Propagation of Periodic Currents over a System of Parallel Wires

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SYNOPSIS: The first section of this paper is devoted to the formal mathematical theory of the propagation of periodic currents over a system of parallel wires energized at its physical terminals only. The theory developed is essentially a generalization of the classical theory of transmission over a single wire (with ground return) or over a balanced metallic circuit. The solution here given furnishes the fundamental formulas and a good deal of information regarding what takes place in a system of parallel wires; for actual calculations, however, the method of treatment is not so well adapted as that developed in the remaining sections of the paper.

The second section deals analytically with the problem of propagation over a line or a circuit exposed throughout its length to an arbitrary impressed field of force. The resulting solution is immediately applicable to problems of crosstalk and interference, and to the theory of the wave antenna.

The last two sections are devoted to the development and application of a more physical or synthetic method of treatment, based on the substitution of 'equivalent electromotive forces' for the arbitrary impressed field. This synthetic treatment, which permits of an intuitive or physical grasp of the various problems, has been found quite useful in dealing with crosstalk and interference, and also with the wave antenna. The method is illustrated (in the last section) by application to two representative problems of a diverse nature.

IN the modern telephone system, transmission takes place over a circuit which is usually in close juxtaposition to a number of parallel circuits, and which may be, and frequently is, exposed to interference from power circuits or other disturbing sources. The mathematical theory of wave propagation over such a circuit involves two problems: (1) propagation over a system of parallel wires, and (2) propagation over a wire or metallic circuit in an arbitrary impressed field of force.

The first Section of this paper is devoted to the formal mathematical theory of the propagation of periodic currents over a system of parallel wires, energized at its physical terminals only.[1] This problem is essentially a generalization of the problem of transmission over a line of uniformly distributed resistance, inductance, capacity and leakage; and involves the formulation and solution of a differential equation which may be termed the *generalized telegraph equation* in contradistinction to the well-known *telegraph equation* which characterizes transmission over a single wire (with ground return) or a balanced

[1] This is the assumption underlying ordinary transmission theory.

metallic circuit. The analysis of this problem, while furnishing the fundamental formulas and a good deal of information regarding what takes place in a system of parallel wires, is not well adapted for actual calculations, except for relatively simple systems; in particular it is not adapted to deal with the important problems of crosstalk and interference.

In Section II the problem of propagation over a circuit or line exposed throughout its length to an arbitrary impressed field of force is taken up. The resulting solution is immediately applicable to interference problems, where the field of the disturbing source is supposed known, and to the theory of the wave antenna. Moreover, as shown in Section II*a*, it is particularly well adapted to the problems of 'crosstalk,' or interference between circuits of the parallel system.

Sections I, II and II*a* furnish the formal analysis and the fundamental formulas. Sections III and IV, constituting the remainder of the paper, are devoted to the development and the application to representative problems of a more physical or synthetic treatment, in which the general theory and formulas are interpreted in terms of 'equivalent electromotive forces'; this concept permits of an intuitive or physical grasp of the various problems, and has been found quite useful in dealing with crosstalk and interference, and also with the wave antenna.

I

Propagation of Periodic Currents over a System of Parallel Wires, with Impressed Field Concentrated at Terminals

The physical system under consideration is supposed to consist of n parallel wires, numbered from 1 to n, which may either be a system of overhead wires parallel to the surface of the earth, or a multi-wire cable enclosed in a sheath. The formal analysis applies equally well to both cases; but the calculation of the circuit parameters is a matter of considerably greater difficulty in the case of the cable, due to the close juxtaposition of the wires. Even in this case, however, the circuit constants are rather easily calculable to a first approximation from the dimensions of the system; and they are, in any case, experimentally determinable.

Let $I_1, I_2 \cdots I_n$ be the currents in the n wires, which are taken as parallel to the x-axis, which is itself parallel to the surface of the earth or to the sheath (in the cable case). A steady state is assumed; that is to say, the currents are sinusoidal and involve the time t only through the common factor $\exp(i\omega t)$, where $\omega/2\pi$ is the frequency and i denotes $\sqrt{-1}$; consequently the differential operator d/dt is replaceable by $i\omega$ in accordance with the usual methods of alternating current theory.

The first equations of the problem are derived by applying the law

$$\text{curl } E = -\mu(dH/dt)$$

to a contour bounded by a length dx in the surface of the jth wire, a corresponding length dx in the surface of the earth, and two lines normal to the axis of the wire and joining the corresponding ends of the two line elements dx. This gives

$$z_{jj}I_j - E_{gj} = -\frac{dV_j}{dx} - \frac{d\phi_j}{dt}, \qquad (j = 1, 2 \cdots n). \tag{1}$$

In this set of equations, z_{jj} denotes the 'internal' impedance per unit length of the jth wire, that is, the ratio of the axial electric force at the surface of the wire to the current I_j. E_{gj} is the electric force, parallel to the axis of the wire, in the earth's surface. V_j is the line integral of the electric force from the wire to the surface of the earth, that is, the 'potential,' or 'voltage,' of the wire. Finally, ϕ_j is the magnetic flux,[2] per unit length, threading the contour.

Now, both ϕ_j and E_{gj} are linear functions of the n currents $I_1, \cdots I_n$; consequently (1) is reducible to the form [3]

$$z_{jj}I_j = -\frac{dV_j}{dx} - \sum_{k=1}^{n} Z_{jk}I_k, \qquad (j = 1, 2 \cdots n). \tag{2}$$

The calculations of the impedance functions Z_{jk} and, in particular, the effect of the finite conductivity of the earth are dealt with in detail in an earlier paper.[4] The internal impedance, z_{jj}, is of the general form $r_{jj} + i\omega l_{jj}$, where r_{jj} is the resistance of the jth wire and l_{jj} its 'internal inductance.' In the ideal non-dissipative system the mutual impedance Z_{jk} is a pure imaginary of the form $i\omega L_{jk}$, where L_{jk} is the mutual inductance between the jth and kth wires; actually, however, due to the finite conductivity of the earth and to 'proximity effect' between the wires, it is always complex and of the form $R_{jk} + i\omega L_{jk}$. A similar statement holds for the self impedance Z_{jj}. The 'proximity effect' [5] is, of course, the increased internal impedance of the wire due to the currents in the neighboring wires. It may be taken as negligible in open wire lines but is quite appreciable, at telephonic frequencies, in cable circuits.

[2] Expressed in 10^{-8} maxwells if the remaining quantities are in 'practical units.'

[3] It is to be noted that Z_{jj} does *not* include the internal impedance z_{jj} of wire j.

[4] 'Wave Propagation in Overhead Wires with Ground Return,' John R. Carson, *B. S. T. J.*, October, 1926.

[5] See 'Wave Propagation over Parallel Wires: The Proximity Effect,' John R. Carson, *Phil. Mag.*, April, 1921. Rigorously the term $z_{jj}I_j$ of equations (1) and (2) should be replaced by $\Sigma z_{jk}I_k$, the additional terms formulating the proximity effect. This effect will not be explicitly included in the following analysis and the term $z_{jk}I_k$ may be regarded as incorporated with $Z_{jk}I_k$.

Let $Q_1, \cdots Q_n$ denote the charges per unit length on the n wires; the potentials and charges are then related by the set of linear equations

$$Q_j = \sum_{k=1}^{n} q_{jk} V_k, \qquad (j = 1, 2 \cdots n), \tag{3}$$

$$V_j = \sum_{k=1}^{n} p_{jk} Q_k, \qquad (j = 1, 2 \cdots n), \tag{4}$$

in which the q and p coefficients are Maxwell's capacity and potential coefficients. They are calculable by the usual methods of electrostatics, on the assumption that all the conductors, including the earth, are of perfect conductivity.

To complete the specification of the system we have the further set of relations

$$i\omega Q_j + I_j' = -\frac{dI_j}{dx}, \qquad (j = 1, 2 \cdots n). \tag{5}$$

Here I_j' is the 'leakage' current from the jth wire; it is, in general, a linear function of the n potentials, that is,

$$I_j' = \sum_{k=1}^{n} g_{jk} V_k, \qquad (j = 1, 2 \cdots n), \tag{6}$$

where the coefficients g_{jk} depend on the geometry of the system and the conductivity of the dielectric medium. From (3), (5) and (6) we have

$$-\frac{dI_j}{dx} = \sum_{k=1}^{n} (i\omega q_{jk} + g_{jk}) V_k, \qquad (j = 1, 2 \cdots n). \tag{7}$$

This system of linear equations, when solved for the potentials, gives

$$V_j = -\frac{d}{dx} \sum_{k=1}^{n} w_{jk} I_k, \qquad (j = 1, 2 \cdots n). \tag{8}$$

For the special case where the dielectric medium surrounding the conductors is homogeneous and isotropic, the coefficient w_{jk}, which in general is obtained by solving (7), is given by

$$w_{jk} = p_{jk}/(i\omega + \delta), \tag{9}$$

where $\delta = 4\pi\sigma/\epsilon\xi$, σ and ϵ being the conductivity and specific inductive capacity of the dielectric, and ξ a constant whose value depends only on the units.[6] In many cases w_{jk} is calculable with sufficient accuracy from equation (9), so that the solution of (7) is then unnecessary.

[6] A derivation of the formula for δ is outlined shortly after equation (18) of Appendix I.

We are now prepared to write down the *generalized telegraph equation*, which is obtained by eliminating $V_1, \cdots V_n$ from equations (2) by aid of (8); it is:

$$z_{jj}I_j = \sum_{k=1}^{n}\left(w_{jk}\frac{d^2}{dx^2} - Z_{jk}\right)I_k, \qquad (j = 1, 2 \cdots n). \tag{10}$$

This system of n equations, which constitutes the *generalized telegraph equation*, will be written as:

$$\begin{aligned} m_{11}I_1 + m_{12}I_2 + \cdots + m_{1n}I_n &= 0, \\ m_{21}I_1 + m_{22}I_2 + \cdots + m_{2n}I_n &= 0, \\ \cdots\cdots\cdots\cdots & \\ m_{n1}I_1 + m_{n2}I_2 + \cdots + m_{nn}I_n &= 0, \end{aligned} \tag{11}$$

where

$$m_{jk} = Z_{jk} - w_{jk}\frac{d^2}{dx^2}, \qquad (j \neq k), \tag{11.1}$$

$$m_{jj} = z_{jj} + Z_{jj} - w_{jj}\frac{d^2}{dx^2}. \tag{11.2}$$

Equations (11) are a system of n homogeneous equations; a finite solution for the currents $I_1, \cdots I_n$ therefore necessitates the vanishing of the determinant of the system, that is,

$$\begin{vmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{21} & m_{22} & m_{23} & \cdots & m_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ m_{n1} & m_{n2} & m_{n3} & \cdots & m_{nn} \end{vmatrix} = 0. \tag{12}$$

In order to solve this equation the operator d^2/dx^2 is to be replaced by γ^2, which is equivalent to the assumption that the n currents $I_1, \cdots I_n$ involve the variable x only through the common factor $\exp(\gamma x)$. With this substitution, equation (12) is of the nth order in γ^2 and its solution gives, in general, $2n$ values of γ, namely, $\gamma_1, \gamma_2, \cdots \gamma_n$ and $-\gamma_1, -\gamma_2, \cdots -\gamma_n$. The general solution of equations (11) is accordingly of the form

$$I_j = \sum_{k=1}^{n}(A_{jk}e^{-\gamma_k x} - B_{jk}e^{\gamma_k x}), \qquad (j = 1, 2 \cdots n). \tag{13}$$

The potentials are then determined from (8) and (13) in terms of the parameters of the system, the propagation constants, and the arbitrary constants of integration A_{jk}, B_{jk}.

By means of the relations (11) obtaining among the currents, it is easy to show that the number of independent arbitrary constants of integration is $2n$. These are determined by the $2n$ boundary conditions to be satisfied at the physical terminals of the system. In general these boundary conditions specify $2n$ relations among the impressed voltages, the terminal impedances, and the line currents and voltages. While the evaluation of the constants of integration from these $2n$ relations is formally straightforward, it is actually a matter of very considerable complexity if the system is composed of a large number of wires; furthermore, the evaluation of the propagation constants $\gamma_1, \cdots \gamma_n$ presents great difficulties in such cases.

The results of the foregoing formal analysis may be summarized as follows: In a system of n parallel wires there are in general n modes of propagation, corresponding to the n roots $\gamma_1, \cdots \gamma_n$ of the generalized telegraph equation; these may be termed the normal modes of propagation. Except when special boundary conditions obtain, the current in each and every wire is made up of component waves of all n modes of propagation, and the distribution of energy among the n modes is determined by the boundary conditions at the terminals of the wires. A characteristic and fundamental property of the normal modes of propagation is that a normal mode of propagation is the type which can exist alone. That is to say, if the boundary conditions have a particular set of values, the currents in all the wires may be made up of one mode only; unless, however, the particular conditions obtain, the currents involve components of all modes.

The existence of n modes of propagation in a system of n parallel wires follows from the fact that the determinant is of the nth order in γ^2 and therefore has n roots. In certain cases of practical importance, however, we may have multiple roots, so that the number of distinct modes of propagation is reduced. For example, in the ideal case of perfect conductors and perfect ground conductivity, $L_{jj}/p_{jj} = L_{jk}/p_{jk} = 1/c^2$ and therefore $\gamma = i\omega/c$, where c is the velocity of propagation in the medium. In this case, only one mode of propagation exists, namely, unattenuated transmission with the velocity of propagation of light in the dielectric medium; thus, for the direct wave,

$$I_j = A_j e^{-i\omega/c} \tag{14}$$

and the n constants $A_1, \cdots A_n$ are independent.

Another case of some interest is that in which the wires are all alike, so that $m_{11} = m_{22} = \cdots m_{nn} = m$ and furthermore $m_{jk} = m'$ (a condition which is partially realizable by a properly designed system of transpositions). In this case equation (12) becomes

$$\begin{vmatrix} m & m' & m' & \cdots & m' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ m' & m' & m' & \cdots & m \end{vmatrix} = 0, \tag{15}$$

and there are only two modes, γ_1 and γ_2, corresponding to $m - m' = 0$ and $m + (n - 1)m' = 0$ respectively. The first mode obviously corresponds to metallic transmission, the second to ground return transmission. It is easily shown that the direct current waves are expressed by

$$I_j = A_j e^{-\gamma_1 x} + B e^{-\gamma_2 x}, \tag{16}$$

$$\sum A_j = 0, \tag{17}$$

with corresponding expressions for the reflected waves. The corresponding potentials are

$$V_j = \tfrac{1}{2} K_1 A_j e^{-\gamma_1 x} + n K_2 B e^{-\gamma_2 x}. \tag{18}$$

Here K_1 is the characteristic impedance of a metallic circuit composed of two wires, and K_2 is the characteristic impedance of the n wires in multiple, with the ground for return.

