Operation of Thermionic Vacuum Tube Circuits

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SYNOPSIS: Given the static characteristic of grid current-grid potential, and plate current-plate potential, for any three element vacuum tube, the general exact equations for the output current when the tube is connected in circuits of any impedance whatsoever, and excited by any variable voltage, are here derived. The method of derivation is illustrated in the special case where resistances only are considered, and the adaptation of complex impedance to use in non-linear equations is shown. Approximations that are allowable in various practical applications are indicated, and the equations are applied in some detail to grid-leak detectors, and in brief to other types of detectors, modulators, amplifiers and oscillators.

Certain repetitions of previous work are contained in these pages, as it is believed that the applications of the novel features introduced are illustrated thereby better than by a description dealing only with new material.

THE equations in use at the present time for the relation between input voltage and output current in thermionic vacuum tubes are those developed by a number of pioneers in Radio Communication. They have been summarized very concisely, and somewhat extended in an important paper by John R. Carson, entitled "A Theoretical Study of the Three Element Vacuum Tube," which appeared in the Proceedings of the Institute of Radio Engineers, April, 1919. For some time past the need of relations which include the effect of the variation of certain quantities, considered constant in Mr. Carson's paper, has been growing. Especially in the case of detection and modulation has this need become pronounced. Moreover, in the special case of grid leak detectors, the need for a general theoretical analysis has not, to the author's knowledge, been completely satisfied.

Purpose

It is, therefore, the purpose of the present paper to derive general exact equations for the output current from a three-element thermionic vacuum tube when it is connected in circuits of general impedance, both on the input and output sides, and to show specific methods of applying these general equations to several special cases, with emphasis on the case of the grid leak detector. It is also proposed to show that, whether used for detectors, modulators, amplifiers, or oscillators, the same fundamental theory applies. It is hoped that the theory and methods given will form a basis upon which a complete rational design of vacuum tube circuits may be built.

Theory

In the derivation of these equations, no limitations whatever should be imposed. Consider a three-element vacuum tube connected in circuits of general impedance on both input and output sides. The grid is allowed to take convection current. The amplification factor, μ, is considered variable, and the effect of plate potential on grid current is included. Under these conditions, the total plate current of the tube can merely be said to be a function of the grid and plate potentials; and the total grid current, likewise, is some other function of the grid and plate potentials. The fundamental relations:

$$I_p = I_p(E_g, E_p) \tag{1}$$

$$I_g = I_g(E_g, E_p) \tag{2}$$

express, the operation of the device. They represent the static characteristics of the tube. It is from these two relations alone that the general theory must be built.

In order to do this, the following notation will be employed:

$$\left.\begin{aligned} I_p &= I_{po} + i_p \\ I_g &= I_{go} + i_g \\ E_p &= E_{po} + e_p \\ E_g &= E_{go} + e_g \end{aligned}\right\} \tag{3}$$

It will be recognized that the lower case letters represent variations in the normal values of the currents and voltages denoted by the zero subscripts. It should be noted, moreover, that all voltages and currents refer to the effect directly on the element of the tube, plate or grid as the case may be.

With the aid of (3), equations (1) and (2) may be written

$$i_p = P_1 e_g + P_2 e_p + \tfrac{1}{2} P_3 e_g^2 + P_4 e_g e_p + \tfrac{1}{2} P_5 e_p^2 + \ldots \tag{4}$$

$$i_g = T_1 e_g + T_2 e_p + \tfrac{1}{2} T_3 e_g^2 + T_4 e_g e_p + \tfrac{1}{2} T_5 e_p^2 + \ldots \tag{5}$$

where the P's and T's have the following significance:

$$\left.\begin{aligned} &P_1 = \frac{\partial I_{po}}{\partial E_g} \quad P_2 = \frac{\partial I_{po}}{\partial E_p} \quad P_3 = \frac{\partial^2 I_{po}}{\partial E_g^2} \quad P_4 = \frac{\partial^2 I_{po}}{\partial E_g \partial E_p} \quad P_5 = \frac{\partial^2 I_{po}}{\partial E_p^2} \\ &T_1 = \frac{\partial I_{go}}{\partial E_g} \quad T_2 = \frac{\partial I_{go}}{\partial E_p} \quad T_3 = \frac{\partial^2 I_{go}}{\partial E_g^2} \quad T_4 = \frac{\partial^2 I_{go}}{\partial E_g \partial E_p} \quad T_5 = \frac{\partial^2 I_{go}}{\partial E_p^2} \end{aligned}\right\} \tag{6}$$

Equations (4) and (5) are obtained directly from the extension of Taylor's Theorem. The P's may be written in more useful form with the aid of (1) and the well-known definitions of the amplification factor, μ, the plate resistance, r_p, and the grid resistance, r_g. Thus, from (1)

$$\left.\begin{aligned} \mu &= \frac{\dfrac{\partial I_p}{\partial E_g}}{\dfrac{\partial I_p}{\partial E_p}} = -\left.\frac{dE_p}{dE_g}\right]_{I_p} \\ \frac{1}{r_p} &= \frac{\partial I_p}{\partial E_p} \\ \frac{1}{r_g} &= \frac{\partial I_g}{\partial E_g} \end{aligned}\right\} \text{(by definition)} \tag{7}$$

Hence

$$\left.\begin{aligned} P_1 &= \frac{\mu}{r_p} \\ P_2 &= \frac{1}{r_p} \\ P_3 &= \frac{1}{r_p}\frac{\partial \mu}{\partial E_g} + \frac{\mu}{r_p}\frac{\partial \mu}{\partial E_p} - \mu^2\frac{r_p'}{r_p^2} \\ P_4 &= \frac{1}{r_p}\frac{\partial \mu}{\partial E_p} - \mu\frac{r_p'}{r_p^2} \\ P_5 &= -\frac{r_p'}{r_p^2} \\ \text{where}\quad r_p' &= \frac{\partial r_p}{\partial E_p}\,. \end{aligned}\right\} \tag{8}$$

In similarly treating the T's, it was found convenient to introduce an entirely new symbol. This has been done with reluctance, for it is realized that considerable difficulty has been experienced in the standardization of symbols already in use. But inasmuch as the simplification of both physical interpretation and mathematical expression which results from the use of this new symbol is enormous, its addition is believed to be warranted.

This new symbol we will call the reflex factor, and will denote it by the symbol, ν. It is analogous in its effect on the grid circuit to the effect of μ on the plate circuit. Its definition is analogous to that of μ. Thus, from (2):

$$\nu = \frac{\dfrac{\partial I_g}{\partial E_g}}{\dfrac{\partial I_g}{\partial E_p}} = -\left.\frac{dE_p}{dE_g}\right]_{I_g} \tag{9}$$

Comparison of (7) and (9) shows that while μ is equal to minus the ratio of the increments of E_p and E_g necessary to maintain the plate current constant, ν is equal to minus the ratio of the increments of E_p and E_g necessary to maintain the grid current constant. On the other hand, while in the case of μ, the ratio

$$\left.\frac{dE_p}{dE_g}\right]_{I_p}$$

is intrinsically negative and occurs in (7) with a negative sign, making μ intrinsically a positive number; in the case of ν, the ratio

$$\left.\frac{dE_p}{dE_g}\right]_{I_g}$$

is usually intrinsically positive, and occurs in (9) with a negative sign; hence ν is usually intrinsically a negative number.

