmission and the increasing complexity of the problems involved calls for a more adequate theoretical guide and a more rigorous transmission theory. This paper gives an account, practically without mathematics, of classical transmission theory and its limitations; of the several ways the problem may be attacked more fundamentally and rigorously, and the lines along which transmission theory must be extended, as the writer has come to view the problem in the light of his own experience.

IN the present paper the term *wire transmission theory* will be understood to mean the mathematical theory of guided wave propagation along a system of parallel conductors; which is supposed to be geometrically and electrically uniform throughout its length. The theory of wave propagation along such a system is of fundamental theoretical and practical importance to the communication engineer and presents some extremely interesting and difficult problems to the mathematician. The development of the elementary or classical theory will first be briefly sketched, after which the rigorous mathematical theory will be discussed together with some of the important unsolved problems.

Historically wire transmission theory goes back to the early work of Kelvin and Heaviside. It is based on the simple idea that a transmission line (say consisting of two similar and equal wires in which equal and opposite currents flow) can be represented as consisting of uniformly distributed series inductance and resistance and shunt capacitance and leakance, these concepts deriving from electrostatics and elementary circuit theory. In accordance with this idea, if X denote the axis of propagation, the current I and voltage V are related by the familiar equations

$$\left(L\frac{d}{dt} + R\right) I = -\frac{\partial}{\partial x} V, \tag{1a}$$

$$\left(C\frac{d}{dt} + G\right) V = -\frac{\partial}{\partial x} I. \tag{1b}$$

Writing these in the usual form

$$\begin{aligned} ZI &= -\frac{\partial}{\partial x} V, \\ YV &= -\frac{\partial}{\partial x} I, \end{aligned} \tag{2}$$

where Z is the uniformly distributed series impedance and Y the shunt admittance per unit length, it is easy to show that I and V satisfy the differential equations

$$\left(\gamma^2 - \frac{\partial^2}{\partial x^2}\right) I = 0,$$
$$\left(\gamma^2 - \frac{\partial^2}{\partial x^2}\right) V = 0, \tag{3}$$

where $\gamma^2 = ZY$. The solution of these equations is

$$I = Ae^{-\gamma x} - Be^{\gamma x},$$
$$V = kAe^{-\gamma x} + kBe^{\gamma x}. \tag{4}$$

$\gamma = \sqrt{ZY}$ is called the propagation constant and $k = \sqrt{Z/Y}$ the characteristic impedance of the line. A and B are integration constants which must be so chosen as to satisfy the boundary conditions (continuity of current and potential at the line terminals). The first term represents a direct wave, the second a reflected wave, their relative values depending on terminal reflections and the terminal impressed electromotive forces.

We see therefore that in accordance with elementary or classical transmission theory, the current and potential waves are both expressible as unique simple exponentially propagated direct and reflected waves, the values of which are determined by the continuity of current and potential at the line terminals. The characteristics of the line appear only through two parameters, the propagation constant γ and the characteristic impedance k.

Generalizing the preceding, consider a system of n parallel wires, parallel to the surface of the earth. The differential equations for such a system, in terms of elementary transmission theory, are [1]

$$\sum_{k=1}^{n} Z_{jk} I_k = -\frac{\partial}{\partial x} V_j \quad (j = 1, 2 \cdots n),$$
$$\sum_{k=1}^{n} Y_{jk} V_k = -\frac{\partial}{\partial x} I_j \quad (j = 1, 2 \cdots n). \tag{5}$$

Here the physical system is represented by the parameters Z_{jk} and Y_{jk}, the Z parameters being the series impedances (self and mutual) and the Y parameters the shunt admittances. If the differential operator $\partial/\partial x$ is replaced by γ, thus confining attention to *exponentially* propagated waves, and if either the potential V or the current I is

[1] See references 9 and 10.

eliminated from (5), we get a set of n homogeneous equations in I or V, the determinant of which must vanish for a non-trivial solution. This determinant is a function of γ^2 and it has in general n roots in γ^2, indicating n possible modes of propagation, with corresponding characteristic impedances k_{jk}. The general solution is then of the form

$$\begin{aligned} I_j &= \sum A_{jk}e^{-\gamma_k x} - B_{jk}e^{\gamma_k x}, \\ V_j &= \sum k_{jk}A_{jk}e^{-\gamma_k x} + k_{jk}B_{jk}e^{\gamma_k x}. \end{aligned} \tag{6}$$

Here A_{jk}, B_{jk} are integration constants, the number of independent constants being $2n$. These are determined by the $2n$ boundary conditions at the physical terminals of the system; that is, the continuity of the n currents and n potentials. The solution represents n direct and n reflected current waves, which in general are propagated with different attenuations and different phase velocities.

