A Study of the Regular Combination of Acoustic Elements, with Applications to Recurrent Acoustic Filters, Tapered Acoustic Filters, and Horns

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SYNOPSIS: The use of combinations of tubes to produce interference between sound waves and a suppression of certain frequencies originates with Herschel (1833), and was applied by Quincke to stop tones of definite pitch from reaching the ear. Following the development of electrical filters, G. W. Stewart showed that combinations of tubes and resonators could be devised which would give transmission characteristics at low frequencies similar to electrical filters. The assumptions made by Stewart in the development of his theory are that no wave motion need be considered in the elements, and that the lengths of the elements employed are small compared to the wave-length of sound.

The present paper considers primarily regular combinations of acoustic elements, such as straight tubes, and shows that the equations for recurrent filters, tapered filters and horns can be obtained in this manner. The assumption of no wave motion in the elements, made by Stewart, is removed and also account is taken of the viscosity and heat conduction dissipation. The principal difference between acoustic and electric filters is that the former have an infinite number of bands. The effect of using filters between varying terminal impedances is also determined.

Studying next the combination of filters having the same propagation characteristics but in which the conducting tube areas increase in some regular manner, it is shown that a tapered filter results which has a transforming action in addition to its filtering properties. It is shown that if straight tubes are employed and the distance between successive changes in areas is made small we obtain the horn equations first developed by Webster. The general combination of acoustic elements is then considered, and a proof of several theorems has been given.

STEWART, in a series of papers,[1] has studied the recurrent acoustic filter as an analogue of the electric filter with lumped constants. If due account is taken of the wave motion occurring in the individual elements themselves, it appears that the nearest electrical analogue of the acoustic filter is a combination of electric lines.

In the present paper we study primarily regular combinations of acoustic elements, such as straight tubes, and show that the equations for recurrent filters, tapered filters, and horns can be obtained in this manner. The effect of viscosity and heat conduction dissipation has been taken into account, and a consideration of the effect of varying terminal impedances has been included.

I. Equations of Propagation of a Plane Wave in a Uniform Tube

The propagation of plane waves of sound in uniform tubes has been discussed in a number of places,[2] but generally the results obtained are

[1] *Phys. Rev.*, 20, 528 (1922); 23, 520 (1924); 25, 90 (1925).

[2] Rayleigh's "Theory of Sound," Vol. II, p. 318. Lamb's "The Dynamical Theory of Sound," p. 193.

only a determination of the propagation constant, that is, a determination of the attenuation and phase change per unit length, or as more often stated, the attenuation and velocity characteristics. If we solve the differential equations in the manner first employed by Heaviside in the solution of the equation of the electric line, we obtain one more parameter, namely, the characteristic impedance of the tube.

The differential equation, given by Rayleigh,[2] for the propagation of plane waves of sound in a tube of uniform cross-section is

$$\left(1+\frac{R}{S}\sqrt{\frac{\mu}{2\omega\rho}}\right)\frac{\partial^2\xi}{\partial t^2}+\frac{R}{S}\sqrt{\frac{\mu\omega}{2\rho}}\frac{\partial\xi}{\partial t}=c^2\frac{\partial^2\xi}{\partial x^2}, \tag{1}$$

where ξ denotes the displacement of the fluid at a distance x from one end of the tube,

$\mu =$ the coefficient of viscosity of the medium,

$\rho =$ the density of the medium,

$R =$ perimeter and $S =$ cross-sectional area of pipe,

$\omega =$ frequency of vibration times 2π,

$C = \sqrt{\frac{P_0\gamma}{\rho}} =$ velocity of sound in medium,

$\gamma =$ ratio of specific heats of medium.

This equation is valid for tube diameters and frequencies such that

$$\sqrt{\frac{\rho\omega}{2\mu}}\frac{S}{R} > 1$$

and hence can be used for all frequencies of interest in connection with acoustic filters.

Kirchoff[3] extended the theory to take account of the losses due to heat conduction in the medium. His results indicate that in order to take account of this effect, the square root of the coefficient of viscosity should be replaced by a quantity γ', given by

$$\gamma' = \sqrt{\mu}+\left(\sqrt{\gamma}-\frac{1}{\sqrt{\gamma}}\right)\sqrt{\upsilon},$$

where υ is the coefficient of heat conductivity of the medium. By the kinetic theory of gases υ has the value $5/2\ \mu$.

The most useful solution for our present purpose is obtained by writing

$$\xi = e^{i\omega t}(A\cosh\alpha x + B\sinh\alpha x), \tag{2}$$

[3] Rayleigh, "Theory of Sound," Vol. II, p. 325.

where A and B are constants and α by analogy with an electric line is the propagation constant of the tube. Substituting (2) in (1), we see that (2) is a solution provided

$$\alpha^2 = -\frac{\omega^2}{C^2}\left[\left(1 + \frac{R}{S}\sqrt{\frac{\gamma'^2}{2\omega\rho}}\right) - i\frac{R}{S}\sqrt{\frac{\gamma'^2}{2\omega\rho}}\right]. \tag{3}$$

Now α can be written $\alpha = a + ib$, where a is the attenuation constant and b the phase constant. If we solve for a and b, assuming

$$\frac{R}{S}\sqrt{\frac{\gamma'^2}{2\omega\rho}}$$

is a small quantity, we obtain

$$\alpha = a + ib = \frac{1}{2}\frac{R}{CS}\sqrt{\frac{\gamma'^2\omega}{2\rho}} + i\frac{\omega}{C}\left[1 + \frac{1}{2}\frac{R}{S}\sqrt{\frac{\gamma'^2}{2\omega\rho}}\right]. \tag{4}$$

We are generally interested in the volume velocity $S\xi = V$, so we can rewrite equation (2) as

$$V = i\omega S e^{i\omega t}[A \cosh \alpha x + B \sinh \alpha x]. \tag{5}$$

To determine one constant of equation (5), let x equal zero. Then

$$V_{x=0} = V_1 = i\omega e^{i\omega t} SA$$

or

$$A = \frac{V_1}{i\omega S e^{i\omega t}}. \tag{6}$$

We have the additional relation

$$P - P_0 = -P_0\gamma\frac{\partial\xi}{\partial x} = p, \tag{7}$$

where p denotes the excess pressure. Substituting (2) in (7), and differentiating, we have

$$p = -P_0\gamma e^{i\omega t}(A\alpha \sinh \alpha x + B\alpha \cosh \alpha x).$$

Putting $x = 0$, we have

$$p_{x=0} = p_1 = -P_0\gamma e^{i\omega t}(B\alpha)$$

or

$$B = -\frac{p_1}{\alpha P_0 \gamma e^{i\omega t}}. \tag{8}$$

Substituting the value of A and B in (5) and (7), we have

$$\left.\begin{aligned} V &= V_1 \cosh \alpha x - \frac{p_1 i\omega S \sinh \alpha x}{P_0\gamma\alpha}, \\ p &= p_1 \cosh \alpha x - V_1 \frac{(P_0\gamma\alpha)}{i\omega S} \sinh \alpha x. \end{aligned}\right\} \quad (9)$$

$(P_0\gamma\alpha)/(i\omega)$ is, by analogy with the electric line, the characteristic impedance [4] per square centimeter of the tube. It is the ratio of $p_1/\dot{\xi}_1$ for an infinitely long tube. For since $\cosh \alpha x = \frac{1}{2}(e^{\alpha x} + e^{-\alpha x})$ while $\sinh \alpha = \frac{1}{2}(e^{\alpha x} - e^{-\alpha x})$, then when x approaches infinity, and dissipation exists in the tube, $\cosh \alpha x$ approaches $\sinh \alpha x$, and both approach infinity. Hence the ratio of P_1/V_1 equals $P_0\gamma\alpha/i\omega S$. The propagation constant α has the physical significance that $e^{-\alpha x}$ equals the ratio of V to V_1 or p to p_1, when we are dealing with an infinitely long tube, as can be seen by substituting $p_1/V_1 = P_0\gamma\alpha/i\omega S$ in (9) and solving for the above ratios. The real part of α, i.e. a, determines the rate at which the linear or volume velocity, or pressure, decreases with distance, while the imaginary part b determines the phase of pressure or velocity with respect to the initial values, and hence is known as the phase constant and gives the phase rotation per unit length of pipe. Now since the velocity of propagation C' is

$$C' = \frac{\omega}{b},$$

we have by equation (4)

$$C' = C\left[1 - \frac{1}{2}\frac{R}{S}\sqrt{\frac{\gamma'^2}{2\omega\rho}}\right].$$

The attenuation constant and the velocity reduce to the familiar Helmholtz formulæ, for circular sections.[5]

We write (9) as

$$\left.\begin{aligned} V &= V_1 \cosh \alpha x - \frac{p_1 S}{Z_L} \sinh \alpha x, \\ p &= p_1 \cosh \alpha x - \frac{V_1 Z_L}{S} \sinh \alpha x, \end{aligned}\right\} \quad (10)$$

where Z_L represents the specific characteristic impedance $P_0\gamma\alpha/i\omega$.

[4] The analogy between pressure and electromotive force, volume velocity and current, and impedance to ratio of pressure and volume velocity was first pointed out by Webster[9]. Another system in which force and e.m.f., and linear velocity and current are related, is very convenient when we are dealing with combinations of mechanical elements such as masses and elasticities and no account has to be taken of the area. In the first system, the total impedance is Z_L (per sq. cm.) divided by S whereas in the second system it is $Z_L S$. We follow the first system expressing, however, the impedance in terms of the impedance per square centimeter, which is the same on either systems of units.

[5] See Lamb, "Dynamical Theory of Sound," p. 193, or Rayleigh, "Theory of Sound," Vol. II, p. 319.

The value of the specific characteristic impedance $P_0\gamma\alpha/i\omega$ becomes on substituting in the value of α

$$Z_L = \sqrt{P_0\gamma\rho}\left[\left(1 + \frac{1}{2}\frac{R}{S}\sqrt{\frac{\gamma'^2}{2\omega\rho}}\right) - i\frac{1}{2}\frac{R}{S}\sqrt{\frac{\gamma'^2}{2\omega\rho}}\right]. \qquad (11)$$

If we assume no dissipation, $\gamma' = 0$ and $Z_L = \sqrt{P_0\gamma\rho}$. In any case at fairly high frequencies Z_L approaches $\sqrt{P_0\gamma\rho}$. For example, for air in a circular tube 1 centimeter in diameter, Z_L departs from its final value $\sqrt{P_0\gamma\rho}$ by less than 5 per cent at 100 cycles. The attenuation constant a increases as the square root of the frequency, while the phase constant b is little affected by the dissipation and at high frequencies approaches the value ω/C.