With the foregoing definition, the T's may be written as follows:

$$\begin{aligned}
T_1 &= \frac{1}{r_g} \\
T_2 &= \frac{1}{\nu r_g} \\
T_3 &= -\frac{r_g'}{r_g^2} \\
T_4 &= \frac{1}{r_g}\frac{\partial}{\partial E_g}\left(\frac{1}{\nu}\right) - \frac{1}{\nu}\frac{r_g'}{r_g^2} \\
T_5 &= \frac{1}{\nu r_g}\frac{1}{\partial E_g}\left(\frac{1}{\nu}\right) + \frac{1}{r_g}\frac{\partial}{\partial E_p}\left(\frac{1}{\nu}\right) - \frac{1}{\nu^2}\frac{r_g'}{r_g^2}
\end{aligned} \tag{10}$$

where $$r_g' = \frac{\partial r_g}{\partial E_g}.$$

The effective value of T_2, when taken over a cycle of sine wave form, has sometimes been called the reflex mutual conductance (L. A. Hazeltine), and has been denoted by g_n. An attempt to adapt this notation to the present purpose has not proved feasible. For reference, it may be noted in the limiting case, where the amplitude of the sine wave approaches zero:

$$g_n = \frac{1}{\nu r_g}.$$

With the relations given thus far, the problem may now be more specifically stated as follows:

It is desired to express i_p, the output current through a general impedance in the plate circuit, as an explicit function of e, a variable voltage applied in series with a general impedance in the grid circuit.

Special Case

The following special case will make the detailed derivation, where complex quantities are considered, more intelligible.

For this special case consider a vacuum tube connected in circuits containing only pure resistances. Let the resistance in the grid circuit be denoted by Q and that in the plate circuit be denoted by Z. Fig. 1 illustrates this circuit. Let i_p and i_g be determined to satisfy the following series:

$$i_p = a_1 e_g + a_2 e_g^2 + \dots \tag{11}$$

$$i_g = b_1 e + b_2 e^2 + \dots \tag{12}$$

(11) and (12) are valid since (4) and (5) are formally power series. As seen from Fig. 1, e represents a variable voltage impressed in series with the resistance, Q, on the grid of the tube.

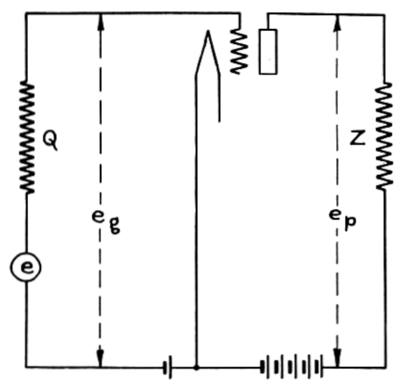

Fig. 1—Fundamental circuit diagram

These equations will give the plate and grid currents as explicit function of the voltages e_g and e, respectively, if we can evaluate the a's and b's. To do this, we have the relation

$$e_p = -i_p Z. \tag{13}$$

Substituting (11) in (4) and equating coefficients of like powers of e_g, we may evaluate a_1 and a_2 and thus express i_p as an explicit function of e_g:

$$i_p = \frac{\mu}{(r_p+Z)} e_g + \frac{1}{2}\left[\frac{-\mu^2 r_p r_p' + \mu \frac{\partial \mu}{\partial E_p}(r_p^2 - Z^2) + \frac{\partial \mu}{\partial E_g}(r_p+Z)^2}{(r_p+Z)^3}\right] e_g^2. \tag{14}$$

In equation (14), when the amplification factor, μ, is considered constant, we have the well-known relation as given in Mr. Carson's paper

$$i_p = \frac{\mu e_g}{(r_p+Z)} - \frac{1}{2}\frac{\mu^2 r_p r_p'}{(r_p+Z)^3} e_g^2 + \cdots \tag{15}$$

Experiments have shown, however, that when the resistance, Z, is not small compared to r_p the modulation resulting from variations of μ amounts to an appreciable part of the total. When the grid is maintained at a negative potential with respect to the filament, (14) may be simplified somewhat by the relation which then holds quite closely[1], namely:

$$\mu \frac{\partial \mu}{\partial E_p} = \frac{\partial \mu}{\partial E_g}.$$

Equation (14) then becomes

$$i_p = \frac{\mu}{r_p+Z} e_g - \frac{1}{2}\left[\frac{\mu^2 r_p r_p'}{(r_p+Z)^3} - \frac{2 r_p \frac{\partial \mu}{\partial E_g}}{(r_p+Z)^2}\right] e_g^2 + \cdots \tag{16}$$

This equation is applicable to the calculation of the output current when e_g is known, and the grid takes no convection current, as is the case in very many circuits met with in practice.

It is instructive to investigate the relative magnitudes of the two components of the second term of (16) in an actual experimental case. For convenience, the contribution of the second component of this term will be called, "μ modulation." A vacuum tube was measured and found to have the following properties under operating conditions

$$r_p = 6400$$
$$r'_p = -61.3$$
$$\mu = 5.84$$
$$\frac{\partial \mu}{\partial E_g} = .05$$

The results of applying these to (16) are shown in the following table:

Z	Total modulation	μ modulation, %
0	$.03341/10^3\ e^2$	23.35
r_p	$.00515/10^3\ e^2$	37.8
$2\ r_p$	$.00186/10^3\ e^2$	46.6
$4\ r_p$	$.000889/10^3\ e^2$	55.0

[1] See appendix I for proof of this.

This illustrates strikingly the importance of the variation of μ in modulators and detectors.

Equation (14) is expressed in terms of e_g, the voltage directly on the grid of the tube. We may derive the expression for i_p in terms of e, a voltage impressed in series with a resistance, Q, in the external grid circuit by noting that

$$e_g = e - i_g Q.$$

Hence, from (12),

$$e_g = (1 - b_1 Q)e - b_2 Q e^2 + \dots \tag{17}$$

Therefore

$$i_p = a_1(1 - b_1 Q)e - [a_1 b_2 Q - a_2(1 - b_1 Q)^2]e^2 + \dots \tag{18}$$

and, as in (13),

$$e_p = -i_p Z.$$

Substituting (17) and (18) into (5) and equating coefficients of like powers of e, we get

$$b_1 = \frac{T_1 - T_2 a_1 Z}{1 + T_1 Q - T_2 a_1 Z Q}, \tag{19}$$

$$b_2 = \frac{[-a_2 Z T_2 + \frac{1}{2} T_3 - a_1 Z T_4 + \frac{1}{2} a_1^2 Z^2 T_5](1 - b_1 Q)^2}{1 + T_1 Q - T_2 a_1 Z Q}. \tag{20}$$

The T's may be expressed in terms of r_g and ν with the aid of (10). The complete solution of this special case for first and second order effects is then given by (18) above, in which we have now evaluated the a's and b's.

