The Rigorous and Approximate Theories of Electrical Transmission Along Wires

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THE theory of electrical transmission along straight parallel guiding conductors is of fundamental importance to the communication engineer. In its original, and largely in its present day form, it involves only relatively simple concepts which go back to the early work of Kelvin and Heaviside. In accordance with these concepts the transmission phenomena are completely determined by the self and mutual impedances of the conductors and the self and mutual capacities (together with the dielectric leakage). As a consequence, the phenomena are completely expressed in terms of the propagation constants and corresponding characteristic impedances of the possible modes of propagation deducible from these underlying concepts.

The elementary theory sketched above is of beautiful simplicity and great value. It is, however, admittedly approximate, and in two respects is not altogether adequate. Its first defect is that it represents the transmission phenomena correctly only at some distance from the physical terminals of the system or at some distance from points of discontinuity. This defect is ordinarily of small practical significance when the conductors all consist of wires of small cross section. When, however, conductors of large cross sections, or the ground, form part of the transmission system, the elementary theory may be quite inadequate. The theoretical questions here involved were briefly discussed by the writer in a previous paper.[1] The mathematics involved in this problem are extremely complicated and the further work of the writer has not as yet been carried to a point which justifies publication.

With the extension of transmission theory discussed in the preceding paragraph the present paper has no concern, and it is to be expressly understood that we are dealing with the transmission phenomena at a sufficient distance from the physical terminals, such that the "end effects" are negligible. The problems here dealt with may be stated as follows: First to investigate the conditions under which the specification of the system by means of its self and mutual impedances is valid and secondly to provide a general method for calculating these circuit parameters from the geometry and electrical constants of the system.

[1] "The Guided and Radiated Energy in Wire Transmission." Trans. A. I. E. E., 1924.

As regards the first phase of this problem it is found that the complete specification of the system in terms of its self and mutual impedances and capacities is only rigorously valid for the ideal case of perfect conductors embedded in a perfect dielectric, and that it becomes quite invalid if either the conductors or the dielectric are too imperfect. Fortunately, however, it is valid to a high degree of approximation for all systems which could be employed for the *efficient* transmission of electrical energy.

Under the circumstances where the approximations discussed in the preceding paragraph are valid it is shown that the electric and magnetic field in both dielectric and conductors are derivable from two wave functions. The first of these is determined as a linear function of the conductor charges by the solution of a well-known two-dimensional potential problem, while the second is determined as a linear function of the conductor currents by the solution of a generalization of the two-dimensional potential problem. The latter problem is believed to be novel, in its general form, and to possess both practical and mathematical interest. For detailed application of the theory to specific problems, the following papers may be consulted.

"Wave Propagation over Parallel Wires: The Proximity Effect." *Phil. Mag.*, April 1921.
"Transmission Characteristics of the Submarine Cable." *Jour. Frank. Inst.*, Dec. 1921.
"Wave Propagation in Overhead Wires with Ground Return." *B. S. T. J.*, Oct. 1926.

I

Maxwell's equations are the set of partial differential equations which formulate the relations between the electric intensity E and the magnetic intensity H in terms of the frequency $\omega/2\pi$ and the electrical constants of the medium. Let λ, μ and k denote the conductivity, permeability and dielectric constant of the medium; let it be supposed that all quantities vary with the time t as $e^{i\omega t}$, and let

$$v = 1/\sqrt{k\mu},$$
$$\nu^2 = 4\pi\lambda\mu i\omega - \omega^2/v^2,$$
$$i = \sqrt{-1}.$$

Then if we introduce the vector

$$M = \mu i\omega \cdot H,$$

Maxwell's equations for a *continuous homogeneous* medium may be written in the compact form [2]

$$\begin{aligned} \text{curl } E &= -M, \\ \text{curl } M &= \nu^2 E, \\ \text{div } E &= 0, \\ \text{div } M &= 0. \end{aligned} \tag{1}$$

From this set of equations it is easily shown that each component of the vectors E and M individually satisfies the *wave equation*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \nu^2\right) f = 0 \tag{2}$$

or in vector notation

$$(\nabla^2 - \nu^2) f = 0.$$

Here f denotes any vector component; thus in Cartesian coordinates f may stand for E_x, E_y, E_z; M_x, M_y, M_z, all of which separately satisfy (2).