II. Effect of a Junction or of a Change in Area of the Conducting Tube

Suppose that we have a straight conducting tube, with a sidebranch as shown in Fig. 1. Let the excess pressure of the incoming plane wave

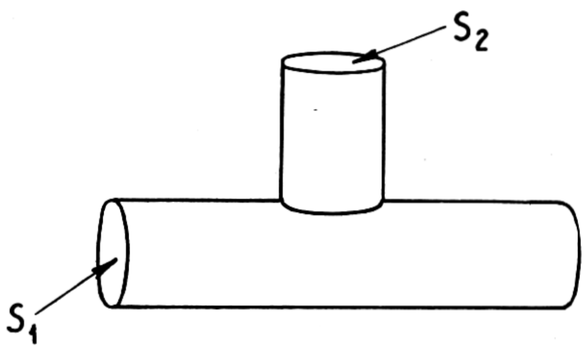

Fig. 1—An acoustic junction

be p_1. The ordinary assumption is that the width of the junction is small compared with a wave-length and hence the pressure is practically constant in the sidebranch, and main branch over the portion in immediate contact with the sidebranch. It states also that the algebraic sum of the volume displacements at a junction of tubes is zero. If S_1 is the area of the main conducting tube, S_2 the area of the branch tube, $\dot{\xi}_1$ the linear velocity of the incoming wave in the conducting tube, $\dot{\xi}_2$ the linear velocity of the outgoing wave from the junction and $\dot{\eta}$ the linear velocity in the branch tube at the junction, we can write the equation

$$\dot{\xi}_1 S_1 = \dot{\xi}_2 S_1 + \dot{\eta} S_2 \qquad \text{or} \qquad V_1 = V_2 + V'.$$

We have now that $\dot{\eta} = p_1/Z_S$ where Z_S is the impedance per unit area of the sidebranch, or the ratio of the excess pressure to the linear velocity. Substituting this value in the above equation, we have

$$\left.\begin{aligned} V_2 &= V_1 - \frac{p_1 S_2}{Z_S}. \\ p_2 &= p_1, \end{aligned}\right\} \tag{12}$$

We have also

where p_2 is the excess pressure in the conducting tube on the outgoing side. The equations are exactly equivalent to Kirchoff's laws, and hence any equation for a combination of acoustic elements will also apply to the combinations of equivalent electric elements.

A slightly better approximation than the above has been obtained by solving completely the case of three pistons placed in the sides of a rectangular box. This corresponds closely to the condition considered here, if we have rectangular tubes, since the waves can be considered plane up to the junction point with little possibility of error. The solution obtained indicates that the main effect of the junction point is to add an end correction to all the tubes entering the junction. For example, we will measure the length of the main conducting tube, between sidebranches, from the center of the sidebranches rather than the edge, as the approximation given first would imply. Also the length of the sidebranch should be measured from the center of the conducting tube, rather than the edge. For other types of junctions, different end corrections will apply to the sidebranch tubes. For example if the width of the junction is large compared to the width of the sidebranch, we should expect Rayleigh's theoretical value of .82 R to apply where R is the radius of the sidebranch tube. Hence the equations for a junction are equivalent to Kirchoff's laws with the additional proviso that end corrections shall be added to tubes entering a junction.

The effect of a change of area of the conducting tube can be obtained with the same assumptions as above. If we have one conducting tube of area S_1, joined to a second of area S_2, we can write

$$\dot{\xi}_1 S_1 = \dot{\xi}_2 S_2 \qquad \text{or} \qquad V_1 = V_2, \tag{13}$$

where $\dot{\xi}_1$ is the linear velocity in the first tube and $\dot{\xi}_2$ in the second tube. We have also that the pressures in the adjoining tubes are equal. Hence

$$p_2 = p_1 \qquad \text{and} \qquad V_2 = V_1. \tag{14}$$

This equation is of the same order of approximation as the second approximation given above for a junction, since we measure the length from one change of area to the next change.

Equation 14 has been found to hold well as long as the change in area is small while equation 12 holds well as long as the length of a junction is less than half of a wave-length.

III. Recurrent Filters

With the aid of equations (10), (12), and (14), we can obtain the propagation characteristics of any structure employing straight tubes, sidebranches, and changes in area of conducting tubes.

Among the simplest of these are recurrent filters. Fig. 2 shows an

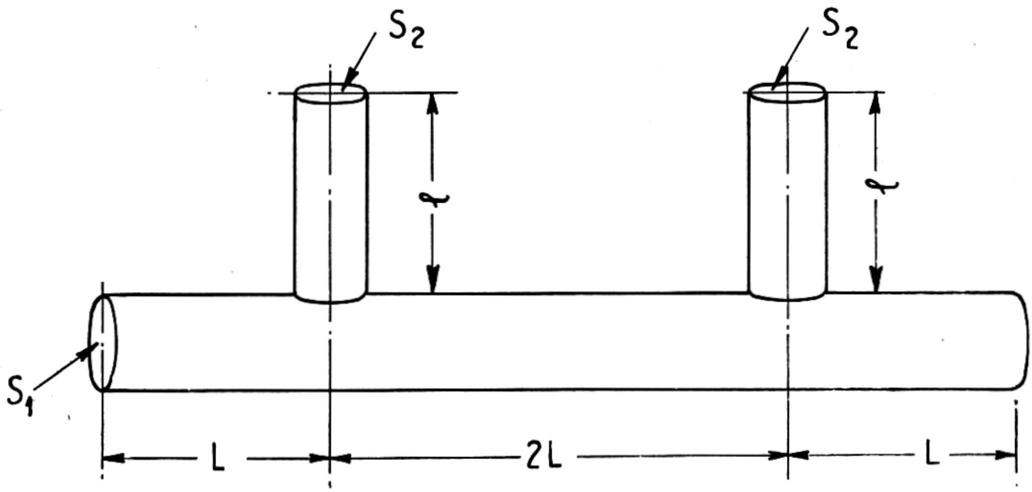

Fig. 2—A typical acoustic filter

example of this type of structure, a main conducting tube, with equally spaced sidebranches. In order to make the structure symmetrical, we let the distance L between one end and the first sidebranch equal one half the distance between two sidebranches. We can then write with regard to the first tube

$$\left.\begin{aligned} V_2 &= V_1 \cosh \alpha_1 L - \frac{p_1}{Z_{L_1}} S_1 \sinh \alpha_1 L, \\ p_2 &= p_1 \cosh \alpha_1 L - V_1 \frac{Z_{L1}}{S_1} \sinh \alpha_1 L, \end{aligned}\right\} \tag{15}$$

where α_1 and Z_{L_1} refer to the conducting tube. For the junction, we have by (12)

$$\left.\begin{aligned} V_3 &= V_2 - \frac{p_2}{Z_S} S_2, \\ p_3 &= p_2. \end{aligned}\right\} \tag{16}$$

Combining with (15), we have

$$\left.\begin{aligned} V_3 = V_1\left(\cosh \alpha_1 L + \frac{Z_{L_1}S_2}{Z_S S_1}\sinh \alpha_1 L\right) \\ - p_1 S_1\left[\frac{\sinh \alpha_1 L}{Z_{L_1}} + \frac{S_2 \cosh \alpha_1 L}{Z_S S_1}\right], \\ p_3 = p_1 \cosh \alpha_1 L - \frac{V_1 Z_{L_1}}{S_1}\sinh \alpha_1 L. \end{aligned}\right\} \quad (17)$$

The pressures and volume velocities p_4 and V_4 at one half the distance between the first and second sidebranches are again

$$\left.\begin{aligned} V_4 &= V_3 \cosh \alpha_1 L - \frac{p_3 S_1}{Z_{L_1}}\sinh \alpha_1 L, \\ p_4 &= p_3 \cosh \alpha_1 L - V_3\frac{Z_{L_1}}{S_1}\sinh \alpha_1 L. \end{aligned}\right\} \quad (18)$$

Combining with (17), we obtain

$$\left.\begin{aligned} V_4 = V_1\left(\cosh 2\alpha_1 L + \frac{Z_{L_1}S_2}{2Z_S S_1}\sinh 2\alpha_1 L\right) \\ - \frac{p_1 S_1}{Z_{L_1}}\left(\sinh 2\alpha_1 L + \frac{Z_{L_1}S_2}{Z_S S_1}\cosh^2 \alpha_1 L\right)\cdot \\ p_4 = p_1\left(\cosh 2\alpha_1 L + \frac{Z_{L_1}S_2}{2Z_S S_1}\sinh 2\alpha_1 L\right) \\ - \frac{V_1 Z_{L_1}}{S_1}\left(\sinh 2\alpha_1 L + \frac{Z_{L_1}S_2}{Z_S S_1}\sinh^2 \alpha_1 L\right)\cdot \end{aligned}\right\} \quad (19)$$

These equations apply to the first section of the filter. By comparison with equation (10) we see that we can write equation (19) as

$$\left.\begin{aligned} V_4 &= V_1 \cosh \Gamma - \frac{p_1 S_1}{Z_0}\sinh \Gamma, \\ p_4 &= p_1 \cosh \Gamma - V_1\frac{Z_0}{S_1}\sinh \Gamma, \end{aligned}\right\} \quad (20)$$

where

$$\cosh \Gamma = \left(\cosh 2\alpha_1 L + \frac{Z_{L_1}S_2}{2Z_S S_1}\sinh 2\alpha_1 L\right),$$

$$Z_0 = Z_{L_1}\sqrt{\frac{1 + \dfrac{Z_{L_1}S_2}{2Z_S S_1}\tanh \alpha_1 L}{1 + \dfrac{Z_{L_1}S_2}{2Z_S S_1}\coth \alpha_1 L}} \quad (21)$$

and

$$\sinh \Gamma = \sinh 2\alpha_1 L \sqrt{\left(1 + \frac{Z_{L_1} S_2}{2 Z_S S_1} \tanh \alpha_1 L\right)\left(1 + \frac{Z_{L_1} S_2}{2 Z_S S_1} \coth \alpha_1 L\right)}.$$

Z_0 and Γ are sometimes called the equivalent line parameters. If we have n sections of the type discussed above, we can write n equations of the kind given by (20). If we eliminate all the terms except for the first and last sections, it can be shown that

$$\left.\begin{aligned} V_n &= V_1 \cosh n\Gamma - \frac{p_1 S_1}{Z_0} \sinh n\Gamma, \\ p_n &= p_1 \cosh n\Gamma - \frac{V_1 Z_0}{S_1} \sinh n\Gamma. \end{aligned}\right\} \quad (22)$$

We see then that Γ represents the propagation constant of one section and Z_0 its specific characteristic impedance. They have the physical interpretation, that Z_0 represents the specific impedance looking into an infinite sequence of these sections, while Γ represents the ratio of excess pressure or volume velocity between one section and the next, when we are dealing with an infinite number of sections, or with a finite number, terminated in the characteristic impedance of the filter.

It is customary in electric filter design to determine the characteristics of a dissipationless filter, and to regard dissipation as causing a slight change in the filter characteristic, which usually occurs most prominently in the pass bands. If we neglect dissipation, equation (21) becomes

$$\begin{aligned} \cosh \Gamma &= \left[\cos\left(\frac{2\omega L}{C}\right) + \frac{i\sqrt{P_0 \gamma \rho} S_2}{2 Z_S S_1} \sin\left(\frac{2\omega L}{C}\right)\right], \\ Z_0 &= \sqrt{P_0 \gamma \rho} \sqrt{\frac{1 + \dfrac{i\sqrt{P_0 \gamma \rho} S_2}{2 Z_S S_1} \tan\left(\dfrac{\omega L}{C}\right)}{1 - \dfrac{i\sqrt{P_0 \gamma \rho} S_2}{2 Z_S S_1} \cot\left(\dfrac{\omega L}{C}\right)}}. \end{aligned} \quad (23)$$

The propagation constant Γ is in general a complex number $A + iB$. The real part represents a diminution of the volume velocity or the pressure, while the imaginary part represents a phase change, as can be seen from the fact that the ratio of pressure or volume velocity is

$$\frac{p_2}{p_1} = e^{-\Gamma} = e^{-(A+iB)} = e^{-A}(\cos B - i \sin B).$$

Now $\cosh \Gamma = \cosh (A + iB) = \cosh A \cos B + i \sinh A \sin B$. Hence we see from equation (23), if Z_S is an imaginary quantity, the expression for $\cosh \Gamma$ is always real, and hence either $\sinh A$ or $\sin B$

is always zero. Hence either the attenuation constant A is zero, or the phase shift is zero, π radians or some multiple of π radians. Now since cosh A can never be less than 1 while cos B must lie between $+1$ and -1, then when the expression for cosh Γ is between -1 and $+1$, the attenuation constant A is zero and cos B equals the expression in (23). When the value of cosh Γ is outside the limits ± 1, the phase shift is 0, π, or some multiple and the attenuation constant A is given by the expression in (23).