A case of greater practical importance is that of n balanced pairs, which is the ideal telephone transmission system. To consider this case let

$$\begin{aligned} m_{11} &= m_{22} = \cdots = m_{nn} = m_{2n,\,2n} = m, \\ m_{jk} &= m' \text{ between wires of the same pair,} \\ m_{jk} &= m'' \text{ between wires of different pairs.} \end{aligned} \tag{19}$$

In this case the determinant becomes

$$\begin{vmatrix} m & m' & m'' & m'' & \cdots & \cdots & m'' \\ m' & m & m'' & m'' & \cdots & \cdots & m'' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ m'' & m'' & m'' & \cdots & \cdots & m & m' \\ m'' & m'' & m'' & \cdots & \cdots & m' & m \end{vmatrix} = 0. \tag{20}$$

There are therefore three modes of propagation γ_1, γ_2, γ_3 corresponding respectively to:

$$\begin{aligned} & m - m' = 0, \\ & m + m' - 2m'' = 0, \\ & m + m' + 2(n - 1)m'' = 0. \end{aligned} \tag{21}$$

The first mode of propagation corresponds to metallic transmission over a pair, the second to transmission over a four wire phantom circuit, and the third to ground return transmission. It may be easily shown that the solution for the direct waves is (writing I_j and I_j' for the currents in the two wires, j and j', of the jth pair, and V_j, V_j' for the corresponding potentials):

$$\begin{aligned} I_j &= \quad A_j e^{-\gamma_1 x} + B_j e^{-\gamma_2 x} + C e^{-\gamma_3 x}, \\ I_j' &= - A_j e^{-\gamma_1 x} + B_j e^{-\gamma_2 x} + C e^{-\gamma_3 x}, \end{aligned} \tag{22}$$

with the further condition $\sum B_j = 0$. The corresponding potentials are:

$$\begin{aligned} V_j &= \quad \tfrac{1}{2} K_1 A_j e^{-\gamma_1 x} + K_2 B_j e^{-\gamma_2 x} + 2nK_3 C e^{-\gamma_3 x}, \\ V_j' &= - \tfrac{1}{2} K_1 A_j e^{-\gamma_1 x} + K_2 B_j e^{-\gamma_2 x} + 2nK_3 C e^{-\gamma_3 x}. \end{aligned} \tag{23}$$

The characteristic impedances K_1, K_2, K_3 correspond to the three modes of propagation; from the foregoing and equation (8) they are found to have the following values:

$$\begin{aligned} K_1 &= 2(w - w')\gamma_1, \\ K_2 &= (w + w' - 2w'')\gamma_2, \\ K_3 &= \frac{1}{2n}(w + w' + 2[n - 1]w'')\gamma_3. \end{aligned} \tag{24}$$

The importance of the case just considered lies in the fact that the conditions of symmetry which obtain are those of the ideal multi-circuit telephone system. Two of the normal modes of propagation, physical and phantom circuit transmission, are those actually employed in telephone transmission and these modes can exist in any physical or phantom circuit without crosstalk or the induction of current in any other circuit. In fact the problem of crosstalk is the designing of the system, by means of transpositions, to approximate the ideal case, and the calculation of the effect of small departures from the ideal conditions of symmetry.

In investigating problems of crosstalk and of induction from foreign disturbing sources, the general formulas developed in the preceding pages do not lend themselves readily to the necessary calculations and interpretations. In the first place, calculation by means of the general formulas involves the location of the n roots of an nth order equation and thus presents the same difficulties as those encountered in the calculation of the transient oscillations of a network of n degrees of freedom; in fact the problems are mathematically the same, the space variable x of the present problem corresponding to the time variable t of the transient problem. In the second place the formulas, as they stand, are inapplicable to the important case where the circuit or line is exposed throughout its length to an arbitrary impressed disturbance. Furthermore, in the problem of crosstalk the departures from the conditions of balanced symmetry are necessarily very small, whereas the formulas are so general as to make it very difficult to introduce the essential simplifications which follow from the condition of small departures. For example, if the foregoing formulas are applied, as they stand, to transposed lines, it is necessary to set up new boundary conditions and evaluate a new set of integration constants at every transposition point, since a transposition point is a discontinuity. The difficulty of such a procedure is very great, aside from the fact that it requires as a preliminary the calculation of the n modes of propagation of the system in each transposition interval. In view of these difficulties a more powerful method of attack is required in the analytical investigation of the problems of crosstalk and of interference in general. Fortunately this is furnished by the solution of the problem dealt with in the next section: the propagation of periodic currents over wires in an arbitrary impressed field of force.

II

Propagation of Periodic Currents Over Wires in an Arbitrary Impressed Field of Force

We shall consider first the simplest case, namely, a single wire with ground return. The impressed or disturbing field is assumed to be periodic, of frequency $\omega/2\pi$, so that the problem is a steady state one and the time is involved only through the factor $\exp(i\omega t)$. The resultant field is made up of two parts: first that due to the primary impressed field; and secondly that due to the current in and the charge on the wire, and the corresponding induced currents and charges in the ground. Let $f(x) = f$ denote the component of the electric force of the primary or impressed field parallel to the axis of the wire at its surface,[7] and $F(x) = F$ the line integral of the impressed or primary field from the surface of the wire to ground. It is then proved in Appendix I that the differential equation of the problem is

$$\frac{K}{\gamma}\left(\gamma^2 - \frac{d^2}{dx^2}\right) I = f, \tag{25}$$

the solution of which is [8]

$$\begin{aligned} I = e^{-\gamma x}\left[A + \frac{1}{2K}\int_v^x dy \cdot f(y) e^{\gamma y} \right] \\ - e^{\gamma x}\left[A' + \frac{1}{2K}\int_v^x dy \cdot f(y) e^{-\gamma y} \right]. \end{aligned} \tag{26}$$

[7] $f(x)$ is assumed to be sensibly constant over the cross-section of the wire.

[8] The lower limit, v, of integration is at our disposal. In case the line begins at $x = 0$, it may be convenient to take $v = 0$.

The corresponding potential of the wire is

$$
\begin{aligned}
V = Ke^{-\gamma x}\left[A + \frac{1}{2K}\int_v^x dy\cdot f(y)e^{\gamma y}\right] \\
+ Ke^{\gamma x}\left[A' + \frac{1}{2K}\int_v^x dy\cdot f(y)e^{-\gamma y}\right] + F(x).
\end{aligned}
\tag{27}
$$

In these equations, γ and K are the *propagation constant* and the *characteristic impedance* of the circuit [9] consisting of the wire with ground return, while A and A' are arbitrary or integration constants which are determined from the boundary conditions. It will be observed that if the arbitrary impressed field is removed ($f = F = 0$), the solution reduces to the usual form. If the terminal impedances are specified, it follows from (26) and (27) that the problem is completely solvable provided that f is specified along the wire and F at its physical terminals.

Two more general cases of practical importance will next be formulated:

(1) Balanced Pair of Wires

Let K_1, γ_1 be the characteristic impedance and propagation constant of transmission over the metallic circuit; and K_3, γ_3 the corresponding quantities for the case of the two wires in multiple, with ground return. Let f_1 and f_2 be the electric force of the primary or impressed field along the surfaces of the wires No. 1 and No. 2 respectively, and I_1 and I_2 the currents in the wires. The solution may then be written as

$$I_1 = a + c, \qquad I_2 = -a + c,$$

where

$$
\begin{aligned}
a = e^{-\gamma_1 x}\left[A + \frac{1}{2K_1}\int_v^x \{f_1(y) - f_2(y)\}e^{\gamma_1 y}dy\right] \\
- e^{\gamma_1 x}\left[A' + \frac{1}{2K_1}\int_v^x \{f_1(y) - f_2(y)\}e^{-\gamma_1 y}dy\right], \\
c = e^{-\gamma_3 x}\left[C + \frac{1}{4K_3}\int_v^x \tfrac{1}{2}\{f_1(y) + f_2(y)\}e^{\gamma_3 y}dy\right] \\
- e^{\gamma_3 x}\left[C' + \frac{1}{4K_3}\int_v^x \tfrac{1}{2}\{f_1(y) + f_2(y)\}e^{-\gamma_3 y}dy\right].
\end{aligned}
\tag{28}
$$

The component a corresponds to transmission over the metallic or physical circuit; while the component c corresponds to transmission over the two wires in multiple, with ground return. A and C are the

[9] It will be observed that in these equations the characteristics of the ground do not appear explicitly. They are, however, implicitly involved in K and γ of the ground return circuit.

integration constants of the direct wave, while A' and C' are those of the reflected wave. The first component, as regards the impressed force f along the wires, depends on the difference $f_1 - f_2$ at the surface of the two wires, while the second depends on the mean value $(f_1 + f_2)/2$. In the case of interference from external sources the latter is usually much the larger and consequently the induction mainly corresponds to the ground return mode of propagation, γ_3.

(2) System of n Balanced Pairs

We shall now write down the expressions for the currents in a system of n balanced pairs ($2n$ wires) when exposed to an arbitrary impressed field. The properties of this system were discussed briefly in the preceding section and formulated in equations (19), $\cdots$ (24). Let I_j and I_j' be the currents in the two wires j and j' respectively of the jth pair, and let f_j and f_j' be the corresponding impressed forces along the surfaces of the two wires, while F_j and F_j' are the corresponding line integrals of the impressed force to ground. By an extension of the previous formulas it is easy to show that the currents are made up of three components:

$$\begin{aligned} I_j &= a_j + b_j + c_j, \\ I_j' &= -a_j + b_j + c_j. \end{aligned} \tag{29}$$

If we write $\bar{f}_j = (f_j + f_j')/2$, the components a_j, b_j, c_j are given by:

$$\begin{aligned} a_j = e^{-\gamma_1 x}&\left[A_j + \frac{1}{2K_1}\int_0^x \{f_j(y) - f_j'(y)\} e^{\gamma_1 y} dy \right] \\ &- e^{\gamma_1 x}\left[A_j' + \frac{1}{2K_1}\int_0^x \{f_j(y) - f_j'(y)\} e^{-\gamma_1 y} dy \right], \end{aligned} \tag{30}$$

$$\begin{aligned} b_j = e^{-\gamma_2 x}&\left[B_j + \frac{1}{2K_2}\int_0^x \{\bar{f}_j(y) - \frac{1}{n}\sum \bar{f}_k(y)\} e^{\gamma_2 y} dy \right] \\ &- e^{\gamma_2 x}\left[B_j' + \frac{1}{2K_2}\int_0^x \{\bar{f}_j(y) - \frac{1}{n}\sum \bar{f}_k(y)\} e^{-\gamma_2 y} dy \right], \end{aligned} \tag{31}$$

$$\sum B_j = \sum B_j' = 0, \tag{32}$$

$$\begin{aligned} c_j = e^{-\gamma_3 x}&\left[C + \frac{1}{4nK_3}\int_0^x \frac{1}{n}\sum \bar{f}_k(y) e^{\gamma_3 y} dy \right] \\ &- e^{\gamma_3 x}\left[C' + \frac{1}{4nK_3}\int_0^x \frac{1}{n}\sum \bar{f}_k(y) e^{-\gamma_3 y} dy \right]. \end{aligned} \tag{33}$$

Here the a component corresponds to transmission over a pair, the b component to transmission over a phantom circuit, and the c component to ground return transmission. It will be observed that, as regards the impressed field, the a component depends on the difference $f - f'$ of the impressed force at the two sides of the circuit, while the c component depends on the mean impressed electric force averaged over the $2n$ conductors of the system; the b, or phantom component, involves the impressed field in a slightly more complicated way, depending on both the mean impressed force averaged for the two conductors of a pair and also averaged over all the $2n$ conductors of the system.

The extension of the preceding analysis to the general case of n parallel wires, in general dissimilar, is straightforward. The resulting formulas are, however, extremely complicated and for this reason, as well as their small practical utility, they will not be written down.

Formulas (25), $\cdots$ (33) are immediately applicable to the wave antenna and to interference problems in general where the impressed disturbance is supposed to be known. Their application to the problem of crosstalk, which will now be taken up, is not immediate in the same sense because here the primary disturbance which sets up crosstalk is itself a function of the unbalances among the wires composing the system. That is to say, the primary disturbance or impressed field causing the crosstalk is implicitly rather than explicitly given.

II*a*

In discussing the theory of crosstalk a representative problem will be dealt with rather than a formulation of the general problem. The types of problem encountered in practice are extremely varied, depending on whether we have to do with 'side-to-side,' 'side-to-phantom' or 'phantom-to-phantom' crosstalk, etc.; and each problem may call for special treatment. The representative problem, however, besides showing the underlying mathematical theory should serve to indicate the correct procedure in other specific problems.

Let us return to the general system of n parallel wires, dealt with in Section I, and let us suppose that two of them, say wires No. 1 and No. 2, constitute a metallic circuit which, for convenience, we shall suppose would be balanced with respect to ground if the other wires were removed. We now suppose that this metallic circuit is energized by an electromotive force impressed at $x = 0$, which in the absence of the other wires would produce a current I^0 in wire No. 1 and an equal and opposite current $- I^0$ in wire No. 2. Our problem is now to calculate the currents induced in the neighboring wires and the additional currents induced in wires No. 1 and No. 2 due to the reactions in the system.

It will be observed that, if the system were ideally balanced, no currents would be induced in the neighboring wires and we should simply have the current I^0 in the metallic circuit; in engineering language there would be no crosstalk. This is the ideal to which the correctly designed telephone system approximates by means of 'transpositions.' It is never, of course, completely realized but the approximation, as regards the neighboring metallic circuits, must be extremely close, since the allowable amount of crosstalk is very small.