Given the electrical constants and geometry of the conducting system and dielectric media, the general problem is to find solutions of (1) and (2) which also satisfy the *boundary conditions* at the surfaces of separation of the different media. These boundary conditions are that the tangential components of E and H shall be continuous over such surfaces of separation. These boundary conditions, as may be seen from (1), necessitate also the continuity of the *normal* components of M and $(\nu^2/\mu)E$.

If we introduce a vector potential $A(A_{x,\,y,\,z})$ and a scalar potential Φ, it is easily shown that (1) may be replaced by

$$\begin{aligned} M &= \text{curl } A, \\ E &= -\dot{A} - \text{grad } \Phi, \end{aligned} \tag{3}$$

with the further relation

$$\text{div } A + \nu^2 \Phi = 0. \tag{4}$$

Φ and the components of the vector A individually satisfy the wave equation; thus

$$\begin{aligned} (\nabla^2 - \nu^2)\Phi &= 0, \\ (\nabla^2 - \nu^2)A &= 0. \end{aligned} \tag{5}$$

In Cartesian coordinates these equations are

[2] Note that in this form the constants of the medium appear explicitly only through the parameter ν^2.

$$
\begin{aligned}
M_x &= \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y, \\
M_y &= \frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z, \\
M_z &= \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x, \\
E_x &= - A_x - \frac{\partial}{\partial x} \Phi, \\
E_y &= - A_y - \frac{\partial}{\partial y} \Phi, \\
E_z &= - A_z - \frac{\partial}{\partial z} \Phi,
\end{aligned}
\tag{6}
$$

and

$$
\frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z + \nu^2 \Phi = 0. \tag{7}
$$

In technical transmission problems we are largely concerned with propagation along a uniform transmission system, composed of straight parallel conductors. That is to say, the transmission system does not vary geometrically or in its electrical constants along the axis of transmission, taken as the axis of Z. It is known that in such transmission systems *exponentially* [3] propagated waves exist. We therefore modify the general equations by assuming that the wave (and all vector components) vary with t and z as $\exp\,(i\omega t - \gamma z)$, γ being entitled the *propagation constant*. As a consequence of this assumption it is easily shown that the vectors E and M are derivable from the wave functions F, Φ, Θ as follows:

$$
\begin{aligned}
M_x &= \frac{\partial}{\partial y} F - \gamma \frac{\partial}{\partial x} \Theta, \\
M_y &= - \frac{\partial}{\partial x} F - \gamma \frac{\partial}{\partial y} \Theta, \\
M_z &= - (\nu^2 - \gamma^2) \Theta, \\
E_x &= - \frac{\partial}{\partial x} \Phi - \frac{\partial}{\partial y} \Theta, \\
E_y &= - \frac{\partial}{\partial y} \Phi + \frac{\partial}{\partial x} \Theta, \\
E_z &= - \frac{\partial}{\partial z} \Phi - F = \gamma \Phi - F.
\end{aligned}
\tag{8}
$$

The *wave functions* F and Φ are not independent but are connected by the relation

$$
\nu^2 \Phi = \gamma F. \tag{9}
$$

[3] This means that the wave involves the axial coordinate z only exponentially.

Another useful formulation of the field equations equivalent to and directly deducible from (8) is

$$
\begin{aligned}
(\nu^2 - \gamma^2)M_x &= -\nu^2 \frac{\partial}{\partial y} E_z + \gamma \frac{\partial}{\partial x} M_z, \\
(\nu^2 - \gamma^2)M_y &= \nu^2 \frac{\partial}{\partial x} E_z + \gamma \frac{\partial}{\partial y} M_z, \\
(\nu^2 - \gamma^2)E_x &= \gamma \frac{\partial}{\partial x} E_z + \frac{\partial}{\partial y} M_z, \\
(\nu^2 - \gamma^2)E_y &= \gamma \frac{\partial}{\partial y} E_z - \frac{\partial}{\partial x} M_z.
\end{aligned} \tag{10}
$$

In this formulation the problem is reduced to the determination of the *wave functions* E_z and M_z, and the *propagation constant* γ.