The specific characteristic impedance Z_0, given in (23), can be shown to be a real quantity within the transmitted band and an imaginary quantity outside the transmitted band.

The type of filter obtained with the structure shown in Fig. 2 depends on the sidebranch impedance Z_S. As long as Z_S is of such a value as to make the expression for cosh Γ greater in magnitude than 1, an attenuation band occurs, while if cosh Γ is less than 1, a pass band occurs. The cut-off frequencies of the band occur when cosh $\Gamma = \pm 1$. From equation (23) the cut-off frequencies occur when

$$Z_S = \frac{i\sqrt{P_0\gamma\rho}S_2}{2S_1}\cot\left(\frac{\omega L}{C}\right) \quad \text{or} \quad Z_S = -\frac{i\sqrt{P_0\gamma\rho}S_2}{2S_1}\tan\left(\frac{\omega L}{C}\right). \qquad (24)$$

A. Low Pass Filter

The model shown in Fig. 2 can be used to obtain the different types of recurrent filters possible by acoustic means. One of the simplest types of filters in the electrical case is the low pass filter. No exact analogue of this filter exists in the acoustic case, as every acoustic filter has more than one band, but a filter which passes low frequencies and attenuates high frequencies can be designed.

Suppose that the sidebranch used is a straight tube closed at one end. Then by equation (10), the impedance Z_S, when the tube is terminated in an infinite impedance, is

$$Z_S = Z_{L_2}\coth\alpha_2 l,$$

where Z_{L_2} and α_2 are respectively the specific characteristic impedance and propagation constant of the sidebranch, and l its length measured to the center of the conducting tube. Substituting this in the expression for cosh Γ and Z_0, we have

$$\left.\begin{aligned} \cosh\Gamma &= \left(\cosh 2\alpha_1 L + \frac{Z_{L_1}S_2\sinh 2\alpha_1 L}{2Z_{L_2}S_1\coth\alpha_2 l}\right), \\ Z_0 &= Z_{L_1}\sqrt{\frac{1+\dfrac{Z_{L_1}S_2\tanh\alpha_1 L}{2Z_{L_2}S_1\coth\alpha_2 l}}{1+\dfrac{Z_{L_1}S_2\coth\alpha_1 L}{2Z_{L_2}S_1\coth\alpha_2 l}}}. \end{aligned}\right\} \qquad (25)$$

If we assume no dissipation, and substitute the values of α_1 and Z_L given in section (I), we have

$$\cosh \Gamma = \left[\cos\left(\frac{2\omega}{C} L\right) - \frac{S_2 \sin\left(\frac{2\omega}{C} L\right)}{2S_1 \cot\left(\frac{\omega}{C} l\right)} \right], \tag{26}$$

$$Z_0 = \sqrt{P_0 \gamma \rho} \sqrt{\frac{1 - \frac{S_2}{2S_1}\left(\frac{\tan \frac{\omega}{C} L}{\cot \frac{\omega}{C} l}\right)}{1 + \frac{S_2}{2S_1}\left(\frac{\cot \frac{\omega}{C} L}{\cot \frac{\omega}{C} l}\right)}}. \tag{27}$$

An example of the type of filter obtained by acoustic means, is given when we let $l = 3L$. Fig. 3 gives a plot of the value of Γ for several ratios of S_2/S_1. Fig. 4 shows the corresponding values of the specific characteristic impedance Z_0.

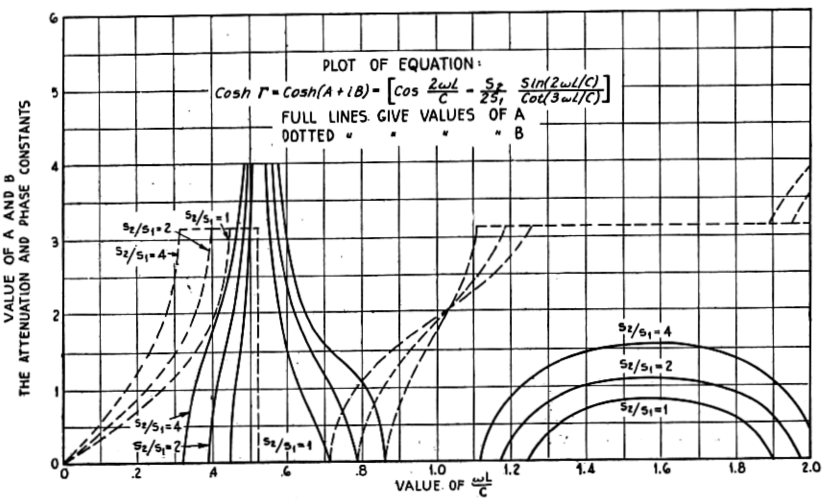

Fig. 3—Propagation constants for a low pass type of filter

A knowledge of Γ will determine the ratio of pressures or volume velocities, if we have an infinite sequence of sections, or if we terminate a finite sequence in the impedance Z_0. If however the terminating impedance is not the characteristic impedance, $e^{-\Gamma}$ no longer represents the ratios of pressures between adjacent sections.

What is generally desired is a knowledge of the effect produced by inserting the filter in a given acoustic system. With the aid of Thévenin's theorem, which is proved for an acoustic system in Appendix I, and equations (20) and (21), this effect can be obtained. Thévenin's

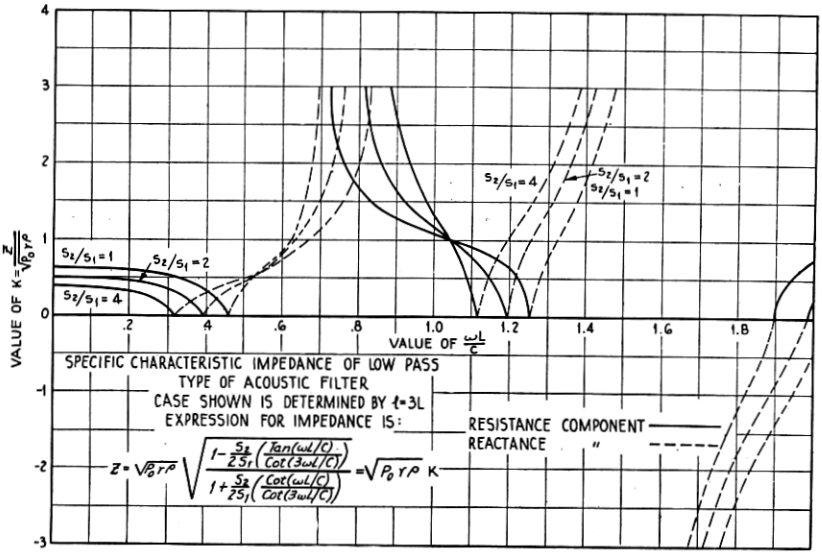

Fig. 4—Specific characteristic impedances for a low pass type of filter

theorem states: If a source of simple harmonic pressure p_0 and of internal impedance Z_T, per square centimeter, is connected to an acoustic system, and if the specific impedance Z_R terminates the system, the volume velocity at the termination of the system will be $p_0'/[(Z_T'/S_1) + (Z_R/S_n)]$, where p_0' is the pressure at the terminating end when this is closed through an infinite impedance, and Z_T' is the impedance per sq. cm. looking back into the acoustic system when this terminated in the impedance Z_T. S_1, and S_n are the areas at the input and output junctions, respectively.

Making use of Thévenin's theorem, the effect of inserting a filter in a given system is the same as the effect obtained by inserting this filter between a source of pressure p_0, with an internal impedance of Z_a/S_1 and a terminating impedance Z_b/S_n, where Z_a/S_1 and Z_b/S_n are respectively the total impedances looking toward the source, and away from the source at the insertion junction of the acoustic system. We have from equation (20)

$$V_2 = V_1 \cosh \Gamma - \frac{p_1 S_1}{Z_0} \sinh \Gamma.$$

$$p_2 = p_1 \cosh \Gamma - \frac{V_1 Z_0}{S_1} \sinh \Gamma,$$

Making use of the above, we can write

$$p_0 = p_1 + \frac{V_1 Z_a}{S_1}.$$

Substituting this, the above equation takes the form

$$\left.\begin{aligned} p_2 &= p_0 \cosh \Gamma - \frac{V_1}{S_1}[Z_0 \sinh \Gamma + Z_a \cosh \Gamma], \\ V_2 &= V_1\left(\cosh \Gamma + \frac{Z_a}{Z_0}\sinh \Gamma\right) - \frac{P_0 S_1}{Z_0}\sinh \Gamma. \end{aligned}\right\} \tag{28}$$

Eliminating V_1 and substituting $V_2 Z_b/S_1$ for p_2, since here the area remains constant at the two junctions, we have

$$V_2 = \frac{p_0 S_1}{\left[Z_b \cosh \Gamma + \frac{Z_a Z_b}{Z_0}\sinh \Gamma + Z_a \cosh \Gamma + Z_0 \sinh \Gamma\right]}.$$

The most useful way of writing this equation is

$$V_2 = \left(\frac{p_0 S_1}{2Z_b}\right)\left(\frac{2Z_0}{Z_0 + Z_a}\right)\left(\frac{2Z_b}{Z_0 + Z_b}\right)(e^{-\Gamma}) \times \left[\frac{1}{1 - e^{-2\Gamma}\left(\frac{Z_0 - Z_a}{Z_0 + Z_a}\right)\left(\frac{Z_0 - Z_b}{Z_0 + Z_b}\right)}\right]. \tag{29}$$

The volume velocity in the termination of the acoustic system, if the filter were not inserted, is obviously $p_0/[(Z_a/S_1) + (Z_b/S_1)]$. Hence the effect of inserting the filter at any junction is to change the volume velocity of the system by the factor

$$\left(\frac{Z_a + Z_b}{2Z_b}\right)\left(\frac{2Z_0}{Z_0 + Z_a}\right)\left(\frac{2Z_b}{Z_0 + Z_b}\right)(e^{-\Gamma}) \times \left[\frac{1}{1 - e^{-2\Gamma}\left(\frac{Z_0 - Z_a}{Z_0 + Z_a}\right)\left(\frac{Z_0 - Z_b}{Z_0 + Z_b}\right)}\right]. \tag{30}$$

A physical interpretation of equation (30) can be obtained in terms of the transmission and reflection factors first introduced by Heaviside.[6] Heaviside showed that at a junction, a reflection of a wave takes place if the impedances looking towards the source and away from the source are not equal. He showed that the current reflected on striking a junction, will be the unmodified current in the line multiplied by the

[6] Heaviside "Electromagnetic Theory" Vol. II, page 79.

factor, $(Z_I - Z_T)/(Z_I + Z_T)$, while the current transmitted to the terminating side of the junction will be the unmodified current in the line multiplied by the factor $2Z_T/(Z_I + Z_T)$ where Z_I and Z_T are respectively the impedances looking towards and away from the source at the junction. We see then that the second and third factors are transmission factors, determining respectively the transmission from the input impedance Z_a to the inserted structure, and from the inserted structure to the output impedance Z_b. The first factor is the inverse of the transmission factor determining the transmission from the impedance Z_a to the impedance Z_b. The fourth factor is the transfer factor and gives the reduction in volume velocity due to attenuation. The fifth factor has been called the interaction factor, and it gives the change in volume velocity in the termination due to repeated reflections of the volume velocity within the structure. All of these factors reduce to 1 except the transfer factor when $Z_a = Z_b = Z_0$. It will be noted that all factors except the transfer factor cancel out if $Z_a = Z_0$, or $Z_b = Z_0$.