Let us now return to the original system of equations for n parallel wires discussed in Section I and let us write $I_1 = I^0 + I_1'$, $I_2 = - I^0 + I_2'$ and replace $I_2, I_3 \cdots I_n$ by $I_2', I_3' \cdots I_n'$ respectively, the primes indicating that the currents are 'unbalance' currents. Similarly write $V_1 = V^0 + V_1'$, $V_2 = - V^0 + V_2'$; $Q_1 = Q^0 + Q_1'$, $Q_2 = - Q^0 + Q_2'$; and for the rest of the wires add primes to the symbols for potential and charge. Equations (2) may then be written as

$$\begin{aligned}
(z_{11} + Z_{11})I_1' + \frac{dV_1'}{dx} &= - Z_{12}I_2' - Z_{13}I_3' - Z_{14}I_4' - \cdots, \\
(z_{22} + Z_{22})I_2' + \frac{dV_2'}{dx} &= - Z_{21}I_1' - Z_{23}I_3' - Z_{24}I_4' - \cdots, \\
(z_{jj} + Z_{jj})I_j' + \frac{dV_j'}{dx} &= - (Z_{j1} - Z_{j2})I^0 - Z_{j1}I_1' \\
&\quad - Z_{j2}I_2' - Z_{j3}I_3' - \cdots, \\
&\qquad (j = 3, 4 \cdots n),
\end{aligned} \tag{34}$$

or, denoting the right hand sides of the equations by f_1', f_2', f_j', respectively,

$$\begin{aligned}
(z_{11} + Z_{11})I_1' + \frac{dV_1'}{dx} &= f_1', \\
(z_{22} + Z_{22})I_2' + \frac{dV_2'}{dx} &= f_2', \\
(z_{jj} + Z_{jj})I_j' + \frac{dV_j'}{dx} &= f_j', \\
&\qquad (j = 3, 4 \cdots n).
\end{aligned} \tag{35}$$

This set of equations in the unbalance currents $I_1', \cdots I_n'$ and unbalance potentials $V_1', \cdots V_n'$ admits of immediate interpretation. This is to the effect that the unbalance currents may be regarded as due to an impressed field characterized by an axial electric intensity

$f_j' - dF_j'/dx$ along the jth wire ($j = 1, 2, \cdots n$), and an impressed potential F_j' (line integral of impressed field from jth wire to ground), where

$$\begin{aligned} F_1' &= p_{12}Q_2' + p_{13}Q_3' + p_{14}Q_4' + \cdots, \\ F_2' &= p_{21}Q_1' + p_{23}Q_3' + p_{24}Q_4' + \cdots, \\ F_j' &= (p_{j1} - p_{j2})Q^0 + p_{j1}Q_1' + p_{j2}Q_2' + p_{j3}Q_3' + \cdots, \\ & \qquad (j = 3, 4 \cdots n). \end{aligned} \tag{36}$$

Consequently if f_j' and F_j' were known, equations (25), $\cdots$ (27) would be immediately applicable to the calculation of the unbalance currents. Inspection of equations (34), $\cdots$ (36) shows, however, that while I^0, V^0, Q^0 are supposed known, the expressions for f_j' and F_j' involve the unbalance currents and charges themselves. The solution of the equations calls therefore for a process of successive approximation, now to be discussed. While this method of solution is theoretically sound and applicable in all cases, its success in practical applications depends largely on the fact that the unbalance currents must be extremely small, compared with the primary current I^0, if the crosstalk is to be kept within tolerable limits.

Returning to equations (34), $\cdots$ (36), the first approximate solution is obtained by (1) ignoring the unbalance currents and charges in their effect on the current in the primary wires (No. 1 and No. 2), and (2) replacing $f_3', \cdots f_n'$ and $F_3', \cdots F_n'$ by

$$\begin{aligned} f_j' &= -(Z_{j1} - Z_{j2})I^0, \\ F_j' &= (p_{j1} - p_{j2})Q^0, \\ & \qquad (j = 3, 4 \cdots n). \end{aligned} \tag{37}$$

Consequently in the first approximate solution the primary current, charge and potential are I^0, Q^0, V^0, which are calculable in terms of the impressed e.m.f. and the terminal impedances for the circuit composed of the primary wires (No. 1 and No. 2) by ignoring the reaction of the other wires.[10] The unbalance currents, charges and potentials of the other wires are then calculable on the supposition that those wires are energized by the known impressed field f_j', F_j', as given by (37), which depends only on I^0 and Q^0.

The second approximate solution is obtainable by substituting the first approximate values of I_j' and Q_j' in the right hand side of equations (34) and (36) and then proceeding precisely as in the first approxi-

[10] It should be clearly understood that this particular procedure is not required and is not always followed in practice. For example, it is customary in calculating the crosstalk induced in a metallic or 'side' circuit to take into account, in the first approximate solution, the reaction between the wires making up the disturbed circuit.

mate solution. This process can, theoretically, be repeated indefinitely and successively closer approximations thereby obtained. Practically, however, even in a system of only a few wires, the process rapidly becomes prohibitively laborious and complicated, so that only the first and perhaps the second approximate solutions are practicable. Theoretically, however, the process is straight-forward and the successive approximate solutions form a convergent sequence. Fortunately, in engineering applications the allowable amount of crosstalk is so strictly limited that higher approximations than the second at most are not usually required.

It is an important and valuable property of the solution by successive approximations that the 'datum configuration' is not uniquely fixed, but is at our disposal, within limits. By 'datum configuration' is meant the assumed distribution from which the first approximate solution is derived. In the preceding the datum configuration for the primary wires is taken as

$$\begin{aligned} I_1 &= I^0 = -I_2, \\ Q_1 &= Q^0 = -Q_2, \end{aligned} \tag{38}$$

while in calculating any I_j' $(j = 3, \cdots n)$ it is assumed that the unbalance currents and charges of the other disturbed wires are zero. From the form of the equations this is certainly the natural configuration with which to start. It does not at all follow, however, that this datum configuration results in the optimum first approximate solution.

Another datum configuration which may be taken and which appears to possess practical advantages in certain cases is the following:[11]

$$\begin{aligned} I_1 &= I^0 = -I_2, \\ Q_1 &= Q^0 = -Q_2, \end{aligned} \tag{39}$$

for the primary wires, while in calculating any I_j' $(j = 3, \cdots n)$ it is assumed that the unbalance currents and *potentials* (instead of *charges*) of the other disturbed wires are zero.[12] Higher successive approximate solutions then follow the same scheme of procedure as in the first case.

The foregoing completes the formal analytical theory. The remaining sections of the paper will be devoted to the interpretation of the fundamental mathematical theory and its formulation along more physical and engineering lines, together with applications to representative problems.

[11] This is essentially the basis of the crosstalk formulas developed, in terms of a different mathematical treatment, by Dr. G. A. Campbell of the American Telephone and Telegraph Co., in his early and fundamental work on crosstalk and transposition theory.

[12] See, however, the preceding footnote as to possible modification of the datum configuration.

III

Representation of Impressed Field by Equivalent Electromotive Forces

In the present section we shall start anew with the problem dealt with in Section II, and attack it by a synthetic method, as distinguished from the analytical method employed there. While the results so derived are all deducible from the analytical theory and formulas of Section II, the synthetic or physical mode of attack has important advantages in engineering applications, in giving a physical picture of the phenomena and an intuitive grasp of the problem. In many cases it enables us to deduce results very simply, when the physical picture is well in mind, whereas the purely analytical solution may be laborious.

The essence of this synthetic method consists in replacing the known electric field impressed on the physical system by a set of equivalent electromotive forces; the current at any point in the system can then be calculated when the transfer admittances between that point and the points where the electromotive forces are situated are known or calculable (as is often the case in practical applications). For, considering any linear system containing any number m of electromotive forces inserted at any points $1, \cdots m$, it is known, from the principle of superposition, that the current I_h at any point h is a linear function of all the electromotive forces, that is,

$$I_h = \sum_{k=1}^{m} A_{hk} E_k. \tag{40}$$

The coefficient A_{hk} is called the 'transfer admittance' from k to h, because A_{hk} is equal to the ratio of I_h to E_k when all of the electromotive forces except E_k are zero. If the system contains any unilateral element (such as a one-way amplifier, for instance), A_{hk} is not in general equal to A_{kh}.

Fundamental Set of Equivalent Electromotive Forces: General Formulation

Consider any system of parallel wires situated in an arbitrary impressed field, with any number of localized admittance bridges between wires or between wires and ground. (Evidently, distributed bridged admittance can be analyzed into infinitesimal elements, and these can be regarded as localized.) The cross-sectional dimensions of the wires are assumed to be small enough so that the axial (longitudinal) impressed electric force is sensibly constant over each cross-section.

The electric constituent of the impressed field is assumed to be specified at every point along the wires by the impressed axial electric force and the impressed potential. At any point x in any wire, h, the impressed axial electric force will be denoted by $f_h(x)$ and the impressed potential by $F_h(x)$; these are to be regarded as arbitrary functions of x, and may even be discontinuous.

The following set of electromotive forces is easily seen to be equivalent to the above-specified arbitrary impressed field, in the sense of producing the same currents and charges. This set will be termed the 'fundamental' set of equivalent electromotive forces; for, from the physical viewpoint of this paper, it is in fact the fundamental set.[13]

(*A*) In each wire a distributed axial electromotive force whose value, per unit length, at each point is equal to the impressed axial electric force there; thus, at any point x in wire h, an electromotive force $f_h(x)dx$ in the differential length dx.

(*B*) At each point where the impressed potential is discontinuous, an axial electromotive force equal to the decrement in the impressed potential there; thus, at any point of discontinuity $x = u$ in any wire h, an electromotive force equal to $-\Delta F_h(u) = F_h(u-) - F_h(u+)$.

(*C*) In each bridge an electromotive force equal to the impressed voltage in that bridge; thus, in a bridge at any point $x = b$, from wire h to any other wire k (or to ground), an electromotive force equal to $F_h(b) - F_k(b)$.

(*D*) In case a point $x = b$ where a bridge is situated coincides with a point $x = u$ where the potential $F_h(x)$ impressed on wire h is discontinuous, the corresponding electromotive forces are as follows: Axial electromotive forces equal to $F_h(b-)$ and $-F_h(b+)$ at points $b-$ and $b+$ respectively in wire h; no electromotive force in the bridge itself, which is connected to the point b situated between $b-$ and $b+$ in wire h.[13a]

[13] For a one-wire line and for a balanced two-wire line, five other sets of equivalent electromotive forces are formulated in a later subsection.

[13a] By supposing points b and u to be not quite coincident, say $b = u-$ or $b = u+$, item (*D*) can be derived by first applying items (*B*) and (*C*) and then applying the 'branch-point theorem' formulated in the second paragraph following equation (75).

A further application of the 'branch-point theorem' yields for item (*D*) the following alternative set of electromotive forces: Axial electromotive forces each equal to $[F_h(b-) - F_h(b+)]/2$ at $b-$ and $b+$ in wire h; an electromotive force equal to $[F_h(b-) + F_h(b+)]/2$ in the bridge at b. Clearly, this set reduces to (*C*) when $F_h(x)$ is continuous at $x = b$, and it reduces to (*B*) when there is no bridge at $x = u$.

A physical verification of the correctness of the foregoing set of equivalent electromotive forces can be obtained by starting with the given system, situated in the specified arbitrary impressed field (but not otherwise energized), and then inserting in the wires and bridges a set of electromotive forces, which will be termed the 'annulling electromotive forces,' such as to annul all currents in the wires and bridges. The resultant axial electric force in the wires will then be zero, and furthermore the wires will be uncharged; hence the inserted axial electric force must be equal and opposite to the impressed axial electric force. Since the wires are uncharged their potentials will be those of the impressed field; hence, since no current flows in the bridges, the electromotive forces inserted in the bridges must be equal and opposite to the voltages of the impressed field at the bridges. Evidently the negatives of the annulling electromotive forces constitute a set of electromotive forces equivalent to the impressed field; for, insertion of the negatives of the annulling electromotive forces restores the system to its original state, in which it is acted on by only the original impressed field.

From the nature of this demonstration it is seen that the 'fundamental set' of equivalent electromotive forces is not limited to a system of parallel horizontal wires. In the general case, where the wires are neither straight nor parallel nor horizontal, x (and hence u and b) is to be interpreted as being the 'intrinsic coordinate' of a point in the particular wire contemplated, that is, the distance measured along that wire from any arbitrary fixed point therein. Thus, for wires h and k respectively, x becomes x_h and x_k, which in general are independent of each other.

For the case of a one-wire line, an analytical derivation of this set of equivalent electromotive forces is given in a later subsection by interpretation of the fundamental differential equations of the line.

A One-Wire Line in an Arbitrary Impressed Field

As indicated by Fig. 1, the line extends from $x = 0$ to $x = s$, and is terminated in impedances Z_0 and Z_s respectively. γ denotes the propagation constant per unit length, and K the characteristic impedance.[14] The direct leakage admittance from the wire to ground, per unit length, is denoted by Y'; this is the generalization of a mere leakage conductance.[31]

The impressed field is specified by the functions $f(x)$ and $F(x)$; $f(x)$ denoting the impressed axial electric force and $F(x)$ the impressed

[14] Given by formulas (12) and (11) of Appendix I.

potential, at any point x of the wire. For generality, $F(x)$ is assumed to be discontinuous at any point $x = u$ by the increment

$$\Delta F(u) = F(u+) - F(u-).$$

The problem is to calculate the current $I(x)$ produced at any point x by the impressed field.

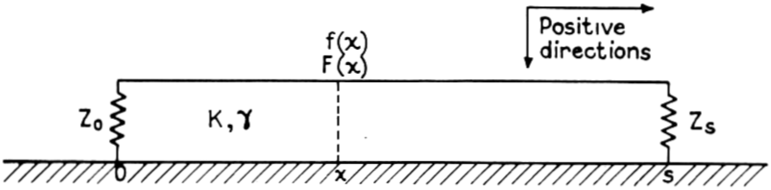

Fig. 1.

Case 1: General Case

The current $I(x)$ is the sum of two constituents: $j(x)$ due to the impressed axial electric force, and $J(x)$ due to the impressed potential. Formulas for these constituents will now be written down by aid of the fundamental set of equivalent electromotive forces formulated in the preceding subsection. Thus [15]

$$j(x) = \int_0^s A(x, y) f(y) dy, \tag{41}$$

$A(x, y)$ denoting the transfer admittance between points x and y. $J(x)$ itself consists of four constituents: $J_0(x)$ and $J_s(x)$, originating in the terminal impedances Z_0 and Z_s respectively, $J_{0s}(x)$ originating in the direct leakage admittance of the whole line 0-s, and $J_u(x)$ originating at the point $x = u$ where $F(x)$ is discontinuous. Thus

$$J_0(x) = - A(x, 0) F(0), \tag{42}$$

$$J_s(x) = A(x, s) F(s), \tag{43}$$

$$J_{0s}(x) = \int_0^s Y' F(y) B(x, y) dy, \tag{44}$$

$$J_u(x) = - A(x, u) \Delta F(u), \tag{45}$$

$B(x, y)$ denoting a current transfer factor representing that fraction

[15] With regard to the analytical evaluation of the integrals, attention should perhaps be called to the fact that the integrand may be discontinuous or may change its functional form at one or more points within the range of integration; whence the integral must be broken up into a sum of integrals.

of the current contribution originating in the direct leakage admittance element $Y'dy$ at y, which reaches point x.