It will be observed that, by virtue of the assumption that the wave functions of (8), (9) and (10) involve t and z only through the common factor $\exp(i\omega t - \gamma z)$, we can write

$$
\begin{aligned}
F &= f(x, y) \cdot \exp(i\omega t - \gamma z), \\
\Phi &= \phi(x, y) \cdot \exp(i\omega t - \gamma z), \\
E &= e(x, y) \cdot \exp(i\omega t - \gamma z), \text{ etc.,}
\end{aligned} \tag{11}
$$

where f, ϕ, e, etc., are two-dimensional functions of x and y alone, and satisfy the two-dimensional wave equations

$$
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f = (\nu^2 - \gamma^2) f, \text{ etc.} \tag{12}
$$

In the following, therefore, we shall regard the wave functions F, Φ, E, etc., as two-dimensional functions with the understanding that the common factor $\exp(i\omega t - \gamma z)$ is omitted for convenience.

II

Before taking up the discussion of the general problem in the light of equations (8) and (10) we shall first consider a type of plane wave propagation to which the transmission phenomena closely approximate in an efficient transmission system. We consider the ideal transmission system composed of any number of straight parallel *perfectly conducting* conductors imbedded in a *perfect* dielectric. For such a system we *assume* the possibility of plane wave propagation by supposing that E_z and M_z are everywhere zero. By virtue of the assumption of perfect conductivity, the electric force must vanish inside the conductors, and at the surface the tangential component must vanish. *In the dielectric* reference to equations (10) shows that if $E_z = M_z = 0$, a finite solution requires that

$$\nu^2 - \gamma^2 = 0$$

or, since $\lambda = 0$ in the dielectric,

$$\gamma = i\omega/v.$$

That is to say, the plane wave is propagated with the velocity of light v, without attenuation.

Reference to equations (8) and (9) shows that the boundary conditions can be satisfied by setting $\Theta = 0$, writing

$$\Phi = \phi \cdot \exp\,(i\omega t - i\omega z/v),$$

and determining the function ϕ which satisfies Laplace's equation in two dimensions,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0,$$

and is constant over the cross section of the conductors.

From the relation $F = (i\omega/v)\Phi$ it is also easily shown that the electric and magnetic forces are both in planes normal to Z and that these vectors are normal to each other and in time phase. The flow of energy is therefore parallel to the Z-axis everywhere. We therefore have a pure plane guided wave of unit power factor; the ideal for the electrical transmission of energy.

III

We now take up the much more complicated problem arising when the conductivity λ of the conductors is finite and when the dielectric media themselves may be dissipative. In attacking this general problem we shall be guided throughout by the fact that the wave solution we are seeking must approximate, more or less closely, to the ideal plane wave [4] if the system is to efficiently transmit electrical energy. We shall therefore introduce *ab initio* approximations which must be valid in all efficient transmission systems. These approximations cannot be all justified *a priori;* their justification must come *a posteriori* from the fact that the final solution satisfies the original assumptions and approximations.

[4] It is to be noted that the solution sought is the *principal wave.* (See "The Radiated and Guided Energy in Wire Transmission," *Trans. A. I. E. E.*, 1924.) This wave does not, in general, completely represent the phenomena, except at a considerable distance from the physical terminals of the transmission system, and then only in the neighborhood of the conductors.

First we have to define what we mean by conductor and by dielectric; the significance of these definitions will appear in the course of the analysis. *A conducting medium* is one in which ω^2/v^2 is very small compared with $4\pi\lambda\mu\omega$; while a *dielectric medium* is one in which $4\pi\lambda\mu\omega$ is very small compared with ω^2/v^2. The intermediate cases will not be discussed in the present paper; in the following it will be assumed that the conductors and dielectrics satisfy these definitions.[5]

The assumptions which we make at the outset in the approximate solution may now be listed and qualitatively justified as follows:

1. The propagation constant γ is an extremely small quantity and its real part is not large compared with its imaginary part. Since $|\gamma|$ is of the order of magnitude of $\omega \cdot 10^{-10}$, it is evident that γ is very small even for frequencies of millions of cycles per second. As regards the second restriction, if the real part of γ is large compared with the imaginary, the wave will be damped out in a few wave-lengths, and the system cannot efficiently transmit energy.