The effect on the pressure due to inserting a filter can be shown to be given also by equation (30).

If the terminating impedances are resistances about equal to an average of the resistance value of Z_0, the effect of these is generally to introduce some loss in the pass band, when the characteristic impedance differs materially from the terminating impedances due to a reflection of the sound wave at the junction points. Since the characteristic impedance of a non-dissipative filter goes either to zero or infinity at the cut-off frequency, the effect of the reflection loss is generally to narrow the pass bands of the filter.

The effect of dissipation, when we take account of the viscosity effects by equations (20) or (21), is two-fold. It changes slightly the position of the band in the frequency range, due to a small change in the velocity of propagation. This is generally negligible. The other effect is to introduce attenuation in the pass band, due to absorption and dissipation of the sound wave.

B. High Pass Filter

An analogous type of high pass filter, which will attenuate the low frequencies and pass the high frequencies, can be made from the structure shown in Fig. 2 by using side tubes which are open on the outer end. The termination at the end of an open tube has been shown by Rayleigh [7] to be a mass with some resistance due to radiation. We could substitute this relation in equation (10) to determine

[7] Rayleigh, "Theory of Sound," Vol. II, p. 106.

the impedance Z_S looking into the sidebranch. Another approximation used with organ pipes is to consider the tube extended by a length .57 times the radius of the tube, and to consider this extended tube terminated in a zero impedance.

The impedance Z_S for this case is from (10)

$$\frac{p_1 S_2}{V_1} = Z_S = Z_{L_2} \tanh \alpha_2 l',$$

where l' is the corrected length of the pipe. Substituting this value in equation (21), we have

$$\cosh \Gamma = \left[\cosh 2\alpha_1 L + \frac{Z_{L_1} S_2 \sinh 2\alpha_1 L}{2 Z_{L_2} S_1 \tanh \alpha_2 l'}\right],$$

$$Z_0 = Z_{L_1} \sqrt{\frac{1 + \dfrac{Z_{L_1} S_2 \tanh \alpha_1 L}{2 Z_{L_2} S_1 \tanh \alpha_2 l'}}{1 + \dfrac{Z_{L_1} S_2 \coth \alpha_1 L}{2 Z_{L_2} S_1 \tanh \alpha_2 l'}}}. \tag{31}$$

For no dissipation these equations become

$$\cosh \Gamma = \left[\cos\left(\frac{2\omega L}{C}\right) + \frac{S_2}{2S_1}\left[\frac{\sin\left(\dfrac{2\omega L}{C}\right)}{\tan\left(\dfrac{\omega l'}{C}\right)}\right]\right],$$

$$Z_0 = \sqrt{P_0 \gamma \rho} \sqrt{\frac{1 + \dfrac{S_2 \tan\left(\dfrac{\omega L}{C}\right)}{2S_1 \tan\left(\dfrac{\omega l'}{C}\right)}}{1 - \dfrac{S_2 \cot\left(\dfrac{\omega L}{C}\right)}{2S_1 \tan\left(\dfrac{\omega l'}{C}\right)}}}.$$

Fig. 5 shows a plot of Γ for several ratios of S_2/S_1, when $l' = 3L$.

C. Band Pass Type of Filter

The high pass type of filter discussed above can also be considered as a band pass type of filter, in that an attenuation occurs at zero frequency, then a pass band, and a second attenuation band. A different arrangement of the pass bands can be obtained from the structure shown in Fig. 2, by inserting two sidebranches at one junction point, one of which is open at the outside end and the other closed.

An example of the type of characteristic obtained, is given by the special case where the lengths of both tubes are the same and equal to

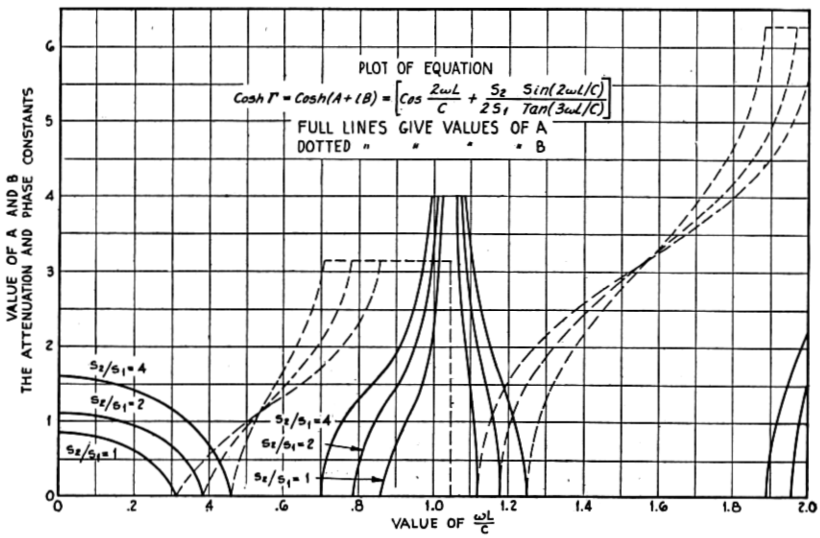

Fig. 5—Propagation constants for a high pass type of filter

$3L$. If S_2 is the area of the open tube and S_3 that of the closed tube, then neglecting dissipation, we find

$$\cosh \Gamma = \cos \frac{2\omega L}{C} + \frac{1}{2S_1}\left[\frac{S_2}{\tan\dfrac{3\omega L}{C}} - \frac{S_3}{\cot\dfrac{3\omega L}{C}}\right] \sin \frac{2\omega L}{C}.$$

A plot of A, the attenuation constant, for several values of S_2/S_1 and S_3/S_1 is given in Fig. 6.

D. Other Types of Sidebranches

We have so far considered only the characteristics obtained where we employ straight tubes. A number of cases can be solved in which the elements employed are not straight tubes although we cannot take account of the viscosity dissipation in these cases. As an example, the characteristics of a filter will be worked out, which employs a straight tube for the conducting tube and conical tubes closed on the end for the sidebranches. We can make use of equation (21) to determine Γ and Z_0, if we insert the proper value of Z_S for the conical tube.

It is evident that for a conical tube, the proper type of wave is a spherical wave, in place of the plane wave employed for a straight tube. For this case we can write [8] for a simple harmonic wave

$$\frac{\partial^2(r\varphi)}{\partial t^2} = C^2\frac{\partial^2(r\varphi)}{\partial r^2}\ ; \quad \dot{\eta} = -\frac{\partial\varphi}{\partial r} \text{ and } \frac{p}{\rho_0} = \dot{\varphi},$$

where φ is the velocity potential, $\dot{\eta}$ the linear velocity for the spherical wave, p the pressure, ρ the average density of the medium, and r the

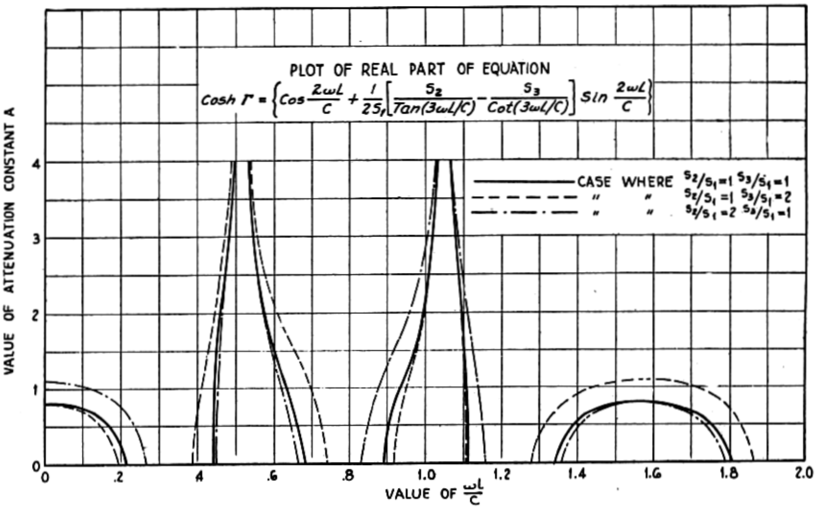

Fig. 6—Attenuation constants for a band pass type of filter

distance from the apex of the cone. The solution for this case is

$$r\varphi = A \sin\frac{\omega}{C}r + B\cos\frac{\omega}{C}r.$$

Hence we can determine $\dot{\eta}$ and p as

$$\dot{\eta} = A\left[\frac{\sin\frac{\omega}{C}r}{r^2} - \frac{\frac{\omega}{C}\cos\frac{\omega}{C}r}{r}\right] + B\left[\frac{\cos\frac{\omega}{C}r}{r^2} + \frac{\frac{\omega}{C}\sin\frac{\omega}{C}r}{r}\right]$$

and

$$p = i\omega\rho\left[\frac{A\sin\frac{\omega}{C}r + B\cos\frac{\omega}{C}r}{r}\right].$$

If now we set $\dot{\eta} = 0$ when $r = x_2$ and determine the ratio of $p/\dot{\eta}$ at

[8] Lamb, "Dynamical Theory of Sound," p. 206. Rayleigh, "Theory of Sound," Vol. II, p. 114.

$r = x_1$, we find

$$Z_S = \frac{p}{\dot{\eta}}$$

$$= -i\sqrt{P_0\gamma\rho}\left[\frac{\cos\frac{\omega}{C}(x_2 - x_1) - \dfrac{\sin\frac{\omega}{C}(x_2 - x_1)}{\frac{\omega}{C}x_2}}{\cos\frac{\omega}{C}(x_2 - x_1)\left[\dfrac{1}{\frac{\omega}{C}x_2} - \dfrac{1}{\frac{\omega}{C}x_1}\right] + \sin\frac{\omega}{C}(x_2 - x_1)\left[1 + \dfrac{1}{\frac{\omega^2}{C^2}x_1x_2}\right]}\right]. \qquad (32)$$

If we substitute this value of Z_S in equation (21), we can readily determine the value of Γ and Z_0. Fig. 7 shows a plot of A and B for this case assuming $(x_2 - x_1) = L$.

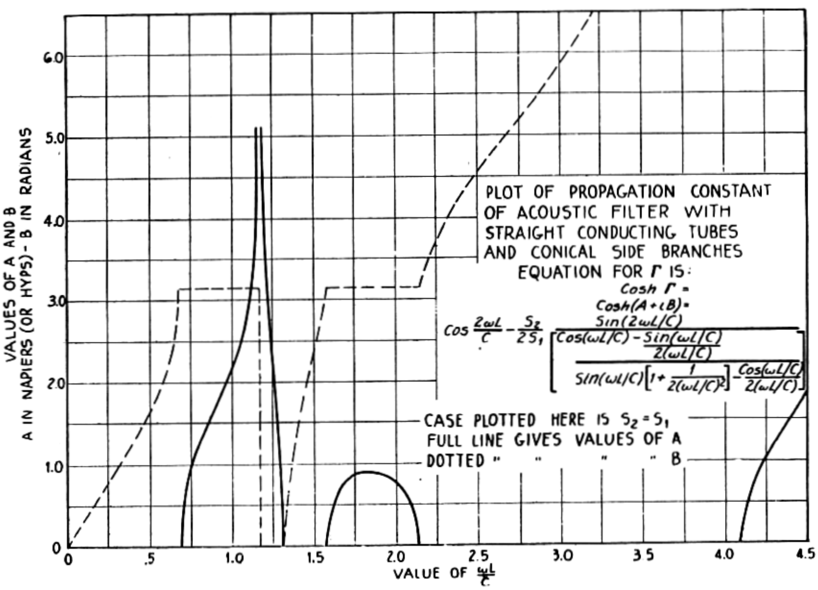

Fig. 7—Propagation constant of a low pass type of filter.