Mathematical Digression

Before the detailed steps in the complete development of the general case, with general impedances instead of resistances, are attempted, the following digression on the use of complex quantities in non-linear equations is apposite. Included at this point, it serves a two-fold purpose; first, the notation to be used is illustrated by means of simple applications; second, it calls to mind the fundamental ideas involved in the representation of impedances by complex quantities.

Consider a current, I. If periodic, this current may be represented by a Fourier series and expressed as the sum of a number of cosine terms. Thus

$$I = I_h\left(\frac{\epsilon^{j(ht+\phi)} + \epsilon^{-j(ht+\phi)}}{2}\right) + I_k\left(\frac{\epsilon^{j(kt+\theta)} + \epsilon^{-j(kt+\theta)}}{2}\right) + \dots \tag{21}$$

where the symbol, j, represents the imaginary, $\sqrt{-1}$. For brevity this may be written

$$I = (i_{1h} + \bar{i}_{1h}) + (i_{1k} + \bar{i}_{1k}) + \dots \tag{22}$$

where the bar over a symbol denotes the conjugate imaginary of the same symbol unbarred. If this current flows through a circuit containing resistance, self-inductance, and capacity in series, we have

$$e=RI+L\frac{dI}{dt}+\frac{1}{C}\int Idt. \tag{23}$$

Substituting for I its equivalent, as given by (21) or (22), we may write the result in abbreviated form as follows:

$$e=(z_h i_{1h}+\bar{z}_h\bar{i}_{1h})+(z_k i_{1k}+\bar{z}_k\bar{i}_{1k})+ \cdots \tag{24}$$

where

$$z_n=R+Ljn+\frac{1}{Cjn},$$

$$\bar{z}_n=R-Ljn-\frac{1}{Cjn}.$$

When the current flows through a network of impedances, we may always write the equivalent series impedance of the network. Hence equation (24) may be extended to cover the general case. It will be noted that lower case z's have been used to represent impedances in the above discussion. Throughout this paper the attempt has been made to employ the lower case letters to denote quantities which involve time, reserving the capitals for those which do not involve time. With this understanding, Z denotes a resistance, while z represents a general impedance, which, of course, varies with the time variation of a voltage impressed on it. With the aid of (24) we are in a position to treat non-linear equations by the complex method. Thus, omitting conjugates e^2 becomes,

$$\begin{aligned} e^2=z_h^2 i_{2(2h)}+z_k^2 i_{2(2k)}+2z_h\bar{z}_h i_{2(0h)}+2z_h z_k i_{2(h+k)}\\ +2z_h\bar{z}_k i_{2(h-k)}+2z_k\bar{z}_k i_{2(0k)}+ \cdots \end{aligned} \tag{25}$$

which may be written

$$e^2=e_{2(2h)}+e_{2(2k)}+e_{2(0h)}+e_{2(h+k)}+e_{2(h-k)}+e_{2(0k)}+ \cdots \tag{26}$$

In (25) and (26) the significance of the double subscript notation is brought out. The first symbol in the subscript refers to the order of the term, and the second refers to the frequency.

In the light of the foregoing discussion, the problem of writing the general equations for the thermionic vacuum tube may be attacked.

General Analysis

Coming back to the detailed problem in hand, we follow out the method illustrated in the special case, but must use the notation developed in the preceding section to take care of a *general* impedance, z, in the plate circuit, and a *general* impedance, q, on the grid circuit. Fig. 1 as before, shows the skeleton circuit, where, however, lower case z and q must be substituted for the capitals. Then

$$e = e_{1h} + \bar{e}_{1h} + e_{1k} + \bar{e}_{1k} + \cdots + e_{1n} + \bar{e}_{1n}. \tag{27}$$

Analogous to (11) and (12):

$$\left. \begin{aligned} i_p &= a_{1h}e_{g1h} + \bar{a}_{1h}\bar{e}_{g1h} + a_{1k}e_{g1k} + \bar{a}_{1k}\bar{e}_{g1k} + \cdots \\ &\quad + a_{2(h-k)}e_{g2(h-k)} + a_{2(h-k)}e_{g2(h-k)} + \cdots \end{aligned} \right\} \tag{28}$$

$$\begin{aligned} i_g &= b_{1h}e_{1h} + \bar{b}_{1h}\bar{e}_{1h} + b_{1k}e_{1k} + \bar{b}_{1k}\bar{e}_{1k} + \cdots \\ &\quad + b_{2(h-k)}e_{2(h-k)} + \bar{b}_{2(h-k)}\bar{e}_{2(h-k)} + \cdots \end{aligned}$$

Hence, analogous to (13) and (17):

$$e_p = -\Sigma[a_{1n}z_{1n}e_{g1n} + \bar{a}_{1n}\bar{z}_{1n}\bar{e}_{g1n} + a_{2m}z_m e_{g2m} + \bar{a}_{2m}\bar{z}_m\bar{e}_{g2m}] \tag{29}$$

$$e_g = \Sigma[(1 - b_{1n}q_n)e_{1n} + (1 - \bar{b}_{1n}\bar{q}_n)\bar{e}_{1n} - b_{2m}q_m e_{2m} - \bar{b}_{2m}\bar{q}_m\bar{e}_{2m}] \tag{30}$$

where the summation refers to terms of different frequencies but of similar form.

From this point on, the procedure is exactly the same as that given in the special case. Coefficients of terms of like order *and frequency* are equated, and the final results are:

$$\left. \begin{aligned} i_p &= \Sigma a_{1h}(1 - b_{1h}q_h)e_{1h} \\ &\quad + \Sigma[(1 - b_{1h}q_h)^2 a_{2(2h)} - a_{1(2h)}q_{(2h)}b_{2(2h)}]e_{2(2h)} \\ &\quad + \Sigma[(1 - b_{1h}q_h)(1 - b_{1k}q_k)a_{2(h+k)} - a_{1(h+k)}q_{(h+k)}b_{2(h+k)}]e_{2(h+k)} \\ &\quad + \Sigma[(1 - b_{1h}q_h)(1 - \bar{b}_{1k}\bar{q}_k)a_{2(h-k)} - a_{1(h-k)}q_{(h-k)}b_{2(h-k)}]e_{2(h-k)} \\ &\quad + \Sigma[(1 - b_{1h}q_h)(1 - \bar{b}_{1h}\bar{q}_h)a_{2(0h)} - a_{1(0h)}q_{(0h)}b_{2(0h)}]e_{2(0h)} \\ &\quad + \cdots \end{aligned} \right\} \tag{31}$$

where the summation refers to terms of different frequencies but of similar form. Note that, having $a_{2(h-k)}$ and $b_{2(h-k)}$, we may readily write the appropriate expressions for the other a_2's and b_2's by reference to the formation in equation (31).