2. *In the conductors* the axial electric intensity E_z is large compared with the component normal to Z. This restriction means that the dissipation in the conductors due to the axial currents is large compared with the dissipation due to the charging currents. Evidently this restriction is necessary for the efficient transmission of energy.

3. *In the dielectric* the axial electric intensity is small compared with the normal electric intensity. The justification of this assumption is as follows: The propagation of energy occurs in the dielectric, and is normal to the direction of the electric intensity. Since the usefully transmitted energy is propagated along the axis of transmission and the propagation normal to the axis simply means dissipation, the axial electric intensity must be small compared with the normal component for efficient transmission.

4. The axial magnetic intensity H_z is everywhere small compared with the normal intensity. The justification of this assumption depends on the same arguments as (3).

As regards (3) and (4) it will be remarked that in the ideal plane wave propagation both E_z and M_z are zero. In the case of imperfect conductors E_z in the dielectric is not zero but may be regarded as a first order small quantity. M_z on the other hand is to be regarded as a second order small quantity because it not only vanishes for the case of perfect conductors but also vanishes for the case of imperfect conductors for the case where the wave is made up of a set of compo-

[5] In accordance with these definitions, conductors and dielectrics depend for their classifications on the frequency, as well as their electrical constants. The definition of *conductor* means that the displacement current is negligible compared with the conduction current.

2

nent radially symmetrical waves oriented on the axes of the conductors; to this the actual wave approximates in important transmission systems.

We shall now introduce the consequences of the foregoing assumptions into the differential equations of the problem.

IV

Referring to equations (10), these may be replaced *in the conductors only* where γ^2 is very small compared with ν^2 and γ is a very small quantity, by the approximation:

$$\begin{aligned} M_x &= -\frac{\partial}{\partial y}E_z, \\ M_y &= \frac{\partial}{\partial x}E_z, \\ E_x &= \frac{\gamma}{\nu^2}\left\{\frac{\partial}{\partial x}E_z + \frac{1}{\gamma}\frac{\partial}{\partial y}M_z\right\}, \\ E_y &= \frac{\gamma}{\nu^2}\left\{\frac{\partial}{\partial y}E_z - \frac{1}{\gamma}\frac{\partial}{\partial x}M_z\right\}, \end{aligned} \tag{13}$$

Therefore *in the conductors* the vector components M_x, M_y are derivable by spatial differentiation from E_z. E_x, E_y are not in general so derivable on account of the factor $1/\gamma$, a very large quantity, which appears with M_z. (It appears that γE_z and M_z may be of comparable orders of magnitude.) We assume, however, for reasons discussed above, that both E_x and E_y are very small compared with E_z *in the conductors*.

In the dielectric, where ν^2 and γ^2 are of comparable orders of magnitude, the foregoing approximations are not valid and the rigorous equations must be employed. Returning to equations (10) and writing for convenience $\gamma^2/\nu^2 = \beta$, we have

$$\begin{aligned} M_x &= -\frac{1}{1-\beta}\left\{\frac{\partial}{\partial y}E_z - \frac{\beta}{\gamma}\frac{\partial}{\partial x}M_z\right\}, \\ M_y &= \frac{1}{1-\beta}\left\{\frac{\partial}{\partial x}E_z + \frac{\beta}{\gamma}\frac{\partial}{\partial y}M_z\right\}, \\ E_x &= \frac{\beta}{\gamma}\frac{1}{1-\beta}\left\{\frac{\partial}{\partial x}E_z + \frac{1}{\gamma}\frac{\partial}{\partial y}M_z\right\}, \\ E_y &= \frac{\beta}{\gamma}\frac{1}{1-\beta}\left\{\frac{\partial}{\partial y}E_z - \frac{1}{\gamma}\frac{\partial}{\partial x}M_z\right\}. \end{aligned} \tag{14}$$