IV. Tapered Filter Structures and Horns

In addition to recurrent filters, other types of filters exist. If, for example, we connect sections with the same propagation constants and characteristic impedances, but whose conducting tube areas increase in some regular manner, a tapered filter is obtained whose characteristics differ from those of a recurrent filter. The distinguishing property introduced by a tapered filter, in addition to its filtering property, is a transformer action which increases the pressure by a given ratio and decreases the volume velocity in the same ratio, or vice versa, thus giving a transforming action and a complete transmission of power over the pass band. This is a useful property, if acoustic systems of different impedances are to be connected together. Horns are the limiting cases of tapered acoustic filters and hence their study has considerable practical importance.

The typical section of a tapered filter considered here is one built up from two symmetrical structures with the same propagation constants and characteristic impedances per square centimeter, but with different cross-sectional areas. If we use any of the recurrent filters discussed in Section III, then, for example, since

$$\cosh \Gamma = \left[\cos\left(\frac{2\omega L}{C}\right) - \frac{S_2 \sin\left(\frac{2\omega L}{C}\right)}{2S_1 \cot \frac{\omega l}{C}} \right]$$

for the low pass filter, to keep the same value of Γ when we vary the conducting tube area it will be necessary to keep the ratio of the areas constant and to leave all values of L and l the same. Similarly for the other types of filters.

If $\Gamma/2$ is the propagation constant of each of the symmetrical structures, Z_0 the characteristic impedance per square centimeter for each structure, S_1 the cross-sectional area of the first structure and S_2 that of the second, we can write three sets of equations for the two structures and the junction point. These are

$$\left.\begin{aligned} p_1' &= p_1 \cosh\frac{\Gamma}{2} - V_1 \frac{Z_0}{S_1} \sinh\frac{\Gamma}{2}, \\ V_1' &= V_1 \cosh\frac{\Gamma}{2} - \frac{p_1 S_1}{Z_0} \sinh\frac{\Gamma}{2}, \end{aligned}\right\}$$

$$p_1'' = p_1'; \quad V_1'' = V_1',$$

$$\left.\begin{aligned} p_2 &= p_1'' \cosh\frac{\Gamma}{2} - V_1'' \frac{Z_0}{S_2} \sinh\frac{\Gamma}{2}, \\ V_2 &= V_1'' \cosh\frac{\Gamma}{2} - \frac{p_1'' S_2}{Z_0} \sinh\frac{\Gamma}{2}. \end{aligned}\right\}$$

Combining these equations, we obtain

$$\left.\begin{aligned} p_2 = p_1\left[\left(\frac{S_1+S_2}{2S_2}\right)\cosh\Gamma + \left(\frac{S_2-S_1}{2S_2}\right)\right] \\ - \frac{V_1Z_0}{S_1}\left(\frac{S_1+S_2}{2S_2}\right)\sinh\Gamma, \\ V_2 = V_1\left[\left(\frac{S_1+S_2}{2S_1}\right)\cosh\Gamma - \left(\frac{S_2-S_1}{2S_1}\right)\right] \\ - \frac{p_1S_1}{Z_0}\left(\frac{S_1+S_2}{2S_1}\right)\sinh\Gamma, \end{aligned}\right\} \tag{33}$$

or for simplicity we write

$$\left.\begin{aligned} p_2 &= p_1A - \frac{V_1Z_0}{S_1}B, \\ V_2 &= V_1C - \frac{p_1S_1}{Z_0}D. \end{aligned}\right\} \tag{34}$$

In order to express the propagation in terms of some known functions we will first obtain some relations between the impedances of the sections and the ratios of p_2/p_1 and V_2/V_1. We can write the above equations as

$$\frac{p_2}{p_1} = A - \frac{Z_0}{Z_1}B, \quad \frac{V_2}{V_1} = C - \frac{Z_1}{Z_0}D,$$

where $Z_1/S_1 = p_1/V_1$. Eliminating Z_1, we have

$$\frac{p_2}{p_1}\frac{V_2}{V_1} - A\frac{V_2}{V_1} - C\frac{p_2}{p_1} = BD - AC = -1 \tag{35}$$

as can be seen by multiplying together the above expressions. Solving for the ratio of V_2/V_1 in terms of p_2/p_1, A and C, we have

$$\frac{V_2}{V_1} = \frac{C\dfrac{p_2}{p_1} - 1}{\dfrac{p_2}{p_1} - A}.$$

Multiplying both sides by p_1/p_2, we have

$$\frac{p_1}{V_1}\frac{V_2}{p_2} = \frac{C - \dfrac{p_1}{p_2}}{\dfrac{p_2}{p_1} - A}.$$

Now since $\frac{p_1}{V_1} = \frac{Z_1}{S_1}$ and $\frac{p_2}{V_2} = \frac{Z_2}{S_2}$, we have

$$\frac{Z_2}{S_2} = \frac{Z_1}{S_1}\left[\frac{\frac{p_2}{p_1} - A}{C - \frac{p_1}{p_2}}\right]. \tag{36}$$

Z_2/S_2 and Z_1/S_1 then are respectively the terminating impedance and input impedance necessary to give a structure specified by the factors A, B, C, D the pressure ratio p_2/p_1. To solve for the input impedance we take the first of equations (34) and obtain

$$Z_1 = \frac{Z_0 B}{A - \frac{p_2}{p_1}}. \tag{37}$$

Hence by virtue of (36), the terminating impedance Z_2/S_2 becomes

$$\frac{Z_2}{S_2} = \frac{\frac{Z_0}{S_1} B}{\frac{p_1}{p_2} - C}. \tag{38}$$

Equations (37) and (38) state that there is a relation between the input impedance and the pressure ratio, and the output impedance and the pressure ratio. When one is specified and the constants of the section Z_0, A, B, C, D are given, the others are known.

Suppose now that we wish to join a second structure of this type to the first, assuming that the cross-sectional area at the junction is the same for both. We must have now Z_2, the specific output impedance of the first section, equal to Z_1', the specific input impedance of the second section. Hence we can write.

$$\frac{\frac{Z_0}{S_1} B}{\frac{p_1}{p_2} - C} = \frac{\frac{Z_0}{S_2} B'}{\left(A' - \frac{p_3}{p_2}\right)} \text{ or } \frac{S_2}{S_1} B\left[A' - \frac{p_3}{p_2}\right] = B'\left[\frac{p_1}{p_2} - C\right],$$

where the primes refer to the constants of the second section and where p_3/p_2 is the pressure ratio of the second section. Substituting in the values of A', B, B', C, we have

$$\left(\frac{(S_1 + S_2)(S_2 + S_3)}{2S_1S_3}\right)\cosh \Gamma + \frac{S_1S_3 - S_2^2}{2S_1S_3} = \left(\frac{S_2 + S_3}{2S_3}\right)\frac{p_1}{p_2} + \left(\frac{S_1 + S_2}{2S_1}\right)\frac{p_3}{p_2}. \tag{39}$$

Equation (39) gives the relationship between p_1/p_2 and p_3/p_2 which must be satisfied if the output impedance of one section equals the input impedance of the next section. If we specify a value of p_2/p_1, then the value of p_3/p_2 is determined. The impedance Z_2' terminating the second section is also determined and hence the pressure ratio of the third section, etc. Hence if we specify a value of p_2/p_1, we also determine the propagation characteristic of any other section in a series of sections. The pressure ratios will not in general be constant from section to section.

We can write $p_2/p_1 = Ke^{-\delta}$ since this will represent any phase or amplitude change. Similarly we can write p_3/p_2 as $K'e^{-\delta'}$. Substituting these values in (39), we have

$$\left(\frac{(S_1 + S_2)(S_2 + S_3)}{2S_1S_3}\right)\cosh\Gamma + \frac{S_1S_3 - S_2^2}{2S_1S_3} = \left(\frac{S_2 + S_3}{2S_3}\right)\frac{e^{\delta}}{K} + \left(\frac{S_1 + S_2}{2S_1}\right)K'e^{-\delta'}. \tag{40}$$

Now if the value of δ remains unchanged from section to section a great simplification results, for in order to determine the overall pressure ratio we have only to multiply the number of sections by δ. Hence it is desirable to determine for what rate of taper this condition is met and also how good an approximation it is for all rates of taper.

If we set $\delta = \delta'$ and multiply through by $e^{-\delta}$, we obtain

$$e^{-2\delta} - \left[\frac{\left(\frac{(S_1 + S_2)(S_2 + S_3)}{2S_1S_3}\right)\cosh\Gamma + \frac{S_1S_3 - S_2^2}{2S_1S_3}}{\frac{S_1 + S_2}{2S_1}K'}\right]e^{-\delta} + \frac{\left(\frac{S_2 + S_3}{2S_3}\right)\frac{1}{K}}{\left(\frac{S_1 + S_2}{2S_1}\right)K'} = 0.$$

Similarly the equation for the next two sections is

$$e^{-2\delta''} - \frac{\left[\left(\frac{(S_2 + S_3)(S_3 + S_4)}{2S_2S_4}\right)\cosh\Gamma + \left(\frac{S_2S_4 - S_3^2}{2S_2S_4}\right)\right]}{\left(\frac{S_2 + S_3}{2S_2}\right)K''}e^{-\delta''} + \frac{\left(\frac{S_3 + S_4}{2S_4}\right)\frac{1}{K'}}{\left(\frac{S_2 + S_3}{2S_2}\right)K''} = 0.$$

If we are to have $\delta = \delta''$, we must have

$$e^{-\delta}\left[\left[\frac{S_2+S_3}{S_3K'}-\frac{S_3+S_4}{S_4K''}\right]\cosh\Gamma + \left[\frac{S_1S_3-S_2^2}{(S_1+S_2)S_3K'}-\frac{S_2S_4-S_3^2}{(S_2+S_3)S_4K''}\right]\right] = \left[\frac{(S_2+S_3)S_1}{S_3(S_1+S_2)KK'}-\frac{(S_3+S_4)S_2}{(S_2+S_3)S_4K'K''}\right]. \quad (41)$$

Since the term on the left is complex, while that on the right is a numeric, each must separately vanish if we are to have this equality. Similarly the terms within the bracket of the left hand side would have to vanish.

We see that the two terms on the left do not vanish simultaneously unless we satisfy the progression equation

$$2S_1S_3^2 - 2S_2^2S_4 + (S_3 - S_2)(S_1S_4 + S_2S_3) = 0. \quad (42)$$

This equation is satisfied by a system whose area increases exponentially with the distance. The terms involving $S_1S_3 - S_2^2$ and $S_2S_4 - S_3^2$ are always very small no matter what the rate of progression. Hence is is desirable to see if neglecting these terms we can still satisfy the above conditions. The most useful value of the two terms on the right hand side of equation (41) is 1. Hence setting each term equal to 1 and solving for K' and K'', we find that

$$K' = \sqrt{\frac{S_2}{S_3}};\quad K'' = \sqrt{\frac{S_3}{S_4}}.$$

We see then that if we neglect second order quantities, we can represent with good approximation the pressure ratio of any tapered filter by the expression

$$\frac{p_2}{p_1} = \sqrt{\frac{S_n}{S_{n+1}}}\,e^{-\delta},$$

where δ is the propagation constant of a tapered structure. For a complete solution, δ is not constant except for a progression which satisfies equation (42).