Case 2: Terminal Impedances Equal to Characteristic Impedances

Here we have

$$Z_0 = Z_s = K, \tag{46}$$

$$A(x, y) = \frac{1}{2K} e^{-\gamma|x-y|}, \tag{47}$$

$$B(x, y) = \mp \tfrac{1}{2} e^{-\gamma|x-y|}, \qquad y \lessgtr x, \tag{48}$$

whence

$$j(x) = \frac{1}{2K} \int_0^s e^{-\gamma|x-y|} f(y)dy \tag{49}$$

$$= \frac{1}{2K} \int_0^x e^{-\gamma(x-y)} f(y)dy + \frac{1}{2K} \int_x^s e^{-\gamma(y-x)} f(y)dy, \tag{50}$$

$$J_0(x) = -\frac{F(0)}{2K} e^{-\gamma x}, \tag{51}$$

$$J_s(x) = \frac{F(s)}{2K} e^{-\gamma(s-x)}, \tag{52}$$

$$J_{0s}(x) = -\frac{Y'}{2} \int_0^x F(y) e^{-\gamma(x-y)} dy + \frac{Y'}{2} \int_x^s F(y) e^{-\gamma(y-x)} dy, \tag{53}$$

$$J_u(x) = -\frac{\Delta F(u)}{2K} e^{-\gamma|x-u|}. \tag{54}$$

A Balanced Two-Wire Line in an Arbitrary Impressed Field

Because the metallic circuit here contemplated is balanced, its treatment can be made formally the same as the treatment of the one-wire line in the preceding subsection.

This fact is immediately evident in two special cases of the impressed electric field (potential and axial electric force): (1) the case where the impressed electric field has equal values at the two wires, and (2) the case where it has equal but opposite values at the two wires.

The general case where the impressed field at the two wires has any values can be treated as a superposition of the two special cases just

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Page 514, Equations (48), (49) and (54) *should read:*

$$B(x, y) = \mp \tfrac{1}{2} e^{-\gamma |x-y|}, \qquad y \lessgtr x, \tag{48}$$

whence

$$j(x) = \frac{1}{2K} \int_0^s e^{-\gamma |x-y|} f(y) dy \tag{49}$$

$$J_u(x) = -\frac{\Delta F(u)}{2K} e^{-\gamma |x-u|}. \tag{54}$$

Page 519, Line 1: *read* "Set 2 (Fig. 7)" *for* "Set 4 (Fig. 8)."

Page 530, Equation (91) *should read:*

$$V_{k\pi} = -\sum_{h=1}^{n} W_{kh} \left(\frac{dI_h}{dx} + \sum_{r=1}^{n} X_{hr} F_r \right), \qquad (k = 1, \cdots n), \tag{91}$$

mentioned, by the simple device of resolving the impressed field at each wire into two constituents one of which has equal values at the two wires while the other has equal but opposite values at the two wires. This resolution is always possible, for it is merely in accordance with the following pair of algebraic identities:

$$\eta_1 = \tfrac{1}{2}(\eta_1 + \eta_2) + \tfrac{1}{2}(\eta_1 - \eta_2), \tag{55}$$

$$\eta_2 = \tfrac{1}{2}(\eta_1 + \eta_2) - \tfrac{1}{2}(\eta_1 - \eta_2). \tag{56}$$

Although η_1 and η_2 may in general denote any two quantities whatever, in the present application they refer to the impressed electric field at the two wires No. 1 and No. 2 of the contemplated two-wire line. It is convenient to introduce the symbols η_c and η_a defined by the equations

$$\eta_c = \tfrac{1}{2}(\eta_1 + \eta_2), \tag{57}$$

$$\eta_a = \eta_1 - \eta_2, \tag{58}$$

so that the resolutions (55) and (56) of η_1 and η_2 can be written in the more compact forms

$$\eta_1 = \eta_c + \tfrac{1}{2}\eta_a, \tag{59}$$

$$\eta_2 = \eta_c - \tfrac{1}{2}\eta_a. \tag{60}$$

η_c and η_a will be termed respectively the mode-*c* and mode-*a* constituents of the impressed field, because they give rise to mode-*c* and mode-*a* effects respectively; mode-*c* effects being defined as those which are equal in the two wires, mode-*a* effects as those which are equal but opposite in the two wires—as discussed in connection with equations (28). From (57) and (58) respectively it will be noted that the mode-*c* effects and the mode-*a* effects depend respectively on the average and on the difference of the impressed fields at the two wires.

As in treating the one-wire line (in the preceding subsection), so also in treating the balanced two-wire line (in the present subsection) it is usually advantageous to deal separately with the axial electric force and the potential of the impressed field. Furthermore, in the case of the two-wire line each of these constituents of the impressed field is to be resolved into two modes, *c* and *a*, in the manner represented by equations (59) and (60) together with (57) and (58).

Owing to the balance (bilateral symmetry) of the assumed two-wire line, the mode-*c* constituent η_c of the impressed field will produce only mode-*c* effects, and the mode-*a* constituent only mode-*a* effects. Thus, η_c will produce equal currents I_c and I_c in the two wires, while η_a will produce equal but opposite currents I_a and $-I_a$ in the two wires. The total mode-c current $2I_c$ along the two wires in parallel is calculable from η_c through the mode-c parameters (γ_c, K_c, and terminal impedances) of the system; while the mode-a or loop current (I_a and $-I_a$ in the two wires respectively) is calculable from η_a through the mode-a parameters (γ_a, K_a, and terminal impedances). The connection of each current constituent with the corresponding field constituent, through the corresponding parameters, is formally the same as for the one-wire line (treated in the preceding subsection).

Finally, it may be remarked that the assumption of balance (bilateral symmetry) for the two-wire line is essential to the above simplicity; for otherwise each mode of the impressed field would produce components of both modes of effects, instead of only the appropriate single mode of effects.

Illustrative Special Case

For illustration it will suffice to choose the simple case of a balanced two-wire line terminated at each end in its mode-c and mode-a characteristic impedances simultaneously. That is, the line consisting of the two wires in parallel, with ground return, is terminated at each end in the mode-c characteristic impedance K_c; while the loop circuit is terminated at each end in the mode-a characteristic impedance K_a. (Evidently these two modes of terminating can be simultaneously accomplished by means either of a balanced T-network or of a balanced Π-network at each end.)

Let $f_1(x)$ and $f_2(x)$ denote the axial impressed electric forces at any point x in wires No. 1 and No. 2 respectively; and let them be resolved into mode-c and mode-a constituents $f_c(x)$ and $f_a(x)$, respectively, such that

$$f_c(x) = \tfrac{1}{2}[f_1(x) + f_2(x)], \tag{61}$$

$$f_a(x) = f_1(x) - f_2(x), \tag{62}$$

in accordance with equations (57) and (58). Similarly, let $F_1(x)$ and $F_2(x)$ denote the impressed potentials at point x; and let them be resolved likewise, so that

$$F_c(x) = \tfrac{1}{2}[F_1(x) + F_2(x)], \tag{63}$$

$$F_a(x) = F_1(x) - F_2(x). \tag{64}$$

Thus, formulas (49), $\cdots$ (54) of the one-wire line are seen to be formally applicable to the balanced two-wire line, for calculating separately the two modes of currents. This is with the understanding that they give the sum of the mode-c currents in the two wires, hence twice the mode-c current in each wire; and that they give the loop current (the current circulating in the metallic loop), which is equal to the mode-a current in one of the wires and hence to the negative of the mode-a current in the other wire.

Six Different Sets of Equivalent Electromotive Forces for a One-wire[16] *Line in an Arbitrary Impressed Field*

The physical system here contemplated (Fig. 2 below) is a one-wire transmission line consisting of a uniform horizontal straight wire situated in an arbitrary impressed field and terminated in any arbitrary impedances to ground. The wire extends from $x = 0$ to $x = s$; the arbitrary terminal impedances[17] are denoted by Z_0 and Z_s. The arbitrary impressed field is specified, at each point of the wire, by the impressed axial electric force $f(x)$ and the impressed potential $F(x)$, as previously.

Six different sets of 'equivalent electromotive forces' are formulated in the early part of this subsection; while their derivations are briefly outlined toward the latter part. Set 1 will be recognized as a particular case of the 'fundamental' set already formulated in the early part of Section III. The five remaining sets are derived from Set 1. In the actual formulations of these various sets of equivalent electromotive forces, the impressed potential $F(x)$ is assumed to be a continuous function of x; the extension to the case where $F(x)$ is discontinuous is a simple matter and is formulated in connection with equations (65) and (66).

In the following diagrams (Figs. 2, $\cdots$ 11) it is found convenient to represent any localized electromotive force by the conventional battery-symbol. This symbol is intrinsically directional; the longer of the two plates is to be regarded as at the higher potential, so that there is an internal rise of potential in passing through the symbol from the shorter to the longer plate.

In some of the figures the actual line is represented as replaced by the corresponding artificial line composed of differential elements, each of length dx. (For clearness, the line is represented as composed of only a small number of such elements.)

The letters Z, Y, Y', Y^0 denote certain line parameters per unit length, as follows: Z and Y respectively denote the 'complete series impedance' and the 'complete shunt admittance' or, briefly, the 'series impedance' and the 'shunt admittance.' These may be regarded as defined by the equations

$$Z = \gamma K, \qquad Y = \gamma / K,$$

[16] The case of a balanced two-wire line is outlined in the next subsection.
[17] See also the remarks under the subheading following shortly after equation (67).

γ denoting the propagation constant of the line per unit length, and K the characteristic impedance. Or they may be regarded as defined by the differential equations

$$-dV/dx = ZI, \qquad -dI/dx = YV,$$

characterizing the line when there is no impressed field present. Y' denotes the 'direct leakage admittance' and Y^0 the 'basic shunt admittance,' the latter defined as being the value of Y when $Y' = 0$, whence $Y = Y^0 + Y'$. On referring to equations (12), (11), (8), (1) of Appendix I, and also to equations (2) and (7) in Section I, it is seen that [18]

$$Z = z + i\omega L, \qquad Y = G + i\omega C,$$
$$G = G^0 + Y', \qquad Y^0 = G^0 + i\omega C.$$

The various sets of equivalent electromotive forces remain valid even when the line parameters are functions $Z(x)$, $Y(x)$, etc., of position x along the system. For the 'fundamental' set this fact can be readily seen by reference to the formulation and verification of the fundamental set, in the early part of Section III.

As indicated by the arrows, the positive axial (longitudinal) direction is the direction of increasing x, and the positive vertical direction is downward.

Six Different Sets of Equivalent Electromotive Forces

Set 1 (Fig. 4)

(*A*) In the wire, a distributed electromotive force, $f(x)dx$ in each differential length dx.

(*B*) In the distributed direct leakage admittance, a distributed electromotive force, $F(x)$ in each differential element $Y' \cdot dx$ of direct leakage admittance.

(*C*) In the termial impedances Z_0 and Z_s, electromotive forces $F(0)$ and $F(s)$ respectively.

From the physical viewpoint of the present paper, Set 1 is the fundamental set of equivalent electromotive forces.

This set is particularly simple when there is no direct leakage admittance ($Y' = 0$), for then it reduces to merely the axial constituents (*A*) and the terminal constituents (*C*).

[18]Thus Z, unsubscripted, includes the internal impedance $z = z_w + z_g$ of the circuit, and hence is to be sharply distinguished from the double-subscripted Z occurring frequently in this paper; for, as remarked in connection with equation (2), Z_{jj} does not include the internal impedance z_{jj} of wire j, whence it is seen that $Z = Z_{jj} + z_{jj}$ for wire j.

Set 4 (Fig. 8)

(A) In the wire, a distributed electromotive force, $[f(x) + (Y'/Y) \times dF(x)/dx]dx$ in each differential length dx.

(B) In the terminal impedances Z_0 and Z_s, electromotive forces $(1 - Y'/Y)F(0)$ and $(1 - Y'/Y)F(s)$ respectively.

Set 2 is distinguished by containing no electromotive forces in the shunt admittance even when the direct leakage admittance Y' is not negligible. Thus Set 2 with the direct leakage admittance not negligible is *formally* as simple as Set 1 with the direct leakage admittance negligible. However, in Set 2 the element of axial electromotive force is a much more complicated function than in Set 1.

Set 3 (Fig. 9)

(A) In the wire, a distributed electromotive force, $[f(x) + dF(x)/dx]dx$ in each differential length dx.

(B) In the distributed basic shunt admittance, a distributed electromotive force, $-F(x)$ in each differential element Y^0dx of the distributed basic shunt admittance.

In Set 3 it should be noted that the electromotive force $-F(x)$ is in the basic shunt admittance element Y^0dx, not in the complete shunt admittance element Ydx.

It will be observed that this set contains no electromotive forces in the terminal impedances.

When the ground is a perfect conductor, so that $f_g(x) = 0$, the differential element of axial electromotive force in this set reduces to merely $-[d\Phi(x)/dt]dx$, as is shown by equation (2) of Appendix I, $\Phi(x)$ denoting the impressed magnetic flux.

Set 4 (Fig. 8)

(A) In the wire, a distributed electromotive force, $[f(x) + (1 + Y'/Y)dF(x)/dx]dx$ in each differential length dx.

(B) In the distributed complete shunt admittance, a distributed electromotive force, $-F(x)$ in each differential element Ydx of the complete shunt admittance.

(C) In the terminal impedances Z_0 and Z_s, electromotive forces, $-(Y'/Y)F(0)$ and $-(Y'/Y)F(s)$ respectively.

In Set 4 it should be noted that the electromotive force $-F(x)$ is in the complete shunt admittance element Ydx.

The differential element of axial electromotive force in this set does not reduce to $-[d\Phi(x)/dt]dx$ when the ground is a perfect conductor ($f_g(x) = 0$) unless also the direct leakage admittance is zero ($Y' = 0$).

Set 5 (Fig. 6)

(A) In the wire, a distributed electromotive force, $f(x)dx$ in each differential length dx.

(B) In the distributed complete shunt admittance, a distributed electromotive force, $(Y'/Y)F(x)$ in each differential element Ydx of the complete shunt admittance.

(C) In the terminal impedances Z_0 and Z_s, electromotive forces $F(0)$ and $F(s)$ respectively.

In Set 5 it should be noted that the electromotive force $(Y'/Y)F(x)$ is in the complete shunt admittance element Ydx.

It will be observed that Set 5 (Fig. 6) is the same as Set 1 (Fig. 4) as regards the axial and the terminal electromotive forces.

Set 6 (Fig. 11)

(A) At any arbitrary fixed point $x = a$ in the wire, an axial electromotive force G_a,

$$G_a = \int_0^s \left[f(x) + \frac{dF(x)}{dx} \right] dx.$$

(B) In each differential element Ydx of the distributed complete shunt admittance, a distributed electromotive force E_x,

$$E_x = \int_0^x \left[f(x) + \frac{Y'}{Y}\frac{dF(x)}{dx} \right] dx, \qquad x < a,$$

$$E_x = -\int_x^s \left[f(x) + \frac{Y'}{Y}\frac{dF(x)}{dx} \right] dx, \qquad x > a.$$

(C) In the terminal impedances Z_0 and Z_s, electromotive forces $(1 - Y'/Y)F(0)$ and $(1 - Y'/Y)F(s)$ respectively.