In equations (13) and (14), x and y may be any orthogonal coordinate system. Let us suppose that they are so chosen that x is tangential to the conductor surface; M_y is therefore the normal component of M at the surface of the conductor and must there be continuous. E_z and $(\partial/\partial x)E_z$ are also continuous. Consequently, we must have by equating M_y as given by (13) and (14),

$$\frac{1}{\gamma}\left(\frac{\partial}{\partial y} M_z\right)_e = -\frac{\partial}{\partial x} E_z, \tag{15}$$

the subscript e indicating the value of $(\partial/\partial y)M_z$ outside the conductor. But from the expression for E_x, as given by (14), this is precisely the condition that makes $E_x = 0$ at the surface of the conductor. Consequently we arrive at the very important proposition that, subject to the approximations involved in (13), *the tangential component of E in the xy-plane vanishes at the conductor surfaces.*

We shall now find it convenient to express the field in the dielectric in accordance with (8) in terms of the wave functions F, Φ, Θ. Writing

$$\Theta = \theta \cdot \exp(i\omega t - \gamma z), \tag{16}$$

θ satisfies the differential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta = (\nu^2 - \gamma^2)\theta. \tag{17}$$

Now, *in the dielectric*, ν^2 and γ^2 are both exceedingly small quantities which are nearly equal, so that $\nu^2 - \gamma^2$ is the difference of two very small and nearly equal quantities. We therefore replace it by zero, so that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta = 0. \tag{18}$$

θ is therefore a two-dimensional potential function. Consequently *a conjugate two-dimensional potential function ψ exists, such that*

$$\begin{aligned} \frac{\partial}{\partial x}\theta &= \frac{\partial}{\partial y}\psi, \\ \frac{\partial}{\partial y}\theta &= -\frac{\partial}{\partial x}\psi. \end{aligned} \tag{19}$$

Writing

$$\Psi = \psi \cdot \exp(i\omega t - \gamma z),$$

equations (8) become

$$M_x = \frac{\partial}{\partial y}(F - \gamma\Psi),$$

$$
\begin{aligned}
M_y &= -\frac{\partial}{\partial x}(F - \gamma\Psi), \\
E_x &= -\frac{\partial}{\partial x}(\Phi - \Psi), \\
E_y &= -\frac{\partial}{\partial y}(\Phi - \Psi), \\
E_z &= \gamma(\Phi - \Psi) - (F - \gamma\Psi).
\end{aligned} \tag{20}
$$

Introducing new wave functions

$$
\begin{aligned}
F' &= F - \gamma\Psi, \\
\Phi' &= \Phi - \Psi,
\end{aligned} \tag{21}
$$

we have (dropping primes)

$$
\begin{aligned}
M_x &= \frac{\partial}{\partial y} F, \\
M_y &= -\frac{\partial}{\partial x} F, \\
E_x &= -\frac{\partial}{\partial x}\Phi, \\
E_y &= -\frac{\partial}{\partial y}\Phi, \\
E_z &= \gamma\Phi - F,
\end{aligned} \tag{22}
$$

where now Φ and F are *independent* wave functions.

If the foregoing analysis has been carefully followed, the important advantage of equations (22) as compared with (8) will be appreciated. The transformation of (8) into (22) is strictly dependent upon and conditioned by the legitimacy of neglecting $\nu^2 - \gamma^2$ in the dielectric, whereby the wave functions are essentially reduced to two-dimensional potential functions. It is evident that the whole engineering theory of transmission involves this approximation.

V

We are now prepared to sketch the general solution of the problem,[6] employing equations (13) *in the conductors*, and equations (22) *in the dielectric*. The procedure is as follows:

1. At the surfaces of the conductors the tangential component E_τ in the xy-plane of E vanishes, as shown above. That is,

$$
E_\tau = -\frac{\partial}{\partial \tau}\Phi = 0 \tag{23}
$$

[6] For detailed applications of this method of solution to specific problems, the published papers referred to in the introduction to this paper may be consulted.

at the surface of each conductor.[7] In the dielectric outside the conductor, the potential Φ satisfies Laplace's equation in two dimensions; hence

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi = 0 \tag{24}$$

in the dielectric; and

$$\frac{\partial}{\partial \tau_j}\Phi = 0, \qquad (j = 1, 2, \cdots n) \tag{25}$$

at the surface of the jth conductor. Also

$$\oint E_n{}^j d\tau_j = -\int \frac{\partial}{\partial n_j}\Phi d\tau_j = \frac{4\pi}{k}Q_j, \qquad (j = 1, 2, \cdots n) \tag{26}$$

the integration being carried around the surface of the jth conductor, Q_j being the charge per unit length on the jth conductor.