A. Exponentially Tapered Filters and Horns

If we assume that the area of a given section is e^{2t} times as large as that of the section preceding, equation (40) reduces to

$$2e^{-t}[\cosh\Gamma\cosh t] = K'e^{-\delta} + \left(e^{-2t}\times\frac{1}{K}\right)e^{\delta}. \quad (43)$$

We choose now $K' = \sqrt{\frac{S_2}{S_3}} = e^{-t}$ and $K = \sqrt{\frac{S_1}{S_2}} = e^{-t}$.
Then

$$\cosh \delta = \cosh \Gamma \cosh t.$$

To show that δ is the propagation constant for an infinite sequence of such sections, it is necessary to show that δ is the same for any two sections. But equation (43) holds good for any two sections, and hence δ is the same, and represents a solution for an infinite sequence of sections. Now

$$e^{-\delta} = \cosh \delta - \sinh \delta = \cosh \Gamma \cosh t - \sqrt{\sinh^2 \Gamma \cosh^2 t + \sinh^2 t}.$$

Hence

$$\frac{p_2}{p_1} = Ke^{-\delta} = e^{-t}[\cosh \Gamma \cosh t - \sqrt{\sinh^2 \Gamma \cosh^2 t + \sinh^2 t}]$$

and

$$\frac{V_2}{V_1} = e^{-t}e^{-\delta}\left[\frac{C - e^t e^\delta}{e^{-t}e^{-\delta} - A}\right]$$
$$= e^t[\cosh \Gamma \cosh t - \sqrt{\sinh^2 \Gamma \cosh^2 t + \sinh^2 t}],$$

and hence the pressure and volume velocity have the same propagation constant δ but an inverse multiplying factor.

The specific impedance Z_1, looking into a given section, is by equation (37)

$$Z_1 = \frac{Z_0 B}{A - Ke^{-\delta}} = Z_0\left[\frac{\sqrt{\tanh^2 t + \sinh^2 \Gamma} - \tanh t}{\sinh \Gamma}\right] \qquad (44)$$

and similarly Z_2, the specific terminating impedance, can be shown equal to Z_1. Hence the impedance per square centimeter at the junction points is the same for each section.

To observe the action of a tapered filter, let us obtain the product of the pressure by the volume velocity and see how these are propagated. Since the specific impedance is the same from section to section, this will represent also the power propagation. Now since

$$\cosh \delta = \cosh \Gamma \cosh t,$$

a pass band occurs when $1 \geqq \cosh \delta \geqq -1$ and hence the band occurs only when Γ is imaginary, since $\cosh \Gamma < 1$ and > -1, or when the filter repeated recurrently is in its pass band. Furthermore the pass band for the tapered structure will not be as wide as that for a similar recurrent structure, since for the tapered structure the band occurs when $\cosh \Gamma = \pm 1/\cosh t$ while in the recurrent structure, the band occurs when $\cosh \Gamma = \pm 1$. One result of this is that no low pass filter exists in exponentially tapered structures.

Considering now the pressure and volume velocity ratios when $1 \geqq \cosh \delta \geqq -1$, the absolute value of $e^{-\delta}$ is 1. Hence over the band the ratios of pressure and of volume velocity from section to section are respectively e^{-t} and e^{t} or $\sqrt{S_1/S_2}$ and $\sqrt{S_2/S_1}$. Hence one section multiplies the pressure by a ratio $\sqrt{S_1/S_2}$, and the volume velocity by the factor $\sqrt{S_2/S_1}$. Therefore a tapered structure of this kind is equivalent to a transformer of turns ratio $\sqrt{S_1/S_2}$, and a filter of somewhat narrower bands than for the filter repeated recurrently.

To specify completely a filter of this type requires three parameters. Two such parameters have been developed above and are δ, the propagation constant of a tapered filter, and Z_{R_1}, the specific recurrent impedance in one direction. These are given by

$$\left.\begin{aligned} \cosh \delta &= \cosh \Gamma \cosh t, \\ Z_{R_1} &= Z_0 \left[\frac{\sqrt{\tanh^2 t + \sinh^2 \Gamma} - \tanh t}{\sinh \Gamma} \right]. \end{aligned}\right\} \tag{45}$$

We take as the third parameter Z_{R_2}, the specific recurrent impedance in the opposite direction. We can readily determine that Z_{R_2}, the impedance looking in the opposite direction from that used to specify Z_{R_1}, but obtained at the same junction point, is

$$Z_{R_2} = \frac{Z_0 B}{\dfrac{p_1'}{p_2'} - A}.$$

It is desirable to have the same propagation constant serve for the two directions, hence we let $p_2'/p_1' = K_1 e^{-\delta}$. Since K represents a transformer change of the pressure in one direction, we find, when going in the opposite direction, that the pressure should change by the inverse of K, so $K_1 = 1/\mathrm{K}$. Substituting these values for p_1'/p_2',

$$Z_{R_2} = \frac{Z_0 B}{K e^{\delta} - A}. \tag{46}$$

Hence for an exponentially tapered filter

$$Z_{R_2} = Z_0 \left[\frac{\sqrt{\tanh^2 t + \sinh^2 \Gamma} + \tanh t}{\sinh \Gamma} \right]. \tag{47}$$

In terms of the parameters, δ, Z_{R_1} and Z_{R_2} we can express p_2, p_1, V_2, and V_1 as

$$p_2 = e^{-t}\left[p_1 \left[\cosh \delta + \left[\frac{Z_{R_2} - Z_{R_1}}{Z_{R_2} + Z_{R_1}} \right] \sinh \delta \right] - \frac{V_1}{S_1}\left[\frac{2Z_{R_1}Z_{R_2}}{Z_{R_1} + Z_{R_2}} \right] \sinh \delta \right],$$
$$V_2 = e^{t}\left[V_1 \left[\cosh \delta + \left[\frac{Z_{R_1} - Z_{R_2}}{Z_{R_1} + Z_{R_2}} \right] \sinh \delta \right] - p_1 S_1 \left[\frac{2 \sinh \delta}{Z_{R_1} + Z_{R_2}} \right]\right]. \tag{48}$$

If now the elements of our structure are non-dissipative straight tubes, instead of a general filter structure, and the length of these tubes between changes of area is made very small, it is evident that the structure reduces to an exponential horn. We now let the ratio $S_1/S_2 = e^{-2t}$ be expressed as

$$\frac{S_1}{S_2} = e^{-2Tl} = e^{-2t},$$

where l is the distance between changes in area and T a new taper constant. Then Γ, for a straight tube, neglecting dissipation, becomes $\Gamma = i\omega l/c$ and hence

$$\begin{aligned}\cosh \delta &= \cosh \frac{i\omega l}{C} \cosh Tl \\ &= \left(1 - \frac{\omega^2 l^2}{2!C^2} + \frac{\omega^2 l^2}{4!C^4} + \cdots\right)\left(1 + \frac{(Tl)^2}{2!} + \frac{(Tl)^4}{4!} + \cdots\right) \\ &= 1 + \frac{l^2\left(T^2 - \frac{\omega^2}{C^2}\right)}{2!} + l^4 \frac{\left(T^4 - 6\frac{\omega^2}{C^2}T^2 + \frac{\omega^4}{C^4}\right)}{4!} \cdots .\end{aligned}$$

This reduces for small values of l to

$$\cosh \delta = \cosh\left(l\sqrt{T^2 - \frac{\omega^2}{C^2}}\right).$$

Hence

$$\frac{p_n}{p_1} = e^{-nTl}e^{-n\delta} = e^{-nl\left(T + \sqrt{T^2 - \frac{\omega^2}{C^2}}\right)} = e^{-L\left(T + \sqrt{T^2 - \frac{\omega^2}{C^2}}\right)}$$

since $nl = L$, the total length of the horn.

As long as $T^2 > (\omega^2/C^2)$, an attenuation band exists, while if $\omega^2/C^2 > T^2$, the expression becomes

$$\frac{p_n}{p_1} = e^{-TL}\left[\cos\left(L\sqrt{\frac{\omega^2}{C^2} - T^2}\right) - i \sin\left(L\sqrt{\frac{\omega^2}{C^2} - T^2}\right)\right]$$

and a pass band occurs.

The complete equation for the horn, equivalent to equation (48), becomes

$$p_2 = e^{-LT}\left\{\left[\cosh\left(L\sqrt{T^2-\frac{\omega^2}{C^2}}\right) + \frac{T}{\sqrt{T^2-\frac{\omega^2}{C^2}}}\sinh\left(L\sqrt{T^2-\frac{\omega^2}{C^2}}\right)\right]p_1 - \frac{iV_1\sqrt{P_0\gamma\rho}\,\frac{\omega}{C}\sinh\left(L\sqrt{T^2-\frac{\omega^2}{C^2}}\right)}{S_1\sqrt{T^2-\frac{\omega^2}{C^2}}}\right\},$$

$$V_2 = e^{+LT}\left\{\left[\cosh\left(L\sqrt{T^2-\frac{\omega^2}{C^2}}\right) - \frac{T}{\sqrt{T^2-\frac{\omega^2}{C^2}}}\sinh\left[\left(L\sqrt{T^2-\frac{\omega^2}{C^2}}\right)\right]\right]V_1 - \frac{iS_1p_1\frac{\omega}{C}\sinh\left(L\sqrt{T^2-\frac{\omega^2}{C^2}}\right)}{\sqrt{P_0\gamma\rho}\sqrt{T^2-\frac{\omega^2}{C^2}}}\right\}. \quad (49)$$

These expressions can be derived from Webster's [9] differential equations for an exponential horn. Exponential horns have also been discussed by a number of writers.[10]

B. Tapered Filters Whose Area Increases as the Square of the Distance

One other example of a tapered filter, for which an approximate solution can be obtained, will be considered because of its bearing on the straight or conical horn. Let us assume that the area S_1 of a typical section of a tapered filter chain is n^2E, while that of the section next to it is equal to $(n+1)^2E$, where E is a small constant. Sub-

[9] A. G. Webster, "Acoustic Impedance, and The Theory of Horns and of the Phonograph," *Nat. Acad. of Science*, Vol. 5, 1919, p. 275. The solution given by Webster for the exponential horn appears to have some typographical errors.

[10] Hanna and Slepian (*Trans. A. I. E. E.*, 43, 1924, p. 393); H. C. Harrison (British Patent No. 213,525, 1925); I. B. Crandall, "Theory of Vibrating Systems and Sound," D. Van Nostrand, 1926, p. 158.

stituting these values in equation (40), we obtain

$$\left[\left(\frac{(n^2+(n+1)^2)((n+1)^2+(n+2)^2)}{2(n)^2(n+2)^2}\right)\cosh\Gamma + \frac{n^2(n+2)^2-(n+1)^4}{2(n)^2(n+2)^2}\right] = K'\left(\frac{n^2+(n+1)^2}{2(n)^2}\right)e^{-\delta} + \left(\frac{(n+1)^2+(n+2)^2}{2(n+2)^2}\right)\frac{e^{\delta}}{K}.$$

If we substitute $K' = \sqrt{\frac{S_2}{S_3}} = \frac{n+1}{n+2}$ and $K = \sqrt{\frac{S_1}{S_2}} = \frac{n}{n+1}$ and neglect 1 as compared with n^3, we have

$$\cosh\delta = \left[\left(\frac{2n^2+1}{2n^2}\right)\cosh\Gamma - \frac{1}{2n^2}\right]. \qquad (50)$$

If again our changes in areas are very small and hence n very large, we can neglect 1 compared with $2n^2$ and obtain

$$\cosh\delta = \cosh\Gamma\,, \quad \text{or} \quad \delta = \Gamma.$$

Either of these solutions will hold for any other pair of sections if we neglect 1 as compared with n^3 for the first of 1 compared with n^2 for the second. Hence for either solution, the propagation constant is little affected for this type of taper. The specific characteristic impedances Z_{R_1} and Z_{R_2} become

$$Z_{R_1} = \frac{Z_0\sinh\Gamma}{\frac{1}{n}+\left(\frac{2n^2}{2n^2+1}\right)\sqrt{\left(\frac{2n^2+1}{2n^2}\cosh\Gamma-\frac{1}{2n^2}\right)^2-1}},$$
$$Z_{R_2} = \frac{Z_0\sinh\Gamma}{-\frac{1}{n}+\frac{2n^2}{2n^2+1}\sqrt{\left(\frac{2n^2+1}{2n^2}\cosh\Gamma-\frac{1}{2n^2}\right)^2-1}}. \qquad (51)$$

If we neglect 1 as compared with n^2, these expressions reduce to

$$Z_{R_1} = \frac{Z_0 n\sinh\Gamma}{1+n\sinh\Gamma}\,; \quad Z_{R_2} = \frac{Z_0 n\sinh\Gamma}{-1+n\sinh\Gamma}. \qquad (52)$$

These impedances represent the impedances per square cm. looking in both directions at the input junction of the filter, whose area is n^2E. As we move in either direction these impedances change since n itself changes. If n becomes sufficiently large and Γ is not zero, the two characteristic impedances approach the value Z_0.