Set 6 is perhaps mainly of academic interest.

Two limiting cases of Set 6 may be noted, corresponding to $a = 0$ and $a = s$ respectively, each characterized by containing no *internal* axial electromotive force: for when $a = 0$ the axial electromotive force G_a can be combined with the terminal electromotive force $(1 - Y'/Y)F(0)$ in the terminal impedance Z_0, and when $a = s$ it can be combined with $(1 - Y'/Y)F(s)$ in Z_s.

Extension to the Case where the Impressed Potential is Discontinuous

In the foregoing formulations of Sets 1, $\cdots$ 6 of equivalent electromotive forces it has been assumed that the impressed potential $F(x)$ is a continuous function of x throughout the length of the line.

Suppose now, for greater generality, that the impressed potential $F(x)$ is discontinuous at any point $x = u$ by the increment

$$\Delta F(u) = F(u+) - F(u-).$$

Then (as shown in the next paragraph), for the particular differential element which contains the point u, the quantities $f(x)dx$ and $[dF(x)/dx]dx$ must be replaced by $-\Delta F(u)$ and $\Delta F(u)$ respectively; that is,

$$f(u)du = -\Delta F(u), \tag{65}$$

$$\frac{dF(u)}{du}du = \Delta F(u). \tag{66}$$

Equations (65) and (66) can be obtained, by a limiting process, from equation (2) of Appendix I, which for the present purpose will be written in the form

$$f(x)dx = -\frac{dF(x)}{dx}dx - \frac{d\Phi(x)}{dt}dx + f_0(x)dx. \tag{67}$$

It will be recalled, from Appendix I, that this equation was derived by applying the second curl law to a differential rectangle extending from x to $x + dx$; but x may equally well be a point within the differential segment dx, and for the present purpose it will be so regarded. The limiting process now consists in letting $dF(x)/dx$ approach infinity while dx approaches zero, but in such a way that the product $[dF(x)/dx]dx$ approaches a preassigned finite value, denoted by $\Delta F(x)$. Then, in the limit, the last two terms on the right side of (67) vanish so that (67) reduces to

$$f(x)dx = -\Delta F(x).$$

Thus we obtain equations (65) and (66), where u denotes, for distinction, the particular value of x at which $F(x)$ is discontinuous.

Remarks on the Terminal Impedances and the Equivalent Electromotive Forces in Them

The arbitrary terminal impedances Z_0 and Z_s (Fig. 2) need not actually be localized. They may, for instance, be the impedances offered by other lines to which the given line 0-s may be connected, and these other lines may themselves be situated in arbitrary impressed fields; in particular, 0-s may be merely a segment, of any length, forming part of a given line in an arbitrary impressed field.

From this broad view, any 'equivalent electromotive forces' situated in the terminal impedances Z_0 and Z_s may advantageously be regarded as being situated in the ends of the line itself (that is, in the end-points $x = 0$ and $x = s$), these electromotive forces being then regarded as pertaining primarily to the line-segment 0-s rather than to the terminal

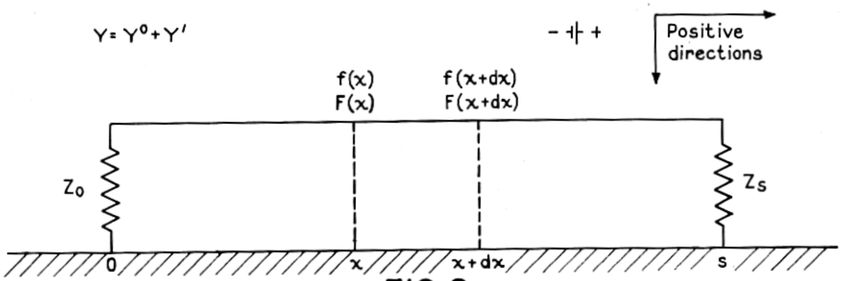

FIG. 2

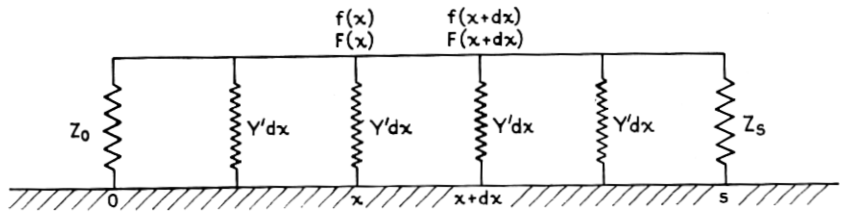

FIG. 3

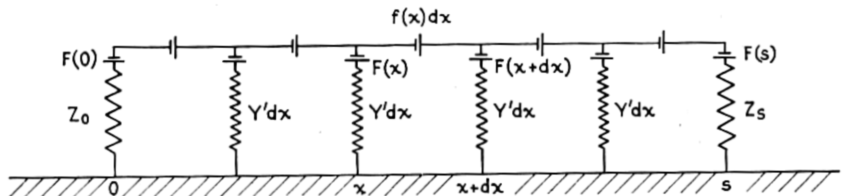

FIG. 4 (SET 1)

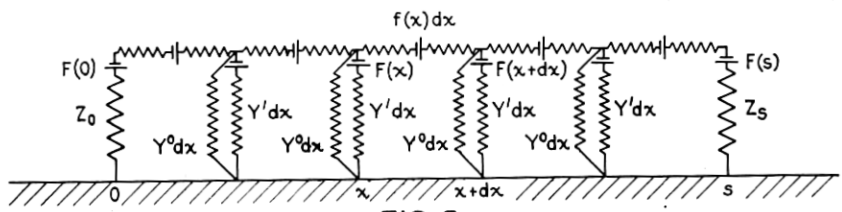

FIG. 5

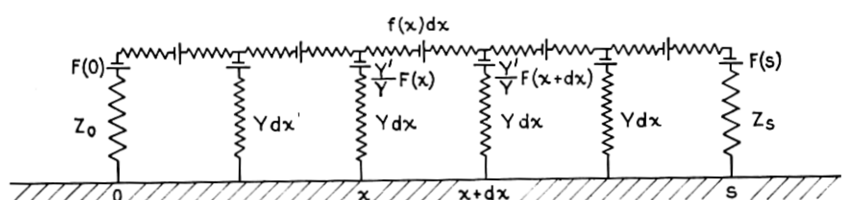

FIG. 6 (SET 5)

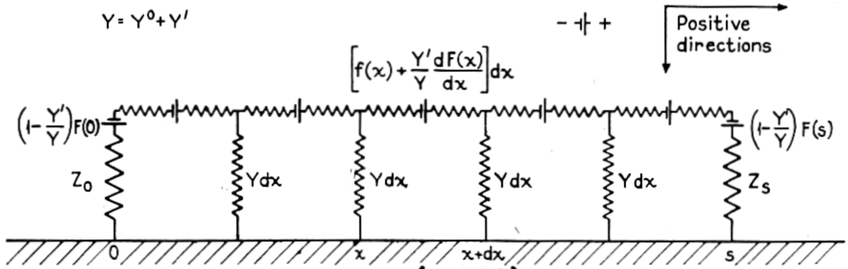

FIG. 7 (SET 2)

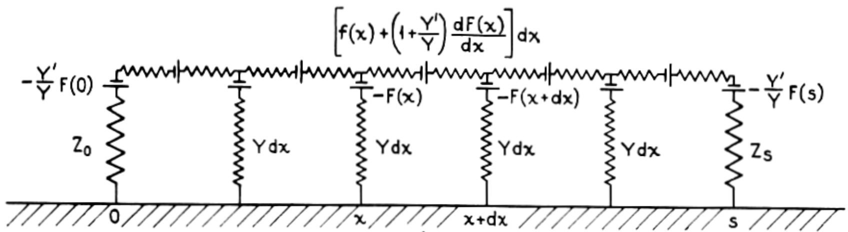

FIG. 8 (SET 4)

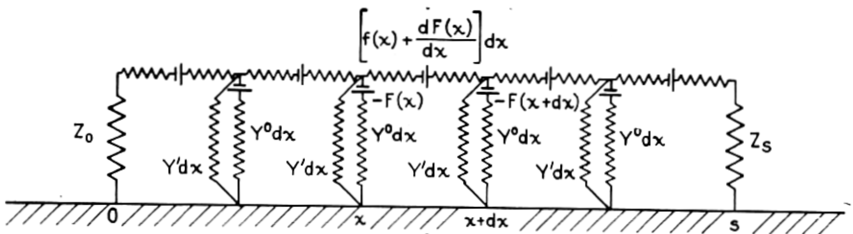

FIG. 9 (SET 3)

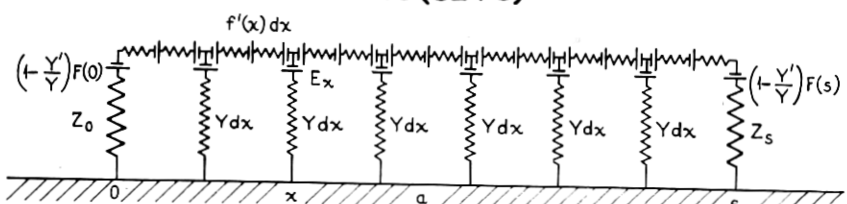

FIG. 10

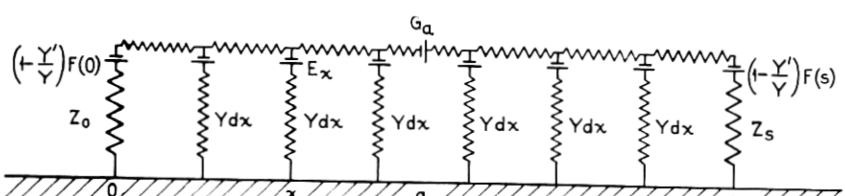

FIG. 11 (SET 6)

impedances. Thus, for instance, in the formulations of Set 1 and Set 5, item (C) would read: '(C) In the ends $x = 0$ and $x = s$ of the line, axial electromotive forces $- F(0)$ and $F(s)$ respectively.' (Observe, here, the negative sign before $F(0)$, in contrast to the positive sign in the original formulation.)

In this way it is readily seen that at a point $x = u$ where the impressed potential $F(x)$ is discontinuous, the equivalent electromotive force is an axial electromotive force equal to the decrement of the impressed potential, that is, equal to $F(u -) - F(u +)$; this agrees with equation (65), and with item (B) in the fundamental set of equivalent electromotive forces formulated in the early part of Section III.

Derivations of Set 1

A synthetic derivation of Set 1 has already been furnished in the early part of Section III. An analytical derivation will now be outlined; it is based on an interpretation of equations (68), (71), (73) below; these equations, in turn, are based on certain equations of Appendix I, as follows:

Combining equations (1) and (2) of Appendix I gives

$$zI + \frac{dV'}{dx} + \frac{d\phi'}{dt} = f, \tag{68}$$

where f denotes f_w; and V' is that part of the potential of the wire due to its charges (and the corresponding opposite charges on the surface of the ground), while ϕ' is that part of the magnetic flux due to the current in the wire (and the corresponding return current in the ground); that is,

$$V' = V - F = Q/C, \tag{69}$$

$$\phi' = \phi - \Phi = LI, \tag{70}$$

so that V' and ϕ' do not include the impressed potential and impressed magnetic flux F and Φ respectively.

By (5) and (7) of Appendix I the equation of current continuity can be written

$$-\frac{dI}{dx} = \frac{dQ}{dt} + \frac{G}{C}Q + Y'F. \tag{71}$$

The actual potential V of the line is of course the resultant of V' and F; that is,

$$V \equiv V(x) = V'(x) + F(x), \tag{72}$$

whence, in particular, at the ends $x = 0$ and $x = s$,

$$\begin{aligned} V(0) &= V'(0) + F(0), \\ V(s) &= V'(s) + F(s). \end{aligned} \tag{73}$$

Returning, now, to a consideration of equations (68), (71), (73), it is seen that they are identically the same as the equations for the same line without any impressed field but containing the set of electromotive forces formulated above under the heading 'Set 1 (Fig. 4)'; for an interpretation of equations (68), (71), (73) yields respectively (A), (B), (C) of Set 1.

It may be noted that equations (68) and (71) can be written in the following more compact forms:

$$-\,dV'/dx = ZI - f, \tag{74}$$

$$-\,dI/dx = YV' + Y'F, \tag{75}$$

whose interpretation yields immediately items (A) and (B) of Set 1.

Outline of Derivations of Sets 2, 3, 4, 5, 6

Synthetic derivations of Sets 2, 3, 4, 5, 6 from Set 1 will now be briefly outlined by aid of the diagrams in Figs. 2, $\cdots$ 11. The physical systems represented by these diagrams are all equivalent in the sense that the currents at corresponding points in all of them are equal.

In the derivation-work extensive use is made of an artifice which, for convenience, will be formulated in what may be termed the 'branch-point theorem,' as follows: *In any network of any number of branches the currents will not be affected by inserting at any branch-point a set of equal electromotive forces, one in each branch, directed either all toward or all from the branch-point.*

Fig. 2 represents the given one-wire line in an arbitrary impressed field, as already specified. For generality the line is assumed to have uniformly distributed leakage admittance of amount Y' per unit length.

Fig. 3 is derived from Fig. 2 by lumping the distributed direct leakage admittance into localized admittances each of amount $Y' \cdot dx$ at intervals of length dx.

Fig. 4 is derived from Fig. 3 by replacing the arbitrary impressed field by Set 1 of equivalent electromotive forces.

Fig. 5 is derived from Fig. 4 by replacing the line, exclusive of the direct leakage admittance Y', by its equivalent artificial line having 'complete series impedance' Z and 'basic shunt admittance' Y^0 per unit length. This replacement of the actual line by the corresponding artificial line is permissible now that the impressed field has been replaced by a set of equivalent electromotive forces (Fig. 4).

Fig. 6 is derived from Fig. 5 by replacing the compound shunt element, consisting of Y^0dx in parallel with $Y'dx$ containing the electromotive force $F(x)$, by the equivalent simple shunt element Ydx containing the electromotive force $(Y'/Y)F(x)$.

Figs. 7, 8, 9 are derived from Figs. 6, 7, 5 respectively by applying the 'branch point theorem.'