The determination of Φ from (24)–(26), when the geometry of the conductors is specified, is a well-known two-dimensional potential problem, for the solution of which very general methods are available. The solution results in the form

$$\Phi = \phi_1(x, y)Q_1 + \phi_2(x, y)Q_2 + \cdots + \phi_n(x, y)Q_n. \tag{27}$$

That is, Φ is a linear function of the conductor charges $Q_1 \cdots Q_n$, and the coefficients $\phi_1 \cdots \phi_n$ are unique functions of the geometry of the transmission system and are determinable by the usual methods of two-dimensional potential theory.

2. The continuity of M_n and $(1/\mu)M_\tau$ at the surfaces of the conductors is analytically formulated by the equations

$$\begin{aligned}\frac{\partial}{\partial \tau}F &= -\frac{\partial}{\partial \tau}E_z,\\ \frac{\partial}{\partial n}F &= -\frac{\mu}{\mu_c}\frac{\partial}{\partial n}E_z,\end{aligned} \tag{28}$$

where μ is the permeability of the dielectric and μ_c that of the conductor. These relations, it will be understood, hold at the surfaces of all the conductors. F is a wave function which satisfies Laplace's equation in two dimensions *in the dielectric;* thus

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)F = 0, \tag{29}$$

[7] In the following, τ and n denote vectors tangential and normal to the conductor surface respectively.

and E_z is a wave function which *in the conductor* satisfies the two-dimensional wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_z = \nu^2 E_z. \tag{30}$$

In addition, E_z and F are connected with the conductor current I by the relations

$$\begin{aligned} I &= \lambda \int E_z dS, \\ 4\pi\mu i\omega \cdot I &= \oint \frac{\partial}{\partial n} F d\tau, \end{aligned} \tag{31}$$

λ here denoting the conductivity of the conductor.

It follows at once that the determination of F and E_z from (28)–(31) is a generalization of the two-dimensional potential problem involved in the determination of Φ from (24)–(26); it may be precisely stated as follows:

The function F satisfies Laplace's equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) F = 0 \tag{32}$$

everywhere outside the n conductors. Inside the jth conductor the electric force E_z^j satisfies the two-dimensional wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_z^j = \nu_j^2 E_z^j, \qquad (j = 1, 2, \cdots n) \tag{33}$$

while at the surface of the jth conductor

$$\begin{aligned} \frac{\partial}{\partial \tau_j} F &= -\frac{\partial}{\partial \tau_j} E_z^j, \\ \frac{\partial}{\partial n_j} F &= -\frac{\mu}{\mu_j}\frac{\partial}{\partial n_j} E_z^j, \qquad (j = 1, 2, \cdots n) \end{aligned} \tag{34}$$

and

$$4\pi\mu i\omega \cdot I_j = \oint \frac{\partial}{\partial n_j} F d\tau_j, \qquad (j = 1, 2, \cdots n). \tag{35}$$

Just as equations (24)–(26) uniquely determine Φ as a linear function of $Q_1 \cdots Q_n$, so equations (32)–(35) uniquely determine the potential function F in the dielectric and the electric intensities $E_z^{(1)} \cdots E_z^{(n)}$ in the n conductors as linear functions of the conductor currents; thus

$$F = f_1(x, y)I_1 + f_2(x, y)I_2 + \cdots + f_n(x, y)I_n, \tag{36}$$

and

$$E_z^j = e_{1j}(x, y)I_1 + \cdots + e_{nj}(x, y)I_n, \qquad (j = 1, 2, \cdots n) \quad (37)$$

the f and e functions depending on the geometry of the conducting system and, through the parameter ν^2, on its electrical constants. They are uniquely determined by the differential equations.