To express p_2 and V_2 in terms of p_1 and V_1 and these three parameters, we can obtain the equations.

$$p_2 = \frac{n}{n+1}\left[p_1\left[\cosh\delta + \left(\frac{Z_{R_2}{}^I - Z_{R_1}{}^I}{Z_{R_2}{}^I + Z_{R_1}{}^I}\right)\sinh\delta\right] - \frac{V_1}{S_1}\left[\frac{2Z_{R_1}{}^I Z_{R_2}{}^I}{Z_{R_1}{}^I + Z_{R_2}{}^I}\right]\sinh\delta\right],$$

$$\begin{aligned} V_2 = \frac{n+1}{n}\Bigg[& V_1\left[\cosh\delta + \left(\frac{Z_{R_1}{}^O - Z_{R_2}{}^O}{Z_{R_1}{}^O + Z_{R_2}{}^O}\right)\sinh\delta\right] \\ & - p_1 S_1\left[\frac{Z_{R_1}{}^I + Z_{R_2}{}^I}{2Z_{R_1}{}^I Z_{R_2}{}^I}\right] \\ & \times\left[\sinh\delta\left[1 - \left(\frac{Z_{R_2}{}^I - Z_{R_1}{}^I}{Z_{R_2}{}^I + Z_{R_1}{}^I}\right)\left(\frac{Z_{R_2}{}^O - Z_{R_1}{}^O}{Z_{R_2}{}^O + Z_{R_1}{}^O}\right)\right] + \cosh\delta \right. \\ & \left. \times\left[\left(\frac{Z_{R_2}{}^I - Z_{R_1}{}^I}{Z_{R_2}{}^I + Z_{R_1}{}^I}\right) - \left(\frac{Z_{R_2}{}^O - Z_{R_1}{}^O}{Z_{R_2}{}^O + Z_{R_1}{}^O}\right)\right]\right]\Bigg], \end{aligned} \tag{53}$$

as can readily be seen by comparing these expressions with the equations, $p_2 = p_1A - (V_1/S_1)Z_0B$; $V_2 = V_1C - (p_1/Z_0)S_1D$. In the above expression the letter I indicates that the impedances Z_{R_1} and Z_{R_2} are to be taken at the input junction, while the letter O indicates that they are to be taken at the output junction.

The effect of this type of tapering is to change the propagation constant scarcely at all, but to lower the characteristic impedances in the neighborhood of the cut-off frequencies. This tends to produce large reflection losses and hence effectively the band is narrowed. A transforming action equivalent to a transformer of turns ratio $\sqrt{S_1/S_2}$ occurs as before.

To obtain the equation for a straight horn, we let S_1, a typical area of the horn, equal

$$S_1 = n^2K = K'(nl)^2 = K'(x_1)^2,$$

where $nl = x_1$, the distance from the apex of the horn, and l the length of an individual section. Γ becomes $i\omega l/C$, and Z_{R_1} and Z_{R_2} are

$$Z_{R_1} = \frac{\sqrt{p_0\gamma\rho}\, in\dfrac{\omega}{C}l}{1 + in\dfrac{\omega}{C}l} = \frac{\sqrt{p_0\gamma\rho}\, i\dfrac{\omega}{C}x_1}{1 + i\dfrac{\omega}{C}x_1},$$

and

$$Z_{R_2} = \frac{\sqrt{p_0\gamma\rho}\, i\dfrac{\omega}{C}x_1}{-1 + i\dfrac{\omega}{C}x_1}. \tag{54}$$

Substituting these values in equation (53), we obtain the equation

$$p_2 = \frac{x_1}{x_2}\left\{ p_1 \left[\cos\frac{\omega}{C}(x_2 - x_1) + \frac{\sin\left(\frac{\omega}{C}(x_2 - x_1)\right)}{\frac{\omega}{C}x_1} \right] - i\frac{V_1}{S_1}\sqrt{P_0\gamma\rho}\sin\frac{\omega}{C}(x_2 - x_1) \right\},$$

$$V_2 = \frac{x_2}{x_1}\left\{ V_1\left(\cos\frac{\omega}{C}(x_2 - x_1) - \frac{\sin\left(\frac{\omega}{C}(x_2 - x_1)\right)}{\frac{\omega}{C}x_2} \right) - i\frac{p_1 S_1}{\sqrt{P_0\gamma\rho}} \times\left[\left(1+\frac{1}{\left(\frac{\omega}{C}\right)^2 x_1x_2}\right)\sin\frac{\omega}{C}(x_2-x_1) + \left(\frac{1}{\frac{\omega}{C}x_2} - \frac{1}{\frac{\omega}{C}x_1}\right)\cos\frac{\omega}{C}(x_2-x_1)\right]\right\}. \quad (55)$$

If we introduce two lengths ϵ_1 and ϵ_2 defined by $\tan(\omega/C)\epsilon_1 = (\omega/C)x_1$ and $\tan(\omega/C)\epsilon_2 = (\omega/C)x_2$ and take account of the fact that the impedance as defined here must be multiplied by $i\omega$ to correspond to the impedance defined by Webster, then it is evident that the above equation corresponds to the relation given by Webster.[9, 10]

It is interesting to compare the relations obtained above involving the assumptions introduced in Section II with the solution involving no assumptions. This can be done for the conical horn, since its solution can be obtained using spherical waves. In Section III–*D*, the impedance looking into a conical horn was obtained when an infinite impedance terminated the horn. If we set $V_2 = 0$ in the last of equations (55) and solve for the ratio of p_1/V_1, it is evident that the impedance agrees with that given in Section III–*D*. Hence it is evident that both methods give the same solution.

Many other types of tapered filters can be solved in a similar manner, but no more will be considered here.

V. General Network Equations and Network Parameters

We can combine a number of symmetrical structures to form a general network. For any symmetrical structure we can write the

[10] The solution for the conical horn has been discussed in more detail by I. B. Crandall, "Theory of Vibrating Systems and Sound," D. Van Nostrand, 1926, p. 152.

equations

$$p_2 = p_1 \cosh \Gamma_1 - V_1 \frac{Z_{0_1}}{S_1} \sinh \Gamma_1,$$

$$V_2 = V_1 \cosh \Gamma_1 - \frac{p_1 S_1}{Z_{0_1}} \sinh \Gamma_1.$$

Suppose then that we wish to join this structure to other structures, with different characteristics and with different area conducting tubes. At the junction of the structures, we have by equation (14)

$$p_3 = p_2, \qquad V_3 = V_2.$$

Combining these with the above, we have

$$p_3 = p_1 \cosh \Gamma_1 - V_1 \frac{Z_{0_1}}{S_1} \sinh \Gamma_1,$$

$$V_3 = V_1 \cosh \Gamma_1 - \frac{p_1 S_1}{Z_{0_1}} \sinh \Gamma_1.$$

Writing a set of equations similar to the above for the second structure and combining, we have

$$\left.\begin{aligned} p_4 = p_1 &\left(\cosh \Gamma_1 \cosh \Gamma_2 + \frac{S_1}{S_2}\frac{Z_{0_2}}{Z_{0_1}} \sinh \Gamma_1 \sinh \Gamma_2 \right) \\ &- V_1 \frac{Z_{0_1}}{S_1} \left(\sinh \Gamma_1 \cosh \Gamma_2 + \frac{Z_{0_2}}{Z_{0_1}}\frac{S_1}{S_2} \cosh \Gamma_1 \cosh \Gamma_2 \right), \\ V_4 = V_1 &\left(\cosh \Gamma_1 \cosh \Gamma_2 + \frac{Z_{0_1}}{Z_{0_2}}\frac{S_2}{S_1} \sinh \Gamma_1 \sinh \Gamma_2 \right) \\ &- \frac{p_1 S_1}{Z_{0_1}} \left(\sinh \Gamma_1 \cosh \Gamma_2 + \frac{S_2}{S_1}\frac{Z_{0_1}}{Z_{0_2}} \cosh \Gamma_1 \sinh \Gamma_2 \right). \end{aligned}\right\}$$

We can also write this in the form

$$p_4 = p_1 \begin{vmatrix} \cosh \Gamma_1 & \frac{S_1}{S_2}\frac{Z_{0_2}}{Z_{0_1}} \sinh \Gamma_2 \\ -\sinh \Gamma_1 & \cosh \Gamma_2 \end{vmatrix} - V_1 \frac{Z_{0_1}}{S_1} \begin{vmatrix} \sinh \Gamma_1 & \frac{S_1}{S_2}\frac{Z_{0_2}}{Z_{0_1}} \sinh \Gamma_2 \\ -\cosh \Gamma_1 & \cosh \Gamma_2 \end{vmatrix}.$$

$$V_4 = V_1 \begin{vmatrix} \cosh \Gamma_1 & \frac{Z_{0_1}}{Z_{0_2}}\frac{S_2}{S_1} \cosh \Gamma_2 \\ -\sinh \Gamma_1 & \cosh \Gamma_2 \end{vmatrix} - \frac{p_1 S_1}{Z_{0_1}} \begin{vmatrix} \sinh \Gamma_1 & \frac{Z_{0_1}}{Z_{0_2}}\frac{S_2}{S_1} \sinh \Gamma_2 \\ -\cosh \Gamma_1 & \cosh \Gamma_2 \end{vmatrix}.$$