Fig. 11 is derived from Fig. 7 by applying the 'branch point theorem' in the manner indicated by Fig. 10, where $f'(x)dx$ denotes, for brevity, the original axial equivalent electromotive force situated between x and $x + dx$ of Set 2 as indicated by Fig. 7, so that

$$f'(x) \equiv f(x) + \frac{Y'}{Y}\frac{dF(x)}{dx}, \tag{76}$$

and a is the coordinate of the contemplated arbitrary point. The E's, of which E_x is typical, are sets of electromotive forces inserted at the branch-points. At first these electromotive forces are arbitrary, except that each set of three accords with the branch-point theorem, so as not to alter the original currents in the system. Next, starting at the ends, it is found that these electromotive forces can be so determined as to annul the original axial electromotive forces $f'(x)dx$ in all of the differential elements dx except in the one containing the point a; the requisite value of E_x and the resulting value of G_a are found to be as formulated in Set 6.

Finally it may be remarked that each of the Sets 2, 3, 4, 5, 6 can be verified against Set 1 by formulating the total current produced at any point x by each of the Sets 2, 3, 4, 5, 6 and then comparing the resulting formula with the sum of formulas (41), (42), (43), (44). Evidently it suffices to do this for the relatively simple case where the terminal impedances are equal to the characteristic impedance of the line; for this case, formulas (41), (42), (43), (44) reduce to (50), (51), (52), (53) respectively.

Sets of Equivalent Electromotive Forces for a Balanced Two-Wire Line in an Arbitrary Impressed Field

The foregoing six sets of equivalent electromotive forces for a one-wire line can be readily extended to a two-wire line after resolving the impressed field into mode-a and mode-c constituents, which are then dealt with separately. For Set 1 this procedure has been fully outlined above in the subsection entitled 'A Balanced Two-Wire Line in an Arbitrary Impressed Field,' and it has found a natural application in the 'Crosstalk Problem' treated below in Section IV.

It is clear that all of the sets of equivalent electromotive forces are immediately applicable to dealing with the mode-c constituent of the impressed field, since this constituent acts on the circuit consisting of the two wires in parallel with each other, with ground return, which is formally the same as a one-wire line with ground return.

All of the sets of equivalent electromotive forces become applicable to dealing with the mode-a constituent of the impressed field by an appropriate interpretation of the diagrams (Fig. 2, $\cdots$ 11), namely, the following interpretation:

1. In each diagram regard the wire-symbol as representing the outgoing wire of the actual two-wire line, and regard the ground-symbol as representing not the ground but the return wire of the two-wire line. (The presence of the earth is then to be regarded as implied, its effects appearing implicitly in the values of the line parameters.)

2. Hence regard Z_0 and Z_s as denoting the mode-a terminal impedances functioning as though connected directly across the two-wire line at its ends $x = 0$ and $x = s$ respectively.

3. Regard Y', Y^0, Y, Z as denoting the mode-a line constants (including implicitly the effects of the earth).

4. Regard $f(x)$, $F(x)$, and $\Phi(x)$ as denoting the mode-a constituents of the impressed field—that is, as denoting the difference of the actual values impressed at the two wires. (In order to maintain the balanced condition of the two-wire line, $f(x)$ is to be regarded as constituted of $f(x)/2$ in the outgoing wire and $-f(x)/2$ in the return wire; and similarly for $F(x)$ and $\Phi(x)$.)

The Electric Field Due to a System of n Parallel Wires in an Arbitrary Impressed Field

Thus far in the present section of this paper the field impressed on the given physical system has been supposed known and the problem has been to calculate the resulting currents. Actually, however, the impressed field is not usually known but has to be calculated—from a knowledge of the currents and charges producing it.

The present subsection deals with the problem of calculating the electric field impressed on a secondary system consisting of a single horizontal wire j by a primary system π consisting of n wires which are parallel to each other and to j. For generality, the primary and secondary systems are supposed to be in an arbitrary impressed field.[19]

Consider at first any parallel geometrical line i, not necessarily in any of the wires; and let $V_i = V_i(x)$ and $E_i = E_i(x)$ denote the

[19] Of course the field produced by any given system is directly due only to the currents and charges of the system, and does not depend directly on any field that may be impressed on the system; but, assuming the system to be energized only by the impressed field, the currents—and thence the charges—are directly due to the impressed field and can (theoretically, at least) be expressed in terms of it.

potential and the axial electric force at any point x in i. Then V_i is analyzable into three parts ($V_{i\pi}$, V_{ij}, F_i) and E_i into three parts ($E_{i\pi}$, E_{ij}, f_i) due respectively to the primary system π, to the secondary system j, and to the arbitrary impressed field; that is,[20]

$$V_i = V_{i\pi} + V_{ij} + F_i, \tag{77}$$

$$E_i = E_{i\pi} + E_{ij} + f_i. \tag{78}$$

In particular, at the secondary wire j the potential V_j and the axial electric force E_j are analyzable in accordance with the equations

$$V_j = V_{j\pi} + V_{jj} + F_j, \tag{79}$$

$$E_j = E_{j\pi} + E_{jj} + f_j. \tag{80}$$

In the present subsection, the problem to be dealt with is the calculation of $V_{j\pi}$ and $E_{j\pi}$, namely the potential and the axial electric force at any point x in the secondary j due directly to the currents and charges of the primary system π.

The n wires of the primary system π will be numbered 1, 2, 3, $\cdots n$. The letters h, k, r will be employed generically: each may denote any one of the designation numbers 1, $\cdots n$—as h in equation (81); or each may run through the whole set 1, $\cdots n$—as in equation (91). (It is hardly necessary to remark that j is *not* a member of the set 1, $\cdots n$, in the notation of this Section (III), where j *always* designates the secondary wire.)

The current at any point x in any wire h of the primary will be denoted by $I_h = I_h(x)$; and the charge on wire h, per unit length, by $Q_h = Q_h(x)$.

The potential V_h and the axial electric force E_h at any primary wire h are analyzable in the manner expressed by the equations

$$V_h = V_{h\pi} + V_{hj} + F_h, \tag{81}$$

$$E_h = E_{h\pi} + E_{hj} + f_h, \tag{82}$$

in accordance with the general equations (77) and (78) respectively. It will be found convenient to call $V_{h\pi}$ the 'systemic potential' and $E_{h\pi}$ the 'systemic axial electric force' at wire h, since $V_{h\pi}$ and $E_{h\pi}$ are due only to the system π of which h is a member, and do not include

[20] Regarding the use here of double subscripts, it will be noted that the first subscript designates the line or the wire where the effect occurs, and the second the wire or the system of wires which produce the effect. Thus, $V_{i\pi}$ is the potential produced in line i by the whole primary system π; the contribution of any one wire h would be denoted by V_{ih}.

any contributions from the secondary system or from the impressed field. (More fully, $V_{h\pi}$ may be termed the 'primary systemic potential' at h and $E_{h\pi}$ the 'primary systemic axial electric force' at h.)

For explicit use below, we may here note the formulas for the systemic potential $V_{k\pi}$ and the systemic axial electric force $E_{k\pi}$ at any wire k of the primary system π:

$$V_{k\pi} = \sum_{h=1}^{n} p_{kh}Q_h, \qquad (k = 1, \cdots n), \tag{83}$$

$$E_{k\pi} = -\sum_{h=1}^{n} \left(Z_{kh}I_h + p_{kh}\frac{dQ_h}{dx} \right), \qquad (k = 1, \cdots n), \tag{84}$$

p_{kh} and Z_{kh} being respectively the mutual potential coefficient and the mutual impedance[3] between wires h and k, per unit length. Equation (84) is obtainable by applying the second curl law to a differential rectangle substantially as in deriving equations (1) and (2); see also Appendix I.

As already stated in connection with equations (79) and (80) the problem to be considered in the present subsection is the calculation of the potential $V_{j\pi}$ and the axial electric force $E_{j\pi}$ produced at the secondary wire j by the primary system π. The fundamental formulas for $V_{j\pi}$ and $E_{j\pi}$ are:

$$V_{j\pi} = \sum_{h=1}^{n} p_{jh}Q_h = \sum_{h=1}^{n} V_{jh}, \tag{85}$$

$$E_{j\pi} = -\sum_{h=1}^{n} \left(Z_{jh}I_h + \frac{dV_{jh}}{dx} \right) = \sum_{h=1}^{n} E_{jh} \tag{86}$$

$$= -\sum_{h=1}^{n} \left(Z_{jh}I_h + p_{jh}\frac{dQ_h}{dx} \right), \tag{87}$$

where V_{jh} and E_{jh} are the contributions of wire h to $V_{j\pi}$ and $E_{j\pi}$ respectively.

With regard to applications of the equations (85) and (87) for the potential $V_{j\pi}$ and the axial electric force $E_{j\pi}$ impressed on the secondary j by the primary π, it will be supposed that all the primary currents $I_1, \cdots I_n$ are known. But the primary charges Q_h and their axial gradients dQ_h/dx (where $h = 1, \cdots n$) are usually not known; and therefore ways will now be indicated for expressing them in terms of quantities which may be known. For that purpose, the presence of the secondary will be entirely ignored, in all respects. (This procedure may be regarded as the first-approximation step in a solution by successive approximations.)

The charges Q_h can be expressed in terms of the systemic potentials $V_{k\pi}$ by solving the set of n equations (83). Thus

$$Q_h = \sum_{k=1}^{n} q_{hk} V_{k\pi}, \qquad (h = 1, \cdots n), \tag{88}$$

where q_{hk} is the Maxwell capacity coefficient between wires h and k; in terms of the potential coefficients, its value is

$$q_{hk} = D_{kh}(p)/D(p), \tag{89}$$

$D(p)$ being the determinant of all the potential coefficients (the p's) in the set of n equations (83) and $D_{kh}(p)$ the cofactor of p_{kh} in $D(p)$.

The systemic potentials $V_{k\pi}$, occurring in (88), can be obtained by solving the equations of current continuity, namely the set of n equations [21]

$$-\frac{dI_h}{dx} = \sum_{k=1}^{n} (Y_{hk} V_{k\pi} + X_{hk} F_k), \qquad (h = 1, \cdots n), \tag{90}$$

Y_{hk} and X_{hk} being of the nature of admittances (per unit length), and F_k the impressed potential at wire k; it is thus found that

$$V_{k\pi} = -\sum^{n} W_{kh} \left(\frac{dI_h}{dx} + \sum_{r=1}^{n} X_{hr} F_r \right), \qquad (k = 1, \cdots n), \tag{91}$$

where the coefficient W_{kh} is the same function of the Y's that q_{kh} is of the p's, that is,

$$W_{kh} = D_{hk}(Y)/D(Y). \tag{92}$$

It is seen that W_{kh} is of the nature of an impedance (per unit length), though it is not a simple impedance.

The charges can now be expressed in terms of the impressed potentials F_r and the axial gradients of the currents by substituting (91) in (88).

The axial gradients of the charges can be expressed in various ways. They can be immediately expressed in terms of the axial gradients of the systemic potentials V_k by merely differentiating (88) with respect to x. Also, they can be expressed in terms of the currents I_r and the systemic axial electric forces $E_{k\pi}$ at the wires, by solving the set of n equations (84); thus

$$\frac{dQ_h}{dx} = -\sum_{k=1}^{n} q_{hk} (E_{k\pi} + \sum_{r=1}^{n} Z_{kr} I_r), \qquad (h = 1, \cdots n), \tag{93}$$

[21] Derived in the latter part of Appendix I.

q_{hk} being the Maxwell capacity coefficient given by (89). Furthermore, the systemic axial electric force $E_{k\pi}$ occurring in (93) is expressible in terms of the current I_k in wire k and the axial electric force f_k impressed on wire k, by the simple relation

$$E_{k\pi} = z_k I_k - f_k, \tag{94}$$

z_k denoting the internal impedance of wire k, per unit length; for, the resultant axial electric force at wire k must be equal to $z_k I_k$ and must also be equal to $E_{k\pi} + f_k$. Thus the axial gradients of the charges can be expressed explicitly in terms of the currents and the impressed axial electric forces at the wires, by substituting (94) in (93). The axial gradients of the charges can be expressed still otherwise by differentiating (88) with respect to x after substituting (91).

Substituting into (85) and (87), the various foregoing expressions for the charge Q_h and its axial gradient dQ_h/dx, and in some cases transforming and rearranging the results, gives the following formulas for the potential $V_{j\pi}$ and the axial electric force $E_{j\pi}$ produced at any point x in the secondary wire j by the primary system π, when the presence of the secondary j is entirely ignored in calculating the currents, charges, and potentials of the primary (in accordance with the statement of the paragraph following equation (87)):

$$V_{j\pi} = \sum_{h=1}^{n} p_{jh} Q_h \tag{95}$$

$$= \sum_{h=1}^{n} T_{jh} V_{h\pi} \tag{96}$$

$$= -\sum_{h=1}^{n} T_{jh} \left[\sum_{k=1}^{n} W_{hk} \left(\frac{dI_k}{dx} + \sum_{r=1}^{n} X_{kr} F_r \right) \right], \tag{97}$$

where T_{jh}, which may be termed a 'potential transfer factor' or 'voltage transfer factor,' has the value

$$T_{jh} = \sum_{k=1}^{n} p_{jk} q_{kh}. \tag{98}$$

$$E_{j\pi} = -\sum_{h=1}^{n} Z_{jh} I_h - \frac{dV_{j\pi}}{dx} \tag{99}$$

$$= -\sum_{h=1}^{n} \left(Z_{jh} I_h + p_{jh} \frac{dQ_h}{dx} \right) \tag{100}$$

$$= -\sum_{h=1}^{n} \left(Z_{jh} I_h + T_{jh} \frac{dV_{h\pi}}{dx} \right) \tag{101}$$

$$= - \sum_{h=1}^{n} \left(Z_{jh} I_h - \sum_{k=1}^{n} T_{jh} W_{hk} \left[\frac{d^2 I_k}{dx^2} + \sum_{r=1}^{n} X_{kr} \frac{dF_r}{dx} \right] \right) \quad (102)$$

$$= - \sum_{h=1}^{n} \left(Z_{jh} I_h - T_{jh} E_{h\pi} - T_{jh} \sum_{r=1}^{n} Z_{hr} I_r \right) \quad (103)$$

$$= - \sum_{h=1}^{n} \left(Z_{jh} I_h - T_{jh} [z_h I_h - f_h] - T_{jh} \sum_{r=1}^{n} Z_{hr} I_r \right). \quad (104)$$

Formula for $E_{j\pi}$ when the Earth is a Perfect Conductor

When the earth is a perfect conductor, all of the external mutual and self impedances (Z_{jk}, Z_{hk}, Z_{kk}, etc.) are pure reactances and are proportional to the corresponding potential coefficients (p_{jk}, p_{hk}, p_{kk}, etc.), the proportionality factor being merely $i\omega/\tau$, where τ is an absolute constant whose value depends only on the units employed. Thence it can be shown that (103) and (104) respectively reduce to the very simple formulas

$$E_{j\pi} = \sum_{h=1}^{n} T_{jh} E_{h\pi}, \quad (105)$$

$$E_{j\pi} = \sum_{h=1}^{n} T_{jh} (z_h I_h - f_h), \quad (106)$$

with T_{jh} given by (98). It is seen that (105) corresponds exactly to (96).