In (31) the a's and b's are given by:

$$\left.\begin{aligned}
a_{1h} &= \frac{\mu}{r_p+z_h} \\
a_{2(h-k)} &= \frac{\frac{1}{2}\left[-\mu^2 r_p r_p' + \mu\frac{\partial\mu}{\partial E_p}(r_p^2 - z_h\bar{z}_k) + \frac{\partial\mu}{\partial E_g}(r_p+z_h)(r_p+\bar{z}_k)\right]}{(r_p+z_h)(r_p+\bar{z}_k)(r_p+z_{h-k})} \\
b_{1h} &= \frac{1-\frac{\mu}{\nu}\frac{z_h}{r_p+z_h}}{r_g+q_h\left(1-\frac{\mu}{\nu}\frac{z_h}{r_p+z_h}\right)} \\
b_{2(h-k)} &= \frac{\left\{\begin{aligned}&\frac{1}{2}\left[-r_g r_g'\left(1-\frac{\mu}{\nu}\frac{z_h}{r_p+z_h}\right)\left(1-\frac{\mu}{\nu}\frac{\bar{z}_h}{r_p+\bar{z}_k}\right) - 2a_{2(h-k)}\frac{r_g^2}{\nu}z_{2(h-k)}\right. \\ &-\frac{\partial}{\partial E_g}\left(\frac{1}{\nu}\right)\left(\frac{\mu z_h r_g^2}{r_p+z_h}+\frac{\mu\bar{z}_k r_g^2}{r_p+\bar{z}_k}-\frac{\mu^2 r_g^2}{\nu}\frac{z_h\bar{z}_k}{(r_p+z_h)(r_p+\bar{z}_k)}\right) \\ &\left.+\frac{\partial}{\partial E_p}\left(\frac{1}{\nu}\right)\left(\frac{r_g^2\mu^2 z_h\bar{z}_k}{(r_p+z_h)(r_p+\bar{z}_k)}\right)\right]\end{aligned}\right\}}{\left[r_g+q_h\left(1-\frac{\mu}{\nu}\frac{z_h}{r_p+z_h}\right)\right]\left[r_g+\bar{q}_k\left(1-\frac{\mu}{\nu}\frac{\bar{z}_k}{r_p+\bar{z}_k}\right)\right]\left[r_g+q_{(h-k)}\left(1-\frac{\mu}{\nu}\frac{z_{(h-k)}}{r_p+z_{(h-k)}}\right)\right]}
\end{aligned}\right\} \quad (32)$$

Discussion of General Equations

Equations (31) and (32) contain the general solution of the problem. The formulas are too long to consider all effects at one time but if we separate (31) into components and consider each component separately, useful applications may be secured.

First taking the component that gives rise to amplification effects, we get

$$\begin{aligned}
i_{p(h)} &= a_{1h}(1-b_{1h}q_h)e_{1h} \\
&= \left(\frac{\mu}{r_p+z_h}\right)\left(\frac{r_g}{r_g+q_h\left(1-\frac{\mu}{\nu}\frac{z_h}{r_p+z_h}\right)}\right)e_{1h}.
\end{aligned} \quad (33)$$

The point to be noted in this relation is that when $q_h << r_g$ we have the well-known relation

$$i_{p(h)} = \frac{\mu}{r_p+z_h}e_{1h}. \quad (34)$$

Since amplifiers are usually operated under the condition that r_g is exceedingly large, the general solution has contributed nothing new to the amplifier equations for conditions where the grid is maintained at a negative potential with respect to the filament. But for positive values of grid potential both q_h and the reflex factor, ν, enter into the calculations. It may be remarked in passing that when the grid and plate are both positive by the same amount, the absolute value of ν is approximately equal to, or somewhat less than, μ. On the other hand, when, as is usually the case, the plate potential is much greater than the positive grid potential, the magnitude of ν is much greater than μ.

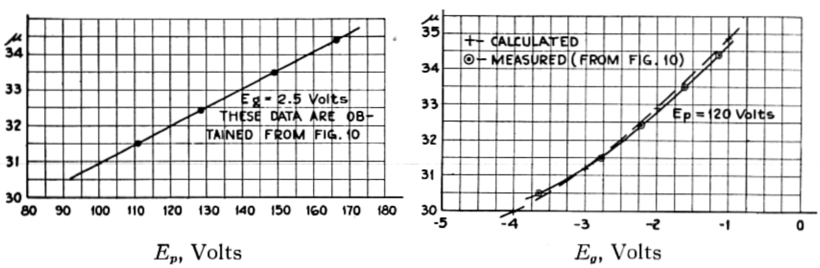

Fig. 2—Change of μ with plate and grid potentials. The points on the calculated curve were obtained as follows: since

$$\frac{\partial \mu}{\partial E_g} = \mu \frac{\partial \mu}{\partial E_p}$$

then

$$\mu = \frac{\mu_o + KE_p}{1 - KE_g}$$

where

$$K = \frac{\partial \mu}{\partial E_p} \div \left(1 + E_g \frac{\partial \mu}{\partial E_p}\right), \ \mu_o = \frac{\mu - E_p \frac{\partial \mu}{\partial E_p}}{1 + E_g \frac{\partial \mu}{\partial E_p}}.$$

From the upper curve, for $E_p = 120$, $E_g = -2.5$

$$\mu = 32, \ \frac{\partial \mu}{\partial E_p} = .05328,$$

whence

$$K = .06146, \mu_o = 29.55$$

We next consider the component of (31) that results in plate curvature detection or modulation. It is given by

$$i_{p2}=(1-b_{1h}q_h)\,(1-\bar{b}_{1k}\bar{q}_k)a_{2(h-k)}\,e_{2(h-k)}$$

$$=\frac{\dfrac{r_g{}^2}{2}\left[-\mu^2 r_p r_p{}'+\mu\dfrac{\partial\mu}{\partial E_p}(r_p{}^2-z_h\bar{z}_k)+\dfrac{\partial\mu}{\partial E_g}(r_p+z_h)(r_p+\bar{z}_k)\right]e_{2(h-k)}}{\left[r_g+q_h\left(1-\dfrac{\mu}{\nu}\dfrac{z_h}{r_p+z_h}\right)\right]\left[r_g+\bar{q}_k\left(1-\dfrac{\mu}{\nu}\dfrac{\bar{z}_k}{r_p+\bar{z}_k}\right)\right](r_p+z_h)(r_p+\bar{z}_k)(r_p+z_{(h-k)})} \qquad (35)$$

For rough calculations, μ may be regarded as a constant. For very careful work, this assumption should never be made without first drawing the curves of $\mu-E_g$ and $\mu-E_p$ and verifying the validity of the assumption under operating conditions. Examples of such curves are given in Fig. 2. When μ may be regarded as constant, and when the grid is maintained negative with respect to the filament, (35) becomes

$$i_{p(h-k)}=\frac{-\frac{1}{2}\mu^2 r_p r_p{}' e_{2(h-k)}}{(r_p+z_h)(r_p+\bar{z}_k)(r_p+z_{h-k})}$$

which may be put into the form given in Mr. Carson's paper, referred to before.