The actual solution of the differential equations (32)–(35) is essentially more difficult than the solution of (24)–(26) involved in the determination of Φ, and they have not been subjected to the exhaustive study accorded to the potential problem. On the other hand, the analogy with the potential problem suggests that extension and modifications of the general and well-known methods of solution available for that problem should be possible.

To summarize the foregoing we have succeeded in expressing the potential function Φ (and therefore E_x, E_y) *in the dielectric* as a linear function of the conductor charges, the coefficients of the conductor charges $Q_1 \cdots Q_n$ being spatial functions of x and y which we determined by the usual methods of two-dimensional potential theory.

Similarly it has been shown that E_z (and therefore the current distribution) *in the conductors*, and the potential function F (and therefore the magnetic field) *in the dielectric*, are expressible as linear functions of the conductor currents $I_1 \cdots I_n$, the determination of the coefficients depending on the solution of a generalization of the two-dimensional potential problem.

3. To complete the solution of the problem, recourse is had to the fact that E_z is continuous at the surfaces of the conductors. At the surface of each conductor we therefore equate E_z, as given by (37), in terms of the currents $I_1 \cdots I_n$ with $\gamma\Phi - F$ (see (22)), Φ being given by (27) in terms of $Q_1 \cdots Q_n$ and F by (36) in terms of $I_1 \cdots I_n$. This gives n equations of the form

$$\begin{aligned} Z_{11}I_1 + \cdots + Z_{n1}I_n &= \gamma\Phi_1 = \gamma(p_{11}Q_1 + \cdots + p_{1n}Q_n), \\ &\cdots\cdots\cdots\cdots \\ Z_{n1}I_1 + \cdots + Z_{nn}I_n &= \gamma\Phi_n = \gamma(p_{n1}Q_1 + \cdots + p_{nn}Q_n). \end{aligned} \quad (38)$$

Here $\Phi_1 \cdots \Phi_n$ are the values of Φ at the surfaces of the n conductors respectively; the p coefficients are Maxwell's potential coefficients of the system, while the Z coefficients are the self and mutual "impedances" of the conductors.

We require a further relation between I and Q; this is furnished by the well-known relation

$$\begin{aligned} i\omega Q &= \gamma I - \lambda \oint E_n ds \\ &= \gamma I + \lambda \oint \frac{\partial}{\partial n} \Phi ds \\ &= \gamma I - \frac{4\pi\lambda}{k} Q, \end{aligned} \tag{39}$$

the integration being carried around the contour of the conductor. (λ is the conductivity of the dielectric and the last term is the "leakage" current.) We have therefore, for a homogeneous dielectric,

$$\left(i\omega + \frac{4\pi\lambda}{k} \right) Q = \gamma I, \tag{40}$$

which furnishes the necessary relation.

Elimination of Q from (38) by means of (40) gives n homogeneous equations in $I_1 \cdots I_n$, the coefficients involving only one unknown quantity, the propagation constant γ. A finite solution necessitates the vanishing of the determinant of the coefficients; equating this to zero gives an nth order equation in γ^2, which determines the n possible values of γ, and therefore the n possible modes of propagation in the system. The formal solution of the problem is thus completed.

In conclusion it is worth while reviewing and summarizing the mathematical restrictions on the solution developed in the foregoing pages; restrictions which have their counterpart in the physical requirements of the system for the efficient guided transmission of electromagnetic energy. The essential restrictions are that (1) *in the conductors* γ^2 is very small compared with ν^2, and (2) *in the dielectric* the wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi = (\nu^2 - \gamma^2)\Phi$$

may be replaced, at least in the neighborhood of the conductors, by

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi = 0.$$

If the conductors are so imperfect, or the dielectric so dissipative that these approximations are not justified, the method of solution given above breaks down, and the problem must be attacked from the rigorous equations. These have never been solved in general, in fact the only rigorous solution known to the writer is for the case of circular symmetry and even this involves the location of the roots of an extremely complicated transcendental equation. Fortunately, in view of these difficulties, the general case of quite imperfect conductors or imperfect dielectric media is of small technical importance for the reason given above.