In fact if we combine η structures of this kind, we can write the equations

$$\begin{aligned} p_\eta &= p_1 A - V_1 \frac{Z_{0_1}}{S_1} B, \\ V_\eta &= V_1 C - \frac{p_1 S_1}{Z_{0_1}} D, \end{aligned} \tag{56}$$

where

$$A = \begin{vmatrix} \cosh \Gamma_1 & \frac{S_1}{S_2}\frac{Z_{0_2}}{Z_{0_1}} \sinh \Gamma_2 & -\frac{S_1}{S_3}\frac{Z_{0_3}}{Z_{0_1}} \sinh \Gamma_3 \cdots \\ -\sinh \Gamma_1 & \cosh \Gamma_2 & \frac{S_2}{S_3}\frac{Z_{0_3}}{Z_{0_2}} \sinh \Gamma_3 \cdots \\ \sinh \Gamma_1 & -\sinh \Gamma_2 & \cosh \Gamma_3 \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cosh \Gamma_\eta \end{vmatrix} \tag{57}$$

$$B = \begin{vmatrix} \sinh \Gamma_1 & \frac{S_1}{S_2}\frac{Z_{0_2}}{Z_{0_1}} \sinh \Gamma_2 & -\frac{S_1}{S_3}\frac{Z_{0_3}}{Z_{0_1}} \sinh \Gamma_3 \cdots \\ -\cosh \Gamma_1 & \cosh \Gamma_2 & \frac{S_2}{S_3}\frac{Z_{0_3}}{Z_{0_2}} \sinh \Gamma_3 \cdots \\ \cosh \Gamma_1 & -\sinh \Gamma_2 & \cosh \Gamma_3 \quad \cdot \quad \cdot \\ -\cosh \Gamma_1 & \sinh \Gamma_2 & \sinh \Gamma_3 \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cosh \Gamma_\eta \end{vmatrix} \tag{58}$$

$$C = \begin{vmatrix} \cosh \Gamma_1 & \frac{S_2}{S_1}\frac{Z_{0_1}}{Z_{0_2}} \sinh \Gamma_2 & -\frac{S_3}{S_1}\frac{Z_{0_1}}{Z_{0_3}} \sinh \Gamma_3 \cdots \\ -\sinh \Gamma_1 & \cosh \Gamma_2 & \frac{S_3}{S_2}\frac{Z_{0_2}}{Z_{0_3}} \sinh \Gamma_3 \cdots \\ +\sinh \Gamma_1 & -\sinh \Gamma_2 & \cosh \Gamma_3 \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cosh \Gamma_\eta \end{vmatrix} \tag{59}$$

$$D = \begin{vmatrix} \sinh \Gamma_1 & \frac{S_2}{S_1}\frac{Z_{0_1}}{Z_{0_2}} \sinh \Gamma_2 & -\frac{S_3}{S_1}\frac{Z_{0_1}}{Z_{0_3}} \sinh \Gamma_3 \cdots \\ -\cosh \Gamma_1 & \cosh \Gamma_2 & \frac{S_3}{S_2}\frac{Z_{0_2}}{Z_{0_3}} \sinh \Gamma_3 \cdots \\ \cosh \Gamma_1 & -\sinh \Gamma_2 & \cosh \Gamma_3 \quad \cdot \quad \cdot \\ -\cosh \Gamma_1 & \sinh \Gamma_2 & -\sinh \Gamma_3 \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cosh \Gamma_\eta \end{vmatrix} \tag{60}$$

Among these four determinants there is one relation

$$AC - BD = 1, \tag{61}$$

as can be seen by multiplying them together.

Hence to completely specify the characteristics of the structure three parameters are required. A number of possible sets of parameters exist whose usefulness depends on the type of structure to which they are applied. The set of parameters having the greatest use in connection with electrical networks are the image parameters which include two image impedances and an image transfer constant. We define these constants as follows for the acoustic case.

If we have a network terminated in impedances Z_{I_1} and Z_{I_2} (per square centimeter of area) at the beginning and at the end of the network, then these impedances are the image impedances of the structure if they terminate the structure in such a way that at either termination junction, the impedance looking in either direction is the same.

The image transfer constant θ may be defined as one half the natural logarithm of the vector ratio of the product of the pressure by the volume velocity, at the input junction point, and this product for the output junction point, when the network is terminated in its image impedances.

Hence

$$\theta = \frac{1}{2} \log_e \frac{p_1 V_1}{p_\eta V_\eta}.$$

To determine the image impedances, we have one set of equations

$$\left.\begin{aligned} p_\eta &= p_1 A - V_1 \frac{Z_{0_1}}{S_1} B, \\ V_\eta &= V_1 C - \frac{p_1 S_1}{Z_{0_1}} D. \end{aligned}\right\} \tag{62}$$

This gives the pressure and volume velocity propagated in one direction. We need also the equation of propagation in the opposite direction. This can evidently be written

$$\left.\begin{aligned} p_\eta' &= p_1' A' - V_1' \frac{Z_{0_\eta}}{S_\eta} B', \\ V_\eta' &= V_1' C' - \frac{p_1' S_\eta}{Z_{0_\eta}} D', \end{aligned}\right\} \tag{63}$$

where p_η' and V_η' represent the pressure and volume velocity at the beginning and V_1' and p_1' at the end of the structure. A' can be obtained from A by cyclically permuting the subscripts. By writing the expansions for these quantities we can show that

$$A' = C; \quad C' = A; \quad B' = \frac{Z_{0_1}}{Z_{0_\eta}} \frac{S_\eta}{S_1} B; \quad D' \doteq \frac{Z_{0_\eta}}{Z_{0_1}} \frac{S_1}{S_\eta} D. \tag{64}$$

Eliminating the ratio V_η/V_1 from (62) and writing $p_1/V_1 = Z_{I_1}/S_1$ and $p_\eta/V_\eta = Z_{I_2}/S_\eta$, we obtain

$$Z_{I_1} Z_{I_2} D + Z_{0_1} \left[Z_{I_1} \left(\frac{S_\eta}{S_1} A \right) - Z_{I_2} C \right] - Z_{0_1}^2 \left(\frac{S_\eta}{S_1} B \right) = 0. \tag{65}$$

From (63) eliminating the ratio V_η'/V_1' and writing $p_\eta'/V_\eta' = Z_{I_1}/S_1$ and $p_1'/V_1' = Z_{I_2}/S_\eta$ and substituting the values in (64), we have

$$Z_{I_1}Z_{I_2}D + Z_{0_1}\left(Z_{I_2}C - Z_{I_1}\frac{S_\eta}{S_1}A\right) - Z_{0_1}{}^2\frac{S_\eta}{S_1}B = 0. \tag{66}$$

Solving (65) and (66) simultaneously, we find

$$Z_{I_1} = Z_{0_1}\sqrt{\frac{BC}{AD}};\;\; Z_{I_2} = Z_{0_1}\frac{S_\eta}{S_1}\sqrt{\frac{AB}{CD}}. \tag{67}$$

From the definition of θ and equations (62), we can show that

$$\cosh\theta = \sqrt{AC}. \tag{68}$$

In terms of these parameters, the effect upon the pressure or volume velocity in the termination of an acoustic system, due to inserting the structure into the system, will be given by multiplying the terminal pressure or volume velocity by the factor

$$\frac{\dfrac{Z_A}{S_1}+\dfrac{Z_B}{S_\eta}}{\dfrac{2Z_B}{S_\eta}} \times \frac{2\sqrt{\dfrac{S_1}{S_n}}\sqrt{Z_{I_1}Z_{I_2}}}{Z_{I_1}+Z_A} \times \frac{2Z_B}{Z_{I_2}+Z_B} \times e^{-\theta} \times \frac{1}{1-\dfrac{Z_{I_2}-Z_B}{Z_{I_2}+Z_B}\times\dfrac{Z_{I_1}-Z_A}{Z_{I_1}+Z_A}\times e^{-2\theta}}, \tag{69}$$

where Z_A and Z_B are respectively the impedances, per square centimeter, of the acoustic system at the insertion junction looking towards and away from the source.

Appendix I. Proof of Thévenin's Theorem for an Acoustic System

The proof of Thévenin's theorem as stated in Section III can be obtained directly from the general network equations given in Section V. These equations are

$$p_2 = p_1A - V_1\frac{Z_0}{S_1}B,$$

$$V_2 = V_1C - \frac{p_1S_1}{Z_0}D,$$

where $AC - BD = 1$. If we connect at the input end a source of pressure p_0, whose specific internal impedance is Z_T, we can write

$$p_0 = p_1 + V_1 \frac{Z_T}{S_1} \cdot$$

Inserting this result in the above equation, we obtain

$$\left.\begin{aligned} p_2 &= p_0 A - \frac{V_1}{S_1}(Z_T A + Z_0 B), \\ V_2 &= V_1\left(C + \frac{Z_T}{Z_0} D\right) - \frac{p_0 S_1}{Z_0} D. \end{aligned}\right\} \tag{70}$$

To obtain the pressure when an infinite impedance is used at the termination, we let $V_2 = 0$, and solving for V_1 we have

$$V_1 = \frac{p_0 D S_1}{Z_0 C + Z_T D} \cdot \tag{71}$$

Substituting this in the first of equations (70), we have

$$p_2 = \frac{p_0 Z_0}{(Z_0 C + Z_T D)} = p_0', \tag{72}$$

which is the terminal pressure for an infinite terminating impedance.

Eliminating V_1 from (70) and substituting $V_2 Z_R / S_\eta = p_2$, we have

$$V_2 = \frac{p_0 Z_0}{(Z_0 C + Z_T D)} \times \frac{1}{\dfrac{Z_R}{S_\eta} + \dfrac{Z_0}{S_1}\left(\dfrac{Z_T A + Z_0 B}{Z_0 C + Z_T D}\right)} \cdot \tag{73}$$

We can show now that

$$\frac{Z_0}{S_1}\left(\frac{Z_T A + Z_0 B}{Z_0 C + Z_T D}\right) = \frac{Z_T'}{S_1},$$

which is the impedance at the terminating junction looking toward the source, when the specific impedance Z_T terminates the input end. From equations (63) and (64), we can write

$$p_2' = p_1' C - V_1' \frac{Z_{0_1}}{S_1} B,$$

$$V_2' = V_1' A - \frac{p_1' S_1}{Z_0} D.$$

Substituting $V_2'(Z_T/S_1) = p_2'$ and solving for the ratio p_1'/V_1', we have

$$\frac{p_1'}{V_1'} = \frac{Z_T'}{S_1} = \frac{Z_0}{S_1}\left(\frac{Z_T A + Z_0 B}{Z_0 C + Z_T D}\right). \tag{74}$$

Hence we can express V_2 in equation (73) as

$$V_2 = \frac{p_0'}{\dfrac{Z_R}{S_\eta} + \dfrac{Z_T}{S_1}}$$

which is Thévenin's theorem.

Appendix II. Determination of Loss for a Constant Volume Velocity Source

Another type of insertion effect desired in some cases is the effect caused by inserting filter structures in an acoustic system in which the source supplies a constant volume velocity. One such acoustic system is the phonograph.

In order to obtain this effect we first prove the theorem: If a source of constant volume velocity V_1 is connected to the input of an acoustic system, and if the impedance Z_R (per square centimeter) is used to terminate the system, the volume velocity V_2 will be $p_0''/[(Z_R/S_1) + (Z_c/S_\eta)]$ where p_0'' is the pressure at the termination of the system when the system is closed through an infinite impedance, and Z_c is the specific impedance of the acoustic system at the output junction looking toward the source when the system is terminated in in an infinite impedance at the input junction. S_1 and S_η are the areas at the input and output junctions, respectively.

To prove this we substitute the value of p_0 given by (71) in the first of equations (70) and obtain for the pressure, with an infinite impedance termination

$$p_2'' = \frac{V_1 Z_0}{S_1 D}. \tag{75}$$

Then eliminating p_0 from equations (70), and inserting the value $p_2 = V_2 Z_R/S_\eta$, we obtain

$$V_2 = \frac{V_1 Z_0}{S_1 D}\left[\frac{1}{\dfrac{Z_R}{S_\eta} + \dfrac{A Z_0}{S_1 D}}\right]. \tag{76}$$

From equation (74) we see that the impedance looking toward the source is $(Z_0 A/S_1 D)$ if we make Z_T approach infinity. Hence

$$V_2 = p_0''\left[\frac{1}{\dfrac{Z_R}{S_\eta} + \dfrac{Z_c}{S_1}}\right].$$

To obtain the insertion loss for a constant current source, then, it is only necessary to substitute Z_c for Z_a in equation (30). One special case of interest is the case where the acoustic filter is connected directly to the source. In this case $Z_c = \infty$ and the insertion effect is determined by the factor

$$\left(\frac{2Z_0}{Z_0 + Z_b}\right) \times e^{-\Gamma} \times \left(\frac{1}{1 + \dfrac{Z_0 - Z_b}{Z_0 + Z_b} e^{-2\Gamma}}\right). \tag{77}$$