The third and last component of (31) is that which produces grid detection or modulation; namely

$$i_{p(h-k)}=-a_{1(h-k)}q_{(h-k)}b_{2(h-k)}e_{2(h-k)}$$

$$=\left(\frac{\mu q_{(h-k)}}{r_p+z_{(h-k)}}\right)\frac{1}{2}\Bigg[-r_g r_g{}'\left(1-\frac{\mu}{\nu}\frac{z_h}{r_p+z_h}\right)\left(1-\frac{\mu}{\nu}\frac{\bar{z}_k}{r_p+\bar{z}_k}\right)$$

$$-\frac{2a_{2(h-k)}r_g{}^2 z_{(h-k)}}{\nu}+\frac{\partial}{\partial E_p}\left(\frac{1}{\nu}\right)\left(\frac{r_g{}^2\mu^2 z_h\bar{z}_k}{(r_p+z_h)(r_p+\bar{z}_k)}\right)-\frac{\partial}{\partial E_g}\left(\frac{1}{\nu}\right)$$

$$\left(\frac{\mu z_h r_g{}^2}{r_p+z_h}+\frac{\mu\bar{z}_k r_g{}^2}{r_p+\bar{z}_k}-\frac{\mu^2 r_g{}^2 z_h\bar{z}_k}{\nu(r_p+z_h)(r_p+\bar{z}_k)}\right)\Bigg]$$

$$\div\left[r_g+q_h\left(1-\frac{\mu}{\nu}\frac{z_h}{r_p+z_h}\right)\right]\left[r_g+\bar{q}_k\left(1-\frac{\mu}{\nu}\frac{\bar{z}_k}{r_p+\bar{z}_k}\right)\right]$$

$$\left[r_g+q_{(h-k)}\left(1-\frac{\mu}{\nu}\frac{z_{(h-k)}}{r_p+z_{h-k}}\right)\right] \qquad (36)$$

In using this relation, ν may nearly always be considered constant. As grid leak detectors are often used, q consists of a resistance, R_g, and a condenser, C, in parallel. The values of R_g and C are so adjusted that the impedance of the combination to the first order frequencies is practically that of the condenser alone and may be neglected, and to the desired second order, or detected, frequency it is practically that of the resistance alone. When this is the case, and when the impedance in the plate circuit is a pure resistance, R_p, we may write (36) as follows:

$$i_{p3}=\frac{1}{2}\frac{\mu R_g\left(r_g r_g'+\frac{2a_{2m}r_g^2 R_p}{\nu}\right)}{(r_p+R_p)r_g^2(r_g+R_g)}e_{2m}$$

and, considering μ constant, in order to obtain a physical view of the result, we get

$$i_{p3}=\frac{\frac{1}{2}\mu R_g\left[r_g r_g'-\frac{\mu^2 r_p r_p'}{(r_p+R_p)^3}\left(\frac{r_g^2 R_p}{\nu}\right)\right]e_{2m}}{(r_p+R_p)r_g^2(r_g+R_g)} \tag{37}$$

This equation shows a condition that is present in many grid-leak detectors and which, it is thought, has not been generally appreciated. The condition referred to is the presence of the term involving the curvature of the plate characteristic in the grid detection component. This effect is *in addition* to the plate detection effect, given by (35). The plate detection component and the grid detection component are opposite in phase. Hence, it would seem that for best operation as a grid-leak detector, the curvature of the plate characteristic should be zero. This means a rather large value of E_b, the plate battery potential. In practice, however, it is usual to operate with fairly low values of E_b. The second term of the numerator of (37) accounts for this. It will be seen that detection resulting from this term and from the first term are in phase, since ν is intrinsically negative. Hence, it is entirely possible in certain cases for the optimum operating point to be such that the effect of the plate curvature is appreciable.

We now combine once more the three components, (33), (35) and (36), under the simplifying assumptions that μ and ν are constant and that ν is large enough so that terms containing ν in the denominator may be neglected. The result is

$$\left.\begin{aligned} i_p=&\frac{r_g}{(r_g+q_h)}\frac{e_{1h}}{(r_p+z_h)}+\frac{r_q}{(r_q+q_k)}\frac{e_{1k}}{(r_p+z_k)}+\ \dots\\ &+\left\{\frac{r_g^2}{(r_g+q_k)(r_g+\bar{q}_k)}\left[\frac{-\frac{1}{2}\mu^2 r_p r_p'}{(r_p+z_h)(r_p+\bar{z}_k)(r_p+z_{h-k})}\right]\right.\\ &\left.+\frac{\mu q_{(h-k)}}{(r_p+z_{h-k})}\left[\frac{\frac{1}{2}r_g r_g'}{(r_g+q_h)(r_g+\bar{q}_k)(r_g+q_{h-k})}\right]\right\}e_{2(h-k)}+\ \dots \end{aligned}\right\} \tag{38}$$

The first two terms of (38) are the amplification terms and represent undistorted reproduction in the plate circuit of the voltage, e, applied in the grid circuit. The third term of (38) represents the second order effects resulting from the curvature of the characteristics of the vacuum tube. The first part of this term represents the effects of so-called plate curvature detection or modulation, and the second part represents the effects of detection and modulation in the grid circuit. It is with this last-named component that the present paper is most concerned.

The Grid-Leak Detector

Fig. 3 shows the usual circuit diagram for a grid-leak detector. It is evident that the impedance, q, in this example is composed of the parallel combination of R_g and C. Suppose that the "h" and "k" frequencies are both radio frequencies, and that, for them, the

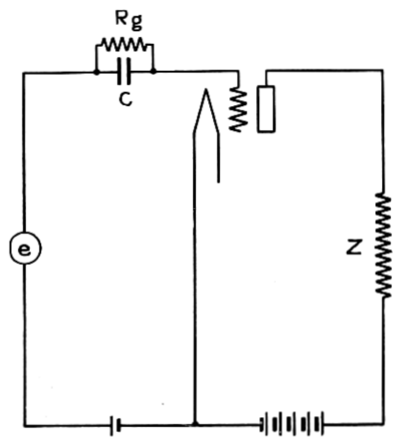

Fig. 3—Grid-leak detector

impedance offered by the resistance and condenser combination is practically that of the condenser, alone. Suppose, further, that practically the only impedance offered by the external circuit to the "$(h\text{-}k)$" frequency is that of the resistance, R_g, alone. This, of course, assumes that the "$h\text{-}k$" frequency is quite low. Then, when E_b, the voltage of the plate battery, is such that r_p' is very small, and when

$$e = A \cos ht + B \cos kt$$

we have, from (38), for second order effects

$$
\left.
\begin{aligned}
i_p = \tfrac{1}{2} r_g r_g' \left(\frac{\mu}{r_p+R_p}\right) \Bigg[& \frac{R_g}{\left(r_g+\frac{1}{jhC}\right)\left(r_g-\frac{1}{jhC}\right)(r_g+R_g)} \frac{A^2}{2} \\
& + \frac{\frac{1}{2jhC}}{\left(r_g+\frac{1}{jhC}\right)^2\left(r_g+\frac{1}{2jhC}\right)} \frac{A^2}{2} \cos 2ht \\
& + \frac{R_g}{\left(r_g+\frac{1}{jkC}\right)\left(r_g-\frac{1}{jkC}\right)(r_g+R_g)} \frac{B^2}{2} \\
& + \frac{\frac{1}{2jkC}}{\left(r_g+\frac{1}{jkC}\right)^2\left(r_g+\frac{1}{2jkC}\right)} \frac{B^2}{2} \cos 2kt \\
& + \frac{\frac{1}{j(h+k)C}}{\left(r_g+\frac{1}{jhC}\right)\left(r_g+\frac{1}{jkC}\right)\left(r_g+\frac{1}{j(h+k)C}\right)} AB \cos (h+k)t \\
& + \frac{R_g}{\left(r_g+\frac{1}{jhC}\right)\left(r_g-\frac{1}{jkC}\right)(r_g+R_g)} AB \cos (h-k)t \Bigg]
\end{aligned}
\right\} \quad (39)
$$

While most of the frequencies in this expression are unimportant in relation to any practical case on hand, they are included here to show the complete result for a given simple case. The last term of the above expression results in what is known as detection.