The conclusions derivable from the classical theory sketched above may be summarized as follows: In a system of n parallel conductors there are in general n modes of propagation, that is, n direct and n reflected waves, which may be termed the normal modes of propagation. The distribution of the wave energy among the n modes of propagation or n component waves, is determined by the boundary conditions, which are essentially the continuity of the currents and potentials of the n wires. The system is supposed to be completely specified by the self and mutual series impedances and the self and mutual shunt admittances of the n conductors, while in the solution for the waves the physical and electrical characteristics of the line enter only through the propagation constants and corresponding characteristic impedances.

Before analyzing the theoretical basis of the preceding elementary theory, and showing its limitations, an interesting and practically important extension will be briefly touched on. The equations of the theory given above presuppose that the *impressed* electromotive forces are concentrated at the terminals of the system, and that in the line itself the electric and magnetic fields are due entirely to the currents and charges of the conductors, and consequently that the distribution of current and charge is determined entirely by their own fields. Suppose, however, that the system is in addition exposed throughout its length to an impressed field, from some disturbing source; then the preceding theory must be modified to take into account the effect of this additional field. To take the simplest case, consider a single wire parallel to the surface of the earth (ground return circuit). Let us suppose that this wire is exposed to an arbitrary field specified by an axial electric force f at the surface of the wire and an impressed potential F (line integral of electric force to ground). The differential equations are then [2]

$$
\begin{aligned}
ZI &= -\frac{\partial}{\partial x} V + f, \\
YV &= -\frac{\partial}{\partial x} I - gF.
\end{aligned}
\tag{7}
$$

(Here g is a shunt admittance. See reference 10.) Writing

$$
\begin{aligned}
\gamma &= \sqrt{(Li\omega + R)(Ci\omega + G)}, \\
k &= \sqrt{\frac{Li\omega + R}{Ci\omega + G}},
\end{aligned}
$$

this reduces to the differential equation

$$
\left(\gamma^2 - \frac{\partial^2}{\partial x^2}\right) I = \frac{\gamma}{k} f + g \frac{\partial}{\partial x} F. \tag{8}
$$

The solution of this equation and its practical significance have been discussed in a recent paper. The resulting analysis is of considerable practical importance in connection with the theory and design of the wave antenna and the problems of 'cross-talk' and induction.

The elementary or classical theory sketched above is essentially based on the simple concepts of electric circuit theory and its beautiful simplicity is a consequence of the fact that it is approximate only. For example the circuit parameters are only approximately calculable from the geometry of the system and its electrical constants and then only when the problem is treated as a two-dimensional one in which the variation of current and charge along the system is ignored as well as the finite velocity of propagation of their fields. Going further, it is by no means evident that even the *form* of the equations is rigorous. (We shall find that the form is rigorously valid only in an ideal case.) In the extension of elementary transmission theory, then, the first problem, as the writer sees it, is to examine the conditions under which the specification of the system by series impedance and shunt admittance parameters is justified; that is, to establish the conditions under which the classical *form* of the differential equations is valid. The second phase of this problem is to formulate a general method for calculating these circuit parameters in terms of the geometry and electrical constants of the system. The investigation of these problems leads to still further problems, arising from the fact

[2] See reference (10).

that the solutions of elementary theory, when valid, are only *particular solutions*, and therefore do not, in general, represent the complete wave.

In taking up this problem it is necessary to discard the simple concepts underlying classical transmission theory and attack the problem, *ab initio*, by aid of Maxwell's equation. Otherwise stated, our problem is to find solutions of the wave equation which satisfy the boundary conditions at the surfaces of the conductors, that is, the continuity of the tangential component of E and H, and therefore represent physically possible waves.

To put the matter otherwise, we shall place ourselves in the position of a mathematician, unacquainted with circuit theory or classical transmission theory, for whom the laws governing propagation of electromagnetic waves are formulated only by Maxwell's equations. His procedure in developing the theory of transmission along wires would be totally different from the way the theory has actually been developed. Starting with Maxwell's equations he would find that the electric and magnetic vectors satisfy a partial differential equation called the *wave equation*. He would then search for particular solutions of the wave equation which satisfy the geometry and electrical constants of the system, and therefore represent physically possible waves. The results of such a mode of approach to the problem are sketched below.