As at least of some academic interest, it may be remarked that equations (105) and (106) hold even when the earth is imperfect, provided

$$\frac{Z_{jh}}{p_{jh}} = \frac{Z_{kh}}{p_{kh}}, \qquad (h = 1, \cdots n;\ k = 1, \cdots n).$$

IV

Practical Applications

For illustrative purposes, the methods presented in the foregoing sections will now be applied to two practical problems of a rather diverse nature. The first application will be to the wave antenna employed in certain important cases of long-distance radio reception,[22] the second, to a problem in crosstalk.

The Wave Antenna

The wave antenna, in its usual form, may be described as a transmission line with ground return, utilized for the reception of radio waves.

[22] Notably in transoceanic radio telephony.

In its simplest form, as here contemplated, the wave antenna consists of a long straight horizontal wire terminated at each end in its characteristic impedance K, as represented by Fig. 12*a*, which gives

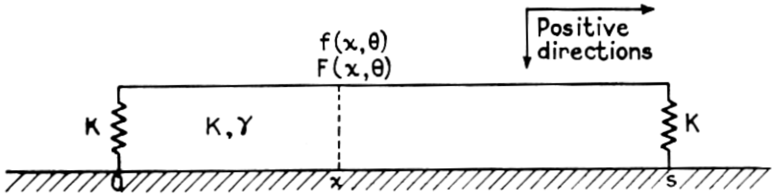

Fig. 12*a*

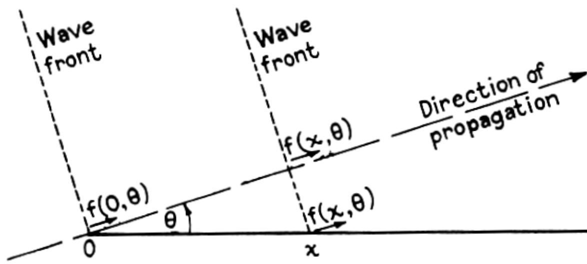

Fig. 12*b*

an elevation view. This is seen to be the same as Fig. 1 when $Z_0 = Z_s = K$; and γ is now the propagation constant, per unit length, of the wave antenna regarded as a transmission line. Hence formulas (46), $\cdots$ (54), pertaining to Fig. 1, are immediately applicable for calculating the current at any point x in the wave antenna of Fig. 12*a*, after the appropriate formulation of the functions $f(x)$ and $F(x)$—namely the impressed axial electric force and the impressed potential, respectively, at any point x in the wave antenna.

These functions can be evaluated by aid of Fig. 12*b*, which gives a plan view representing a train of plane radio waves (whose magnetic component is horizontal) incident on the wave antenna at an arbitrary angle θ measured horizontally from the wave antenna to the direction of propagation of the wave train along the earth's surface. By the 'direction of propagation' is here meant the horizontally specified direction of a vertical plane which is normal to the plane of the wave front. $f(x, \theta)$ denotes the horizontal component of electric force in the impressed waves at any point x of the wave antenna, and $F(x, \theta)$ the potential of the impressed waves there.[23] Then the axial electric

[23] The presence of θ in the functional symbols is of course to allow for a possible

force $f(x)$ and the potential $F(x)$ impressed at point x on the wave antenna by the radio waves are given by the equations

$$f(x) = f(x, \theta) \cos \theta, \tag{107}$$

$$F(x) = F(x, \theta). \tag{108}$$

It is convenient to take one end, say $x = 0$, as a fixed reference point, and then to express $f(x, \theta)$ and $F(x, \theta)$ in terms of their values $f(0, \theta)$ and $F(0, \theta)$ at $x = 0$. For this purpose, it will be assumed that the radio waves are propagated in a simple exponential manner, so that

$$\frac{f(x, \theta)}{f(0, \theta)} = \frac{F(x, \theta)}{F(0, \theta)} = e^{-\Gamma x \cos \theta}, \tag{109}$$

Γ denoting the propagation constant of the radio waves, per unit length measured horizontally along the direction of their propagation. Then the equations (107) and (108) become

$$f(x) = f(0, \theta) \cos \theta \, e^{-\Gamma x \cos \theta}, \tag{110}$$

$$F(x) = F(0, \theta) e^{-\Gamma x \cos \theta}, \tag{111}$$

wherein $f(0, \theta)$ and $F(0, \theta)$ may be supposed known. In this connection it should be remarked that $f(0, \theta)$ and $F(0, \theta)$—and, more generally, $f(x, \theta)$ and $F(x, \theta)$—are not in phase.[24]

On substitution of (110) and (111), equations (49), $\cdots$ (53) now become applicable for calculating the current $I(x)$ at any point x of the wave antenna; this current will be written $I(x, \theta)$ because it depends on the incidence-angle θ, even when $f(x, \theta)$ and $F(x, \theta)$ are independent of θ. In the engineering of wave antennæ, we are usually concerned merely with the current $I(s, \theta)$ received at the end $x = s$. In general there will be four constituents of $I(s, \theta)$, corresponding to equations (50), (51), (52), (53) when $x = s$. From the discussion of the corresponding more general equations (41), (42), (43), (44), it will be recalled that the current-constituent $j(s, \theta)$ is due to the impressed axial electric force acting throughout the length of the wave antenna, $J_0(s, \theta)$ is due to the impressed voltage $F(0, \theta)$ acting at the end $x = 0$, $J_s(s, \theta)$ is due to the corresponding impressed voltage $F(s, \theta) = F(0, \theta)e^{-\Gamma s \cos \theta}$ acting at the end $x = s$, and $J_{0s}(s, \theta)$

dependence on θ. It may be noted that, in the calculation of the ordinary polar diagram representing the directional selectivity of a wave antenna, the functions $f(x, \theta)$ and $F(x, \theta)$ are regarded as independent of θ, in accordance with the very definition of the directional selectivity.

[24] The ratio of the horizontal electric force $f(x, \theta)$ to the vertical electric force $F(x, \theta)/H$—where H here denotes the height of the wave antenna above the earth's surface—is a complex number whose value depends on the conductivity, dielectric constant, and permeability of the ground, and on the frequency.

is due to the distributed impressed voltage acting in the leakage admittance from the wave antenna to ground (this leakage admittance being regarded as uniformly distributed). By substituting the values $f(y)$ and $F(y)$ given by (110) and (111) when x is replaced by y, then carrying out the indicated integrations, and finally transforming the results somewhat, the constituents corresponding to (50), (51), and (52) are found to have the following formulas:

$$j(s,\ \theta) = \frac{sf(0,\ \theta)\cos\theta}{2K}\,\frac{\sinh[(\gamma - \Gamma\cos\theta)s/2]}{(\gamma - \Gamma\cos\theta)s/2}\,e^{-(\gamma+\Gamma\cos\theta)s/2}, \quad (112)$$

$$J_0(s,\ \theta) = -\frac{F(0,\ \theta)}{2K}\,e^{-\gamma s}, \quad (113)$$

$$J_s(s,\ \theta) = \frac{F(0,\ \theta)}{2K}\,e^{-\Gamma s\cos\theta}. \quad (114)$$

The fourth constituent, $J_{0s}(s,\ \theta)$, corresponding to (53), will be omitted, because it is relatively unimportant and also because its formula is found to be somewhat lengthy.

The valuable directional selectivity of a wave antenna resides mainly in the directional properties of the admittance $j(s,\ \theta)/sf(0,\ \theta')$ whose value is found by dividing equation (112) through by $sf(0,\ \theta')$, where θ' denotes some fixed value of θ (usually $\theta' = 0$). This ratio may properly be termed a 'directional admittance.' The corresponding admittances obtained by dividing (113) and (114) through by $sf(0,\ \theta')$ are not usefully directional, the former being entirely non-directional, and the latter only directional as regards its phase angle—not as regards its absolute value. By suitable choice of the length of the wave antenna, the constituent represented by (112) can be made to have high directional selectivity, while the constituents corresponding to (113) and (114) become relatively unimportant (except over a few narrow ranges of the incidence angle θ).[25]

A Crosstalk Problem

This problem is concerned with the derivation of formulas for the first-order crosstalk between two simple open-wire telephone circuits of which one is non-transposed and the other is once-transposed, as represented in plan view by Fig. 13.

The once-transposed circuit is taken as the primary, and the non-transposed as the secondary. Each extends from $x = 0$ to $x = s$; and the primary is transposed at its mid-point $x = s/2$.

[25] For a detailed study of the wave antenna, the reader is referred to the well-known paper by Beverage, Rice, and Kellog entitled 'The Wave Antenna' in *J. A. I. E. E.* beginning with March, 1923.

The primary is energized by an alternating electromotive force E_0 inserted at $x = 0$. Stated precisely, the problem here contemplated is the derivation of formulas for the currents produced in the two ends, $x = 0$ and $x = s$, of the secondary circuit by the primary circuit, when all reactions of the secondary on the primary are neglected.

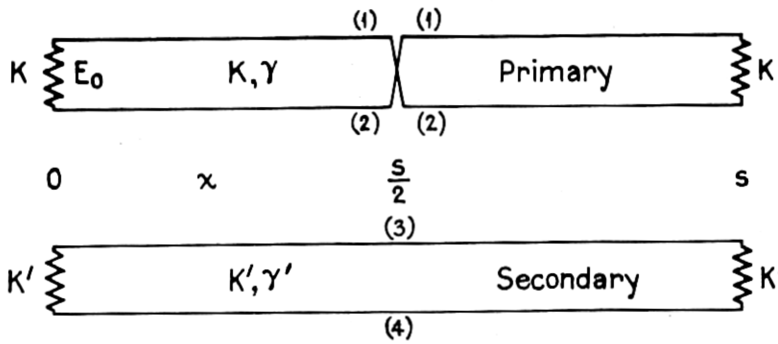

Fig. 13

The wires of the secondary circuit are numbered 3 and 4. The primary wires are numbered 1 and 2, in the sense that 1 and 2 designate the *positions* the wires would occupy if non-transposed; in this sense, wires 1 and 2 are each discontinuous at $x = s/2$, the transposition cross thus being regarded as extraneous to the wires.[26]

Each circuit is terminated at each end in its mode-a characteristic impedance—K for the primary, K' for the secondary. The mode-a propagation constants of the primary and secondary, per unit length, are denoted by γ and γ' respectively.

The earth is assumed to be a perfect conductor. This assumption is effectively a good approximation because the contemplated circuits are such that the distance between the two wires of each circuit is small compared with their height above ground; at the same time the assumption greatly facilitates and simplifies the solution.

Evidently the first step is to formulate the primary current and the primary systemic potential at any point x. The second step is to formulate the electric field impressed on the secondary by the primary. The third and final step is to formulate the currents produced in the secondary by the impressed field of the primary; this third step will be carried through by means of the synthetic method, employing the set of equivalent electromotive forces formulated in the early part of Section III.

[26] The transposition cross may conveniently be regarded as merely a particular kind of transducer (four-terminal network) inserted in the primary, namely a reversing transducer.

Let $I_r(x) = I_r$ and $V_r(x) = V_r$ denote the current and the potential, respectively, at any point x of wire r, where $r = 1, 2, 3, 4$. Then, evidently, for the primary currents and potentials we have:

$$I_1 = -I_2, \qquad V_1 = -V_2, \tag{115}$$

$$V_1 - V_2 = KI_1. \tag{116}$$

For $x \lesseqgtr s/2$:

$$I_1 = \pm \frac{E_0}{2K} e^{-\gamma x}, \tag{117}$$

$$V_1 = \pm \frac{E_0}{4} e^{-\gamma x}. \tag{118}$$

Thus the primary currents, $I_1(x)$ and $I_2(x)$, and the corresponding primary potentials, $V_1(x)$ and $V_2(x)$, are each discontinuous at $x = s/2$ (by reason of the transposition there).

The electric field impressed on the individual wires of the secondary circuit by the primary circuit can be formulated by means of equations (106) and (96). Thus

$$E_3 = (T_{31} - T_{32})zI_1, \tag{119}$$

$$E_4 = (T_{41} - T_{42})zI_1, \tag{120}$$

$$V_3 = (T_{31} - T_{32})V_1, \tag{121}$$

$$V_4 = (T_{41} - T_{42})V_1, \tag{122}$$

where $z = z_1 = z_2$ is the internal impedance of each wire of the primary, per unit length, and the T's are 'voltage transfer factors' given by (98).

Evidently the secondary circuit constitutes a balanced two-wire line in an arbitrary impressed field (the field due to the primary), and hence is amenable to the treatment already fully described and formulated in the subsection following equation (54). Thus the current at any point x in the secondary consists of two modes, a and c. However, as already indicated, we shall ultimately be concerned only with the currents in the ends $x = 0$ and $x = s$ of the secondary; evidently these are mode-a currents, for at each end the mode-c currents must be zero, since the circuit is insulated from ground at each end.

As we shall be concerned only with the mode-a currents produced in the secondary, the next step is to formulate the mode-a constituents of the electric field impressed by the primary. If E' and V' denote the mode-a constituents of the axial electric force and of the potential impressed by the primary, then

$$E' = E_3 - E_4 = 4TzI_1, \tag{123}$$

$$V' = V_3 - V_4 = 4TV_1, \tag{124}$$

where

$$T = (T_{31} - T_{32} - T_{41} + T_{42})/4. \tag{125}$$

Remembering that I_1 and V_1 are discontinuous at $x = s/2$, in accordance with equations (117) and (118), it is seen that E' and V' are discontinuous at $x = s/2$ in accordance with the following equations.[27] For $x \lesseqgtr s/2$:

$$E' = \pm E_0 T \frac{2z}{K} e^{-\gamma x}, \qquad V' = \pm E_0 T e^{-\gamma x}. \tag{126}$$

We are now prepared to formulate the mode-a currents produced in the secondary (3, 4) by the field arising from the primary (1, 2). Since the secondary constitutes a balanced two-wire line in an arbitrary impressed field, it is amenable to the treatment formulated in the subsection following equation (54); thence equations (41), $\cdots$ (45) and equations (50), $\cdots$ (54) are formally applicable.