Let us consider this component in more detail as regards detection of an incoming modulated radio wave of the form

$$e = A\ (1+B \cos qt)\cos pt. \quad (40)$$

They may be written

$$e = A \cos pt + \frac{AB}{2} \cos (p+q)t + \frac{AB}{2} \cos (p-q)t. \quad (41)$$

If we identify the "p" frequency with "h," and let "k" have the values $(p+q)$ and $(p-q)$ in turn, the detection term of (39) gives

$$
\begin{aligned}
i_d = \tfrac{1}{2} r_g r_g' \left(\frac{\mu}{r_p+R_p}\right) \Bigg[& \frac{R_g}{\left(r_g-\frac{1}{jpC}\right)\left(r_g+\frac{1}{j(p+q)C}\right)(r_g+R_g)} \\
& + \frac{R_g}{\left(r_g+\frac{1}{jpC}\right)\left(r_g-\frac{1}{j(p-q)C}\right)(r_g+R_g)} \Bigg] \frac{A^2B}{2} \cos qt.
\end{aligned} \quad (42)
$$

Reference to the mathematical digression will make clear the formation of the impedances in this expression. (42) is an important relation since it shows that there is a possibility that the amplitude of the detected current may be affected by the phase displacements of the side bands of the original wave which occur during the detection. For an ideal grid-leak detector, the magnitudes of the quantities $\frac{1}{pC}$, $\frac{1}{(p+q)C}$ and $\frac{1}{(p-q)C}$ are very small compared with r_g. Equation (42) then becomes

$$i_d = \frac{r_g'}{r_g}\frac{\mu}{(r_p+R_p)}\frac{R_g}{(r_g+R_g)}\frac{A^2B}{2}\cos qt. \tag{43}$$

In (43) we have the simplest possible form of the equation for a grid leak detector. The next step is to show methods for evaluating the quantities r_g and r_g'. As may be seen from the relations given in (7) and (8)

$$\frac{1}{r_g} = \frac{\partial I_{go}}{\partial E_g}, \qquad r_g' = \frac{\partial r_g}{\partial E_g},$$

and, since the action of the grid-leak detector depends upon r_g', it is evident that r_g is not a constant but varies with the value of E_g. We may obtain r_g by direct dynamical measurements or by drawing tangents to the static grid-potential grid-current curve of the tube under consideration. The value of r_g thus obtained applies only to a given value of E_g. Now E_g is a function of the voltage, e, as will be shown:

When e has the form given in (41), one of the resulting currents in the plate circuit is a direct current given by

$$i_{pd} = \frac{1}{2}\frac{r_g'}{r_g}\frac{\mu R_g}{(r_g+R_g)(r_p+R_p)}\left(\frac{A^2}{2}+\frac{A^2B^2}{2}\right).$$

This means that a constant voltage given by

$$e_{gd} = \frac{1}{2}\frac{r_g'}{r_g}\frac{R_g}{(r_g+R_g)}\left[\frac{A^2}{2}+\frac{A^2B^2}{2}\right] \tag{44}$$

must have appeared on the grid in order to produce the constant component of the plate current. This constant voltage is in addition to that which we have denoted by E_{go}, since it is part of e_g. Moreover, its intrinsic value is usually negative, since r_g' is usually negative. This means that the "effective" E_{go} has been reduced by the amount given in (44). However, r_g is slightly different at this new value of E_{go} and hence e_{gd} is not quite what a first calculation would lead one to believe. The method of arriving at the correct value for e_{gd}, and hence for r_g and r_g' is one of trial and error, for, after several recalculations of e_{gd} have been made, it will be found that check results are secured. Then r_g and r_g' may be determined from this resulting value of E_{go}.

In actually making these measurements, a dynamical method of measuring r_g will usually be found superior to the method of drawing tangents to the static characteristic, for the grid-potential grid-current characteristic of any tube is rather elusive because of the

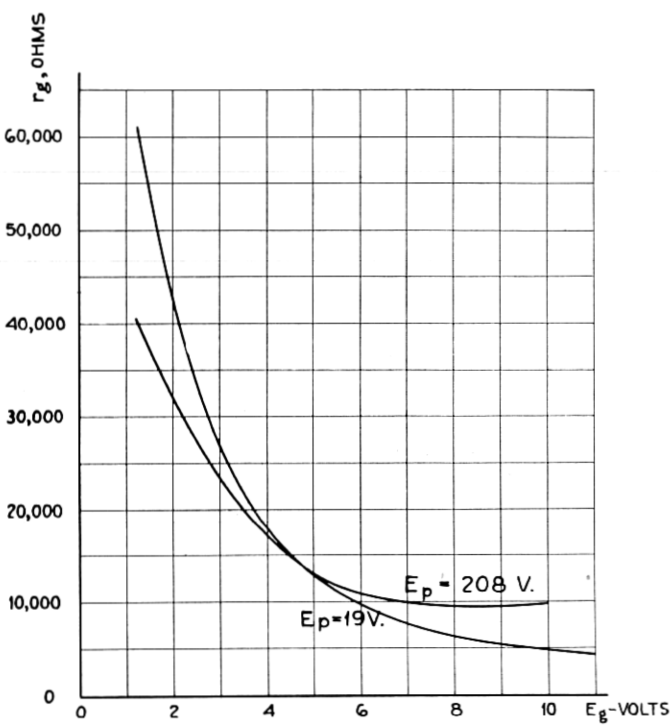

Fig. 4—Grid resistance

very small values of current involved. In the dynamic method a Wheatstone bridge circuit excited by a high frequency buzzer will be found convenient. The value of r_g' is, of course, obtained by drawing tangents to the r_g curve. Several examples of $r_g - E_g$ curves are shown in Fig. 4.

It must be recognized that, for large values of buzzer excitation, the dynamic value of r_g differs somewhat from that found by drawing tangents to the static characteristic. The dynamic value more nearly approaches the value r_g would assume with large signal inputs than does the static value. Hence, if a large signal input, e, is to be used, the amplitude of the buzzer excitation voltage on the grid should equal this amplitude as nearly as possible.

When the method of drawing tangents to the static characteristic is employed, a very close approximation to the value of r_g to use for large signal amplitudes may be obtained by drawing, not true tangents but secant lines to the static characteristic, which join points on the characteristic corresponding to the extreme, or peak, values of e_g.

When either method is used to obtain r_g, the value of r_g' must be obtained by drawing tangents to an E_g-r_g curve.