To formulate the problem concretely, consider a system of n parallel conductors, parallel to the (plane) surface of the earth, and extending along the positive X axis. The conductors may have any cross-sectional shape desired, but it is expressly assumed that they do not vary electrically or geometrically along the axis of transmission X (except at points of discontinuity or the terminals); that is to say, the transmission system is *uniform* along the axis of transmission.

Now in any medium of conductivity σ, permeability μ and dielectric constant ϵ, the electric and magnetic vectors satisfy the *wave equation*[3]

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\nu^2\right)F=0, \tag{9}$$

where

$$\begin{aligned} \nu^2 &= 4\pi\sigma\mu i\omega - \omega^2/v^2,\\ v &= 1/\sqrt{\epsilon\mu},\\ \omega &= 2\pi \text{ times the frequency},\\ i &= \sqrt{-1}, \end{aligned} \tag{10}$$

and F may be any component electric or magnetic vector.

[3] See reference (11).

We now suppose that solutions of the type

$$F = f(y, z)e^{(i\omega t - \gamma x)} \tag{11}$$

exist, where $f(y, z)$ is a two-dimensional wave function satisfying the two-dimensional wave equation

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f = (\nu^2 - \gamma^2) f. \tag{12}$$

In other words we search for *exponentially* propagated waves of this type; that is, waves which involve the spatial coordinate x only exponentially. It is well known that solutions of this type exist when the transmission system is uniform along the X axis.

The mathematical analysis of the problem outlined above is dealt with in detail in my paper 'The Rigorous and Approximate Theories of Electrical Transmission along Wires' (ref. 11) and the outstanding conclusions of that analysis are as follows:

The *form* of the differential equations of classical transmission theory is rigorously valid, that is, the system is specified rigorously by its self and mutual series impedances and shunt admittances, only for the ideal case of a system consisting of perfect conductors embedded in a perfect dielectric. In this case $\nu^2 - \gamma^2 = 0$ in the dielectric; $\nu^2 = \infty$ in the conductors, and the propagation constant γ is $i\omega/v$, indicating unattenuated transmission with the velocity of light, $v = 1/\sqrt{\epsilon\mu}$. The wave is a pure plane guided wave, and the electric and magnetic fields are derivable from two wave functions, one a linear function of the conductor charges and the other a linear function of the conductor currents, the determination of which, in terms of the geometry of the system, is reduced to the solution of a well-known potential problem.

Such a system, the ideal for guided wave transmission, is of course unrealizable, since there are always losses in both conductors and dielectric. For efficient transmission, however, the losses must be small and the guided wave must approximate the plane wave of the ideal case. Let us suppose, therefore, that the losses in the system are so small that *in the dielectric*, in the neighborhood of the conductors, we can set $\nu^2 - \gamma^2 = 0$, and that *in the conductors* the conductivity is so high that $\nu^2 - \gamma^2$ may be replaced by ν^2 without appreciable error. Under the circumstances where these approximations are valid it is found that the electric and magnetic fields in the dielectric and the current distribution over the cross-sections of the conductors are likewise derivable from two wave functions which are linear functions

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of the conductor charges and currents respectively. The first of these is determined in terms of the geometry of the system by the solution of the same two-dimensional potential problem as in the ideal case, while the second is determined in terms of the geometry and electrical constants of the system, by a generalized two-dimensional potential problem.[4] Otherwise stated, to the approximations explained above the system may be regarded as specified by self and mutual series impedances and self and mutual shunt admittances, and these are calculable by the solution of the two-dimensional potential problems. The solution of the differential equations leads, precisely as in the classical theory, to an nth order equation in γ^2, indicating n modes of propagation. Moreover, the n corresponding waves, which will, for reasons explained below, be termed the *principal waves*, are *quasi-plane*. This means that, in the dielectric the axial electric intensity is in general small compared with the electric intensity in the plane normal to the axis of transmission; or, more broadly stated, the departure of the waves from true planarity is due entirely to dissipation in conductors and dielectric. A plane wave is here understood to mean a wave in which $E_x = H_x = 0$.

Now it is important to observe that in arriving at the foregoing result we have introduced at the outset approximations and assumptions regarding the order of magnitude of the propagation constant γ which depend on the assumption that the transmission losses are small. Fortunately these assumptions are justified, and the resulting approximate solutions are valid to a high degree of accuracy, in those systems which can be employed for the *efficient* guided transmission of electromagnetic energy; otherwise stated, the mathematical restrictions correspond to the actual requirements for efficient transmission. If, however, either the conductors or the dielectric become sufficiently imperfect, the approximations introduced and the resulting wave solutions become increasingly inaccurate and unreliable.