If $I_3(x)$ and $I_4(x)$ denote the mode-a currents at any point x in the secondary wires 3 and 4 respectively, then

$$I_3(x) = -I_4(x) = I(x), \text{ say.} \tag{127}$$

On referring to the subsection containing equations (41), $\cdots$ (45) and applying it to the mode-a effects in the present problem, it will be seen that $I(x)$ is the sum of the five mode-a constituents $j(x)$, $J_0(x)$, $J_s(x)$, $J_{0s}(x)$, $J_u(x)$, corresponding to equations (41), (42), (43), (44), (45) respectively. From the discussion in connection with those equations and from the analysis of the impressed field into two modes, a and c, as described and formulated in the subsection following equation (54), it will be seen that $j(x)$ is due to the mode-a axial electric force $E_3(y) - E_4(y)$ acting at all points y of the secondary, $J_0(x)$ is due to the mode-a impressed voltage $V_3(0) - V_4(0)$ acting at the end $y = 0$, $J_s(x)$ is due to the mode-a impressed voltage $V_3(s) - V_4(s)$ acting at the end $y = s$, $J_{0s}(x)$ is due to the mode-a impressed voltage $V_3(y) - V_4(y)$ acting at all points y in the leakage admittance[28] between the secondary wires, and $J_u(x)$ is due to the discontinuity $V'(u -)$

[27] From mere physical considerations, it is evident that the whole primary field is reversed at $x = s/2$.

[28] That is, the 'mutual' leakage admittance (equal to the 'direct' leakage admittance between wires plus one half of the 'direct' leakage admittance from each wire to ground).

$-\ V'(u+)$ in the mode-a impressed voltage $V'(y) \equiv V_3(y) - V_4(y)$ at $y = u = s/2$. (In what follows, the constituent $J_{0s}(x)$ will be omitted because it is relatively unimportant.)

Thus, for the formulation of the mode-a effects the functions $f(y)$ and $F(y)$, representing the electirc forces and potentials in equations (41), $\cdots$ (45) and (50), $\cdots$ (54), have the following mode-a values:

$$f(y) = E_3(y) - E_4(y) = E'(y), \tag{128}$$

$$F(y) = V_3(y) - V_4(y) = V'(y), \tag{129}$$

whence, in particular,

$$F(0) = V'(0), \tag{130}$$

$$F(s) = V'(s), \tag{131}$$

$$F(u-) - F(u+) = V'(u-) - V'(u+). \tag{132}$$

Substituting these values into equations (50), (51), (52), (54), and carrying out the indicated integrations [15] when $x = 0$ and when $x = s$, and finally dividing each equation by the value of the primary current $I_1(0) = E_0/2K$ at $x = 0$, we obtain the following formulas (134), $\cdots$ (137) for the four mode-a current ratios at $x = 0$, and the formulas (141), $\cdots$ (144) for those at $x = s$. Also, there are included formulas for $J(x)/I_1(0)$ at $x = 0$ and at $x = s$, $J(x)$ denoting the sum of the mode-a current constituents due to the impressed potential, that is,

$$J(x) = J_0(x) + J_s(x) + J_u(x) \tag{133}$$

since $J_{0s}(x)$ is neglected.

At $x = 0$ the formulas for the four current-ratios are

$$\frac{j(0)}{I_1(0)} = T\frac{K}{K'}\frac{sz}{K}\frac{[1 - e^{-(\gamma+\gamma')s/2}]^2}{(\gamma + \gamma')s/2}, \tag{134}$$

$$\frac{J_0(0)}{I_1(0)} = -\ T\frac{K}{K'}, \tag{135}$$

$$\frac{J_s(0)}{I_1(0)} = -\ T\frac{K}{K'}e^{-(\gamma+\gamma')s}, \tag{136}$$

$$\frac{J_u(0)}{I_1(0)} = 2T\frac{K}{K'}e^{-(\gamma+\gamma')s/2}. \tag{137}$$

The sum of the last three is

$$\frac{J(0)}{I_1(0)} = -\ T\frac{K}{K'}[1 - e^{-(\gamma+\gamma')s/2}]^2. \tag{138}$$

On dividing (134) by (138) the ratio of $j(0)$ to $J(0)$ is found to have the simple value

$$\frac{j(0)}{J(0)} = -\frac{z}{K(\gamma + \gamma')/2}. \tag{139}$$

In particular, when the two circuits have equal propagation constants $(\gamma' = \gamma)$,

$$\frac{j(0)}{J(0)} = -\frac{z}{Z}, \tag{140}$$

where $Z = \gamma K$ is the mode-a 'complete series impedance'[18] of the primary, per unit length; it will be recalled that z is the 'internal impedance' of each primary wire, per unit length. The ratio z/Z is ordinarily a very small fraction.

At $x = s$ the formulas for the four current-ratios are

$$\frac{j(s)}{I_1(0)} = T\frac{K}{K'}\frac{sz}{K}\frac{[1 - e^{-(\gamma-\gamma')s/2}]^2}{(\gamma - \gamma')s/2}e^{-\gamma's}, \tag{141}$$

$$\frac{J_0(s)}{I_1(0)} = -T\frac{K}{K'}e^{-\gamma's}, \tag{142}$$

$$\frac{J_s(s)}{I_1(0)} = -T\frac{K}{K'}e^{-\gamma s}, \tag{143}$$

$$\frac{J_u(s)}{I_1(0)} = 2T\frac{K}{K'}e^{-(\gamma+\gamma')s/2}. \tag{144}$$

The sum of the last three is

$$\frac{J(s)}{I_1(0)} = -T\frac{K}{K'}[1 - e^{-(\gamma-\gamma')s/2}]^2 e^{-\gamma's}. \tag{145}$$

On dividing (141) by (145) the ratio of $j(s)$ to $J(s)$ is found to have the simple value

$$\frac{j(s)}{J(s)} = -\frac{z}{K(\gamma - \gamma')/2}. \tag{146}$$

When the absolute value of $(\gamma - \gamma')s/2$ is small compared to unity, equations (141) and (145) become approximately

$$\frac{j(s)}{I_1(0)} = T\frac{K}{K'}\frac{sz}{K}\frac{(\gamma - \gamma')s}{2}e^{-\gamma's}, \tag{147}$$

$$\frac{J(s)}{I_1(0)} = -T\frac{K}{K'}\left[\frac{(\gamma - \gamma')s}{2}\right]^2 e^{-\gamma's}. \tag{148}$$

Thus:

$$\text{When} \quad \gamma' = \gamma: \quad j(s) = 0, \qquad J(s) = 0.$$

For some cases, particularly those where the attenuation is neglected, it is advantageous to express the square-bracketed factors in equations (134), (138), (141), (145) partially in terms of hyperbolic sines.

Appendix I

Derivations of Equations (25) and (90)

Equation (25)

Let the primary or impressed field of force be specified by an electric intensity f_a parallel to the axis of the wire (and to the surface of the earth), and an electric intensity f_n normal to the surface of the earth and measured downward. We denote by f_w the value of f_a at the axis of the wire,[29] and by f_g its value at the surface of the ground in the plane which is normal to the ground and which includes the axis of the wire. The impressed or primary potential F of the wire, due to the impressed field, is then

$$F = \int_0^h f_n dy,$$

where h is the height of the wire above ground and the integral is taken along the vertical from the wire ($y = 0$) to ground ($y = h$).

Due to the impressed field, specified above, a current I flows in the wire and a corresponding *superposed* current distribution is induced in the ground. The *resultant* axial electric intensity at the surface of the wire is then $z_w I$ (where z_w is the internal impedance of the wire, per unit length); correspondingly the *resultant* electric intensity along the surface of the ground is $f_g - z_g I$. Application of the second curl law to a contour composed of two verticals from the wire to ground and the line segments dx in the surfaces of the wire and ground gives

$$(z_w + z_g)I - f_g + \frac{dV}{dx} = -\frac{d\phi}{dt},$$

which is preferably written as

$$zI - f_g + \frac{dV}{dx} = -\frac{d\phi}{dt}, \tag{1}$$

where $z = z_w + z_g$ is the internal impedance of the circuit, per unit length; V is the *resultant* potential of the wire; and ϕ is the *resultant* magnetic flux threading the contour, per unit length. But we have

[29] f_w is assumed to be sensibly constant over the cross-section of the wire.

also

$$f_w - f_g + \frac{dF}{dx} = -\frac{d\Phi}{dt}, \tag{2}$$

where Φ denotes the impressed magnetic flux threading the contour, per unit length. Subtracting (2) from (1) and observing that

$$V - F = \frac{1}{C}Q, \qquad \phi - \Phi = LI, \tag{3}$$

we get

$$zI + L\frac{dI}{dt} + \frac{1}{C}\frac{dQ}{dx} = f_w, \tag{4}$$

where, of course, Q is the charge, C the capacity to ground and L the external inductance, per unit length of the wire.

To eliminate Q from (4) we make use of the equation of current continuity, namely

$$-\frac{dI}{dx} = \frac{dQ}{dt} + I', \tag{5}$$

where I' is the leakage current per unit length of the wire. If the wire is embedded in a homogeneous leaky medium, then

$$I' = \frac{G^0}{C}Q = G^0(V - F), \tag{6}$$

where G^0 is proportional to the conductivity of the medium.[30] If, furthermore, there is direct leakage admittance from the wire to ground (as at poles and insulators) of amount Y' per unit length,[31] when regarded as uniformly distributed, then

$$I' = \frac{G^0}{C}Q + Y'V = \frac{G}{C}Q + Y'F, \tag{7}$$

where

$$G = G^0 + Y'. \tag{8}$$

On substituting the last value of I' into (5), setting $d/dt = i\omega$, then differentiating with respect to x, and finally substituting the resulting value of dQ/dx into (4), we get

$$(z + i\omega L)I - \frac{1}{G + i\omega C}\frac{d^2I}{dx^2} = f_w + \frac{Y'}{G + i\omega C}\frac{dF}{dx}, \tag{9}$$

[30] A formula for G^0 is equation (18) derived below.

[31] While G^0 is merely a pure conductance, Y' is in general an admittance (leakage admittance), because the insulators and poles have capacity as well as conductance. Hence G, defined by (8), is an admittance.

which can be written

$$\frac{K}{\gamma}\left(\gamma^2 - \frac{d^2}{dx^2}\right) I = f_w + \frac{Y'K}{\gamma}\frac{dF}{dx}, \tag{10}$$

where

$$K = \sqrt{\frac{z + i\omega L}{G + i\omega C}}, \tag{11}$$

and

$$\gamma = \sqrt{(z + i\omega L)(G + i\omega C)}. \tag{12}$$

Thus K is the characteristic impedance and γ the propagation constant of the transmission system composed of the overhead wire with ground return; it is to be noted that $G = G^0 + Y'$, in accordance with (8), and hence that G is in general an admittance—not a pure conductance.

If we define f' by the equation

$$f' = f_w + \frac{Y'K}{\gamma}\frac{dF}{dx}, \tag{13}$$

then (10) becomes

$$\frac{K}{\gamma}\left(\gamma^2 - \frac{d^2}{dx^2}\right) I = f', \tag{14}$$

which is formally the same as equation (25) of the text. There, however, it is assumed that the term $(Y'K/\gamma)dF/dx$ is negligible; probably this is usually the case but circumstances may arise where it is not negligible. Its inclusion, however, introduces no formal modification of the analysis.

The foregoing derivation has been given in detail because prior derivations known to the writers have not been entirely satisfactory. Their chief defect has been that no explicit consideration was given to the finite conductivity of the ground (except that it produces a tangential component f_a). In the derivation given above, the effect of ground conductivity is expressly recognized and in the final equation appears implicitly in the values of K and γ. These parameters, it will be observed, are experimentally determinable, and are the only parameters besides Y' appearing in the final differential equation.

A formula for the quantity G^0 occurring in equations (6) and (7) can be derived by application of Gauss' theorem, as follows: Let E_r denote the radial component of the total electric force at the surface of the wire, dS a differential element of the surface of the wire, and σ and ϵ the conductivity and specific inductive capacity of the medium, which is homogeneous and isotropic by assumption. Then the leakage current I' flowing outward, per unit length of the wire, is given by

$$I' = \int \sigma E_r dS = \frac{\sigma}{\epsilon}\int \epsilon E_r dS, \tag{15}$$

the surface integral being taken over the unit length of the wire. But, by Gauss' theorem (ξ being a constant whose value depends only on the units),

$$\int \epsilon E_r dS = 4\pi Q/\xi, \tag{16}$$

the resultant axial electric flux from the ends of the element being negligible compared with the radial electric flux from the lateral surface. Thus

$$I' = \frac{4\pi\sigma}{\epsilon\xi} Q, \tag{17}$$

and comparison of this equation with (6) gives the result

$$G^0/C = 4\pi\sigma/\epsilon\xi. \tag{18}$$

In this connection it may be noted that the leakage current represented by (15) does not directly depend on the impressed field, but only on the field produced by the wire itself. This is because the assumed medium is homogeneous and isotropic; hence σ in (15) can be taken outside the sign of integration, and then the conclusion follows from (15) by noting that one of the constituents of E_r is the impressed radial electric force f_r, and that

$$\int f_r dS = \int \operatorname{div} f \cdot dv = 0,$$

since the divergence of the impressed electric force must be zero. The conclusion would not follow, in general, if the medium were either heterogeneous or æolotropic. It may be noted that a homogeneous isotropic medium surrounding a wire and containing direct leakage admittance paths from the wire to ground may be regarded as a heterogeneous æolotropic medium.

The value given for δ in equation (9) of the text, namely $\delta = 4\pi\sigma/\epsilon\xi$, is readily derivable by combining equations (4), (10), (5) of the text with (17) of this Appendix.

Equations (90)

If there were no impressed potential at the primary wires ($F_k = 0$), the equations of continuity would be merely

$$-\frac{dI_h}{dx} = \sum_{k=1}^{n} Y_{hk} V_{k\pi}, \qquad (h = 1, \cdots n), \tag{19}$$

where, in accordance with equations (7) in Section I,

$$Y_{hk} = g_{hk} + i\omega q_{hk}. \tag{20}$$

It should here be remarked that Y_{hk} depends not only on the geometry of the system and on the conductivity of the medium but also on any direct leakage admittance existing between the wires themselves and also on any between the wires and ground. The direct leakage admittance between wires h and k, per unit length, will be denoted by Y'_{hk} and that between wire h and ground, by Y'_{hh}; these are regarded as being uniformly distributed along the system.

When there is present an impressed potential, the existence of the direct leakage admittances gives rise to the following supplementary terms for the right side of equation (19):

$$F_h Y'_{hh} + \sum_{\substack{k=1 \\ k \neq h}}^{n} (F_h - F_k) Y'_{hk} = \sum_{k=1}^{n} X_{hk} F_k,$$

where

$$\begin{aligned} X_{hk} &= -\, Y'_{hk} \quad \text{for} \quad k \neq h, \\ X_{hh} &= \sum_{k=1}^{n} Y'_{hk}. \end{aligned} \tag{21}$$

It is seen that Y_{hk} and X_{hk} are of the nature of admittances (per unit length), although they are not 'direct admittances.' Their precise meanings are readily deducible from equations (90).

When the medium itself is of zero conductivity, g_{hk} reduces to X_{hk}.