With the precautions just given, and when the assumptions made in equation (43) are justifiable, an accuracy within 10% is easily obtained. While this is not very exact, nevertheless, it is a real advance over calculations made without taking the precautions just discussed for measuring r_g.

In many vacuum tubes the value of r_g is so high that the input impedance of the tube, resulting from the interelectrode capacities of the elements cannot justifiably be neglected. In order to include this effect, the following relations are applicable.

Consider the circuits shown in Fig. 5. This gives the equivalent circuit diagram for a vacuum tube with general impedances, z_1 and

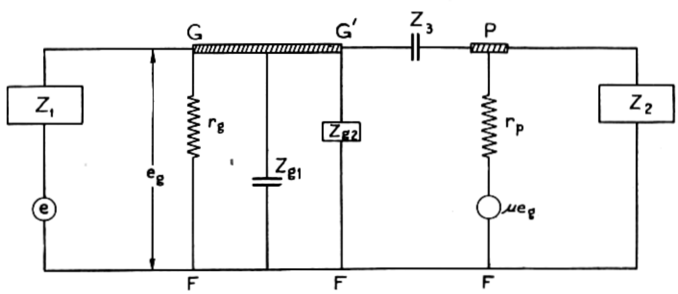

Fig. 5—Equivalent network

z_2, attached to the grid and plate, respectively. The plate to filament capacity may conveniently be included in z_2. The impedance, z_{g_2}, is the effective impedance of the network looking to the right from the point $G'F$. z_{g_1} is the grid to filament capacity of the tube, and z_3 is the grid to plate capacity.

We may write

$$z_g = \frac{z_{g_1} z_{g_2}}{z_{g_1} + z_{g_2}}.$$

In order to apply the general equations we must evaluate z_n and q_n.

To Evaluate z_n.

From the general equations, we have

$$i_p = \frac{\mu e_g}{r_p + z_n}.$$

Hence we may write Kirchoff's law for the plate circuit. This gives

$$i_p = \frac{e_g[(\mu+1)z_2 + \mu z_3]}{r_p z_2 + z_3(r_p + z_2)}.$$

Upon equating the two expressions for i_p, there results

$$z_n = \frac{z_2(\mu z_3 - r_p)}{(\mu+1)z_2 + \mu z_3}.$$

To Evaluate q_n.

By the general equations, we have

$$i_g = \frac{e}{r_g + q_n x}$$

where x stands for

$$\left(1 - \frac{\mu}{\nu}\frac{z_n}{r_p + z_n}\right).$$

This may be written

$$i_g = \frac{\dfrac{e}{x}}{\dfrac{r_g}{x} + q_n}$$

which says that Kirchoff's law may be applied to the grid circuit provided we use a modified voltage, $\frac{e}{x}$, and a modified grid resistance, $\frac{r_g}{x}$. Hence

$$i_g = \frac{\dfrac{e}{x} z_g}{(z_1 + z_g)\left(z_g + \dfrac{r_g}{x}\right) - z_g^2}.$$

Upon equating the two expressions for i_g there results

$$q_n = \frac{z_1}{z_g}\left(\frac{r_g}{1 - \dfrac{\mu}{\nu}\dfrac{z_n}{r_p + z_n}} + z_g\right).$$

To sum up; the following relations are applicable when interelectrode capacities or other coupling impedances are to be included:

$$z_g = \frac{z_{g1} z_{g2}}{z_{g1} + z_{g2}} \tag{45}$$

$$z_{g1} = \frac{1}{j\omega C_{gf}} \tag{46}$$

$$z_{g2} = \frac{z_2 z_3 + r_p(z_2 + z_3)}{z_2(\mu+1) + r_p} \tag{47}$$

$$z_n = \frac{z_2(\mu z_3 - r_p)}{(\mu+1) z_2 + \mu z_3} \tag{48}$$

$$q_n = \frac{z_1}{z_g}\left(\frac{r_g}{1 - \frac{\mu}{\nu} \frac{z_n}{r_p + z_n}} + z_g\right). \tag{49}$$

With the aid of (45), (46), (47), (48), (49), equation (42) may be modified to include all cases where the plate current resulting from detection or modulation in the grid circuit is desired, provided an accuracy greater than about 10% is not required. Where greater accuracy is essential, curves must be made to give the effect of the small terms in the numerator of the expression for b_{2m} in equation (36).

Before leaving the subject of grid-leak detectors, we will discuss briefly one of the physical aspects of grid-leak detection that the example just given, and the equations on which it is based, have emphasized. This is the fact that the fiction of the time-constant of the grid-leak and condenser combination is not a necessary physical interpretation of the phenomena which occur in the grid circuit. Indeed, in many cases, the time constant method of calculating the leak and condenser gives quite erroneous and misleading results. These cases occur when the impedance looking into the vacuum tube is of such value, as it often is, that the magnitudes and forms of q_{1n} and q_{2m} are materially changed from those which they would have if z_g were neglected, and when r_g is not large compared with q_n and q_m. Equation (38) shows that, for greatest plate current resulting from grid detection, q_h and q_k should be as small as possible, while $q_{(h-k)}$ should be as large as possible. It is, then, a filter problem, and if treated as such, will give reliable results both as to physical interpretation and numerical values.

In the special case when the input and detected frequencies are $\frac{h}{2\pi}$ and $\frac{s}{2\pi}$, respectively, and where $z_g \gg r_g$:

$$q_h = \frac{1}{jhC}$$

$$q_s = \frac{\frac{R}{jsC}}{R + \frac{1}{jsC}}$$

R = leak resistance

C = capacity in parallel with R

Then, the optimum size for the condenser, C, is easily shown to be

$$C^2 = \frac{\sqrt{2}(R + r_g)}{hsRr_g^2}, \text{ (approx.)} \tag{50}$$

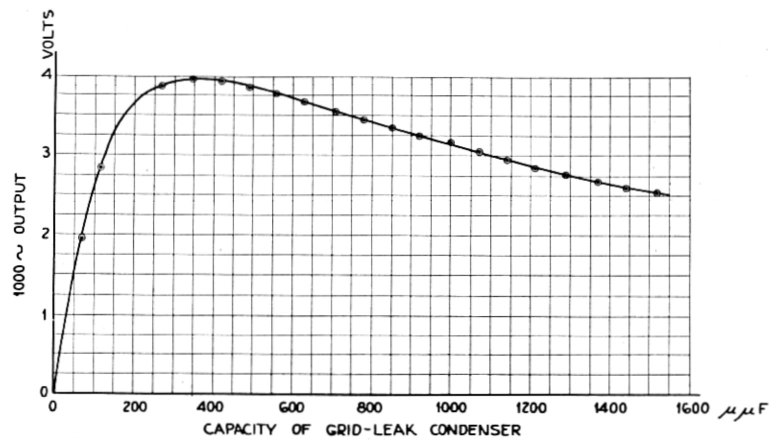

Fig. 6—Optimum size of grid-leak condenser

Experimental Conditions:

$h = 2\pi \times (30000 \pm 500)$

$s = 2\pi \times (1000)$

Grid-leak $= R = 10^6$ ohms

$r_g = 10^5$ ohms

Calculation Conditions:

$z_g \gg r_g$

$$q_h = \frac{i}{jhC}$$

$$q_s = \frac{\frac{R}{jsC}}{R + \frac{i}{jsC}}$$

Then the optimum size of the grid-leak condenser, C, is:

$$C^2_{opt} = \frac{\sqrt{2}(R + r_g)}{hsR\, r_g^2}$$

or;

$$C_{opt} = 361 \mu\mu \text{ farad}$$

Fig. 6 illustrates the agreement between this relation and an actual circuit where the above conditions were closely approximated.