Suppose now that we attack the problem in a still more fundamental way: discard the assumptions regarding the order of magnitude of γ, introduced above, and attempt to deal with the problem and the solution of the wave equation in its general form. The case then is entirely different and vastly more complicated. In general, the solution can not be carried out, but a few simple systems have been studied and the results of this analysis may be generalized as follows:[5] in a system of n parallel conductors there exist, in addition to the n principal modes of propagation, an n-fold infinity of other modes of propagation,

[4] See reference (8).
[5] See reference (5).

which will be termed *complementary* modes of propagation. In general, the corresponding *complementary* waves differ from the *principal* waves in that they are not quasi-plane and are very rapidly attenuated. Consequently it appears that as regards the currents and charges, and the fields *near the conductors*, the effect of the complementary waves is usually appreciable only in the neighborhood of the physical terminals of the system so that at a distance from the terminals, usually small, they are represented with sufficient accuracy by the *principal* waves alone. At a great distance from the conductors, however, it appears that the errors resulting from ignoring the fields of the *complementary* waves may be large; in fact the complementary waves must be expressly included to take into account the phenomena of radiation.

The practical as distinguished from the theoretical importance of the foregoing resides in the fact that the principal waves corresponding to those of elementary theory represent the transmission phenomena accurately only at some distance from the physical terminals of the line and then only in the neighborhood of the wires. This defect may be of small practical consequence when the conductors all consist of wires of small cross section. When, however, conductors of large cross sections, or the ground, form part of the transmission system, the theory may be quite inadequate for some purposes. In particular, in calculating inductive disturbances in neighboring transmission systems at a considerable distance it may lead to large errors.

The discussion given above is based in part on a mathematical analysis of simple representative systems, in part on inferences from physical considerations. Unfortunately a direct frontal attack and rigorous solution of the general problem appears impossible. For example, in addition to finding the infinitely many modes of propagation the corresponding infinitely many complementary waves must be so chosen as to satisfy the boundary conditions at the physical terminals. In the classical theory these boundary conditions are simply the continuity of currents and potentials; in the rigorous formulation of the problem they are the continuity of E_y, E_z, H_y, H_z throughout the entire boundary plane ($x = 0$). Even to formulate these conditions involves specifying the impressed field throughout the plane and this is never given explicitly in technical transmission problems. While, therefore, the theory sketched above leads to inferences and conclusions of importance, the writer is convinced that some more powerful and indirect mode of attack on the problem must be devised; a rather hopeful possibility along this line will be briefly described.

As stated above, it is a reasonable inference from the general theory, that the complementary waves modify the *current* and *charge* waves appreciably only in the immediate neighborhood of the physical terminals, at least in most actual transmission systems. The essence of the method to be described consists of taking advantage of this fact and directly calculating the fields of the principal current and charge waves by means of their retarded potentials,[6] instead of employing for calculations the principal wave fields as given by the solution of the wave equation. This will now be explained in more detail.

In any transmission system energized by impressed forces introduced through terminal networks, the electromagnetic field may be analyzed as follows: (1) the impressed field, (2) the field of the terminal currents and charges, and (3) the field of the line currents and charges proper. The impressed field may be supposed to be concentrated in the terminal network, and the field of the terminal currents and charges may be supposed to be relatively unimportant except in the neighborhood of the terminals; what we are essentially concerned with is the field of the line currents and charges. Now let us suppose that we have calculated the principal wave in the system in the usual manner; corresponding to the resulting current and charge distribution, there will then be an unique corresponding field distribution determined by the solution of the wave equation, and this field is propagated in precisely the same way as the currents. But now suppose that we calculate the field of this current and charge distribution directly by means of their retarded potentials. We will find that the field so calculated is analyzable into two components: (1) a field identical with that given by the solution of the wave equation, and propagated in the same manner as the currents and charges, and (2) an additional field propagated in an entirely different way and for systems of small dissipation much more rapidly attenuated at least in the neighborhood of the conductors. We find further that the field of the principal current and charge wave does not correctly satisfy the boundary conditions at the surfaces of the conductors, which indicates that there must exist a compensating current and charge distribution. However, it appears that this compensating distribution will be relatively small and concentrated in the neighborhood of the terminals, so that we infer that its field, as calculated from its retarded potentials, can be ignored. Under such circumstances the inductive field (and the radiation field) is calculable by means of the retarded potentials in terms of the principal wave of current and charge alone.

[6] See reference (7).

To recapitulate this mode of attack, first determine the distribution of line currents and charges by means of elementary theory; that is, determine the principal wave distribution of currents and charges. Secondly, calculate the field of this current and charge distribution by means of the retarded potentials. This will give in addition to the field calculable from elementary theory an additional field the existence of which is not recognized by elementary theory. In brief, this mode of attack is based on the argument that the actual distribution of current and charge in the system is given with sufficient accuracy by elementary theory, but that in calculating the field at a distance, corrections must be introduced.

As might be expected this mode of attack presents formidable difficulties particularly when the ground plays an important rôle in the transmission phenomena. On the other hand, the analysis of a few of the simplest cases has been quite encouraging and leads one to hope that the method may at least be successfully applied to calculating the orders of magnitude of corrections which must be introduced in such important problems as, for example, inductive disturbances, in neighboring transmission systems.

The foregoing may appear to many as highly academic and theoretical. The writer's actual experience with practical transmission problems has convinced him, however, that the extension of wire transmission theory along the lines indicated above is urgently needed.

References

The papers listed below represent recent work which deals directly or indirectly with the problems discussed in the text. The relatively large number of the writer's own papers which are listed merely reflects the fact that very few specialists are working on the advanced problems of wire transmission theory.

1. "Radiation from Transmission Lines." (Carson, *Jour. A. I. E. E.*, Oct., 1921.)
2. "Radiation from Transmission Lines." (Manneback, *Trans. A. I. E. E.*, 1923.)
3. "A Generalization of the Reciprocal Theorem." (Carson, *B. S. T. J.*, July, 1924.)
4. "Das Reziprotät Theorem der drahtlosen Telegraphie." (Sommerfeld, *Jahrb. d. drahtl. Tel. u. Tel.*, 1925.)
5. "The Guided and Radiated Energy in Wire Transmission." (Carson, *Trans. A. I. E. E.*, 1924.)
6. "Über das Feld einer Unendlich langen Wechselstromdurchflossenen Einfachleitung." (Pollaczek, *E. N. T.*, 3, 1926.)
7. "Electromagnetic Theory and the Foundations of Electric Circuit Theory." (Carson, *B. S. T. J.*, Jan., 1927.)
8. "A Generalized Two-Dimensional Potential Problem." (Carson, *Bull. Am. Math. Soc.*, May–June, 1927.)
9. "Electromagnetic Waves, Guided by Parallel Wires." (Levin, *Trans. A. I. E. E.*, 1927.)
10. "Propagation of Periodic Currents over a System of Parallel Wires." (Carson and Hoyt, *B. S. T. J.*, July, 1927.)

11. "The Rigorous and Approximate Theories of Electrical Transmission along Wires." (Carson, *B. S. T. J.*, Jan., 1928.)
12. As a general reference, the treatise "Electrical and Optical Wave Motion," by Bateman, published by the Cambridge University Press, may be consulted with profit.

Appendix

The mode of attack outlined in the latter part of the text will be illustrated by an application to the simplest possible case.

Let the transmission system consist of a wire of radius a whose axis coincides with the X axis, and a coaxial cylinder of internal radius b. Both conductors are supposed to be perfectly conducting, while the dielectric in the space between $(a \leq \rho \leq b)$ is supposed to be perfect. For this system we know that the principal wave is transmitted without attenuation with the velocity of light c; that is to say, $\gamma = i\omega/c$, where ω is 2π times the frequency.

We suppose that the system extends for an indefinite distance along the positive X axis so that reflected waves are absent. The principal current and charge waves are then:

$$I = I_0 e^{-i\beta x}, \qquad Q = Q_0 e^{-i\beta x}, \tag{1a}$$

where β denotes ω/c, and $i = \sqrt{-1}$. From the relation

$$\frac{1}{c}\frac{\partial}{\partial t} Q = -\frac{\partial}{\partial x} I$$

it follows that

$$Q = I. \tag{2a}$$

Now by definition the retarded potentials are

$$\Phi = \int \frac{q}{r} e^{-i\beta r} dv \quad \text{(Scalar)},$$

$$A = \int \frac{u}{r} e^{-i\beta r} dv \quad \text{(Vector)},$$

where q and u denote the charge and vector current density respectively, r is the distance between the contributing element dv and the point at which the potential is to be calculated, and the integration is extended over the entire system of currents and charges. In terms of the retarded potentials the magnetic and electric intensities E and H are given by

$$\begin{aligned} H &= \text{curl } A, \\ E &= -\text{ grad } \Phi - i\beta A. \end{aligned} \tag{3a}$$

To formulate the retarded potentials of the system under consideration we have recourse to the Sommerfeld integral

$$\frac{e^{-i\beta r}}{r} = \int_0^\infty J_0(\rho\lambda) e^{-|x-x'|\sqrt{\lambda^2-\beta^2}} \frac{\lambda d\lambda}{\sqrt{\lambda^2-\beta^2}}, \tag{4a}$$

where $\rho = \sqrt{y^2 + z^2}$ and J_0 is the Bessel function in the usual notation.

Applying this integral to the system of currents and charges under consideration, and remembering that they are surface currents and charges at $\rho = a$ and $\rho = b$ respectively, we get without difficulty, for $x \geqq 0$,

$$\begin{aligned} \Phi = Q_0 \int_0^\infty J_0(\rho\lambda)[J_0(a\lambda) - J_0(b\lambda)] e^{-x\sqrt{\lambda^2-\beta^2}} \frac{\lambda d\lambda}{\sqrt{\lambda^2-\beta^2}} \\ \times \int_0^x e^{x'[\sqrt{\lambda^2-\beta^2}-i\beta]} dx' \\ + Q_0 \int_0^\infty J_0(\rho\lambda)[J_0(a\lambda) - J_0(b\lambda)] e^{x\sqrt{\lambda^2-\beta^2}} \frac{\lambda d\lambda}{\sqrt{\lambda^2-\beta^2}} \\ \times \int_x^\infty e^{-x'[\sqrt{\lambda^2-\beta^2}+i\beta]} dx', \end{aligned} \tag{5a}$$

which reduces to

$$\begin{aligned} \Phi = 2Q_0 e^{-i\beta x} \int_0^\infty J_0(\rho\lambda)[J_0(a\lambda) - J_0(b\lambda)] \frac{d\lambda}{\lambda} \\ - Q_0 \int_0^\infty J_0(\rho\lambda)[J_0(a\lambda) - J_0(b\lambda)] \frac{e^{-x\sqrt{\lambda^2-\beta^2}}}{\sqrt{\lambda^2-\beta^2} - i\beta} \frac{\lambda d\lambda}{\sqrt{\lambda^2-\beta^2}}. \end{aligned} \tag{6a}$$

Since the currents are entirely axial, we have also $A_y = A_z = 0$, and from (2a)

$$A_x = \Phi. \tag{7a}$$

The first integral in Φ represents a potential wave propagated along the X axis in precisely the same way as the current and charge; it will therefore be termed the *homogeneous* potential wave. We find further that the field derivable from the *homogeneous* potentials is precisely the *principal wave* field, as given by the particular solution of Maxwell's equation, corresponding to $\gamma = i\beta$.

The second integral in Φ represents a potential wave propagated in an entirely different manner, and dying away for sufficiently large values of x. The corresponding field may be called, for want of a better term, the *heterogeneous* field, since its mode of propagation is quite different from that of the current and charge. It is this field which represents the correction which must be added to the field of elementary theory.

It is beyond the scope of this brief appendix to discuss this solution in detail. It may be said, however, that, while the integrals representing the heterogeneous field can not be solved in finite terms, their properties can be approximately and qualitatively deduced without much difficulty.

One point of interest may be noted; the homogeneous wave is plane, that is, the axial electric intensity is everywhere zero. If we apply to the preceding formulas the relation

$$E_x = -\frac{\partial}{\partial x}\Phi - i\beta A_x$$
$$= -\left(\frac{\partial}{\partial x} + i\beta\right)\Phi,$$

we get, for the heterogeneous field,

$$E_x = -Q_0\int_0^\infty J_0(\lambda\rho)[J_0(\lambda a) - J_0(\lambda b)]e^{-x\sqrt{\lambda^2-\beta^2}}\frac{\lambda d\lambda}{\sqrt{\lambda^2-\beta^2}}.$$

The integral term in this expression is simply the retarded potential of a ring of point sources located on the circle $x = 0$, $\rho = a$ minus the retarded potential of a corresponding ring of point charges located on the circle $x = 0$, $\rho = b$. Since this field does not vanish at the conductor surfaces $\rho = a$ and $\rho = b$, it is clear that a compensating charge and current distribution must exist.