Plate Curvature Detection

In discussing this phase of the problem we refer to equation (35). In addition to the remarks made in connection with that equation it is necessary only to add a few words on the evaluation of r_p and r_p'. In general, these quantities are susceptible to the same method of treatment that was suggested in dealing with r_g and r_g'. Two fundamental circuits for plate curvature detectors are in use. In the first the plate battery is placed in series with the load impedance. In the case when the load impedance contains appreciable resistance the normal or effective value of E_p must be obtained in the manner described for finding E_g. In the second circuit the plate battery potential is introduced through a low resistance, high impedance, choke, and the normal value of E_p is then equal to E_b. Especially in dealing with resistance coupled units these points should be borne in mind.

Amplification

Equation (33) gives the general amplification relation. The remarks made under the heading of the "Grid-Leak Detector" concerning the evaluation of the z's and q's are applicable here, as in all other vacuum tube relations. The special points to be brought out are the methods of applying the equations to so-called improper amplifiers of Class III. In this type of amplifier the grid swings negative further than the plate current cut-off point each cycle. Experience has shown that even in this event, to find the tube resistances, the approximation of using the secant line joining two points on the characteristic corresponding to the extreme values of the input voltage, is often justifiable. If greater accuracy is desired, the corrections given by the curve, Fig. 7, should be applied. These corrections are based on the assumption of a sine wave input and a characteristic that follows the square law, and to that extent are themselves in error. For modulated waves the dotted curves give values found by interpolation between the two points shown.

Modulation

The detection equations apply equally well to modulation effects. The only case in which a question may arise is that in which one of the input frequencies is introduced into the plate circuit of the tube

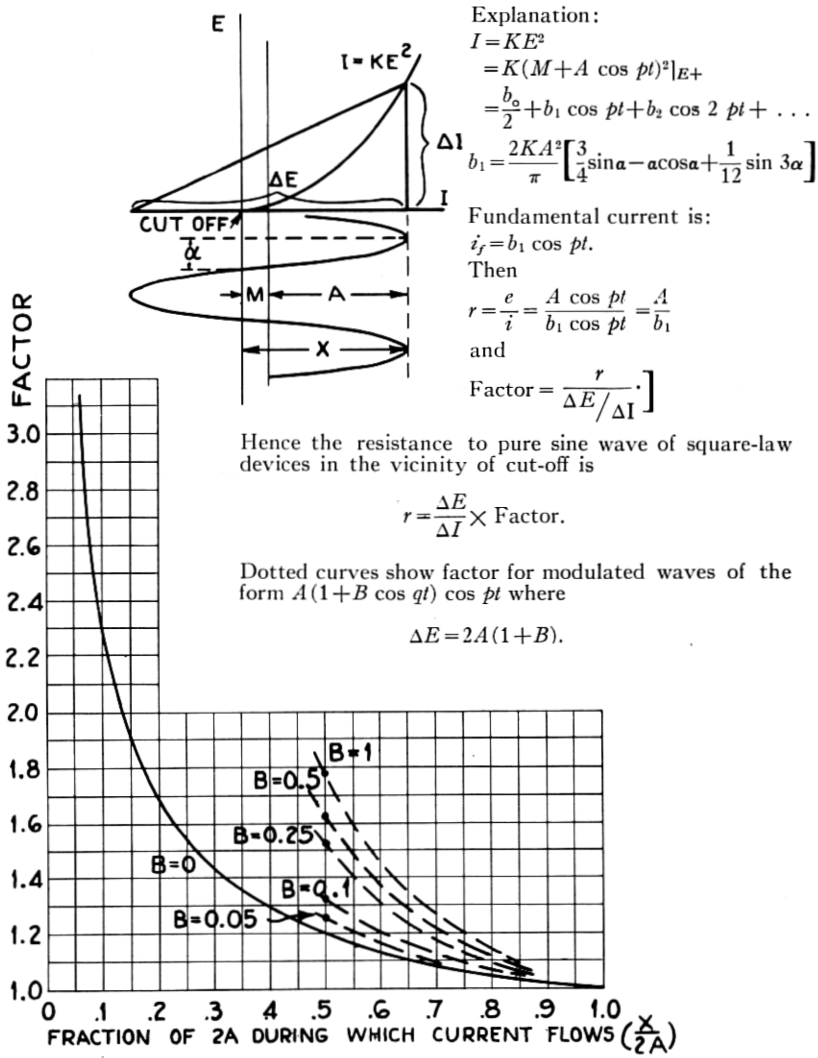

Fig. 7—Correction factor for resistance of non-linear device

while the other is introduced into the grid circuit. To analyze this condition for the general case, (see Fig. 8) let lower case e's refer to the driving voltage impressed directly on the grid. Let the E's refer to the driving voltage in series with an impedance in the plate circuit. We then have the series

$$i_p = a_1(E+e) + a_2(E+e)^2 + \ldots$$

which, in accordance with the complex quantity notation may be written

$$i_p = a_{1h}E + a_{1k}e + a_{2(2h)}E^2 + a_{2(2k)}e^2 + 2a_{2(OE)}E\bar{E} + 2a_{2(h+k)}Ee$$
$$+ 2a_{2(h-k)}E\bar{e} + 2a_{2(Oe)}e\bar{e} + \ldots$$

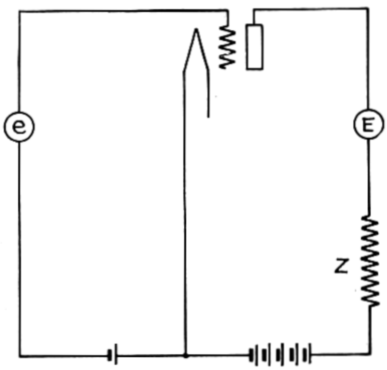

Fig. 8—Plate circuit modulation

Then, with the aid of (4), upon equating coefficients of like powers of e, E, and Ee, we get

$$a_{1h} = \frac{1}{r_p + z_h} \qquad a_{1k} = \frac{\mu}{r_p + z_k}$$

$$a_{2(2h)} = \frac{-\frac{1}{2}r_p r_p'}{(r_p + z_h)^2 (r_p + z_{2h})}$$

$$a_{2(2k)} = \frac{\frac{1}{2}\left[\frac{\partial\mu}{\partial E_g}(r_p + z_k)^2 + \mu\frac{\partial\mu}{\partial E_p}(r_p^2 - z_k^2) - \mu^2 r_p r_p'\right]}{(r_p + z_k)^2 (r_p + z_{2k})}$$

$$a_{2(OE)} = \frac{-\frac{1}{2}r_p r_p'}{(r_p + z_h)^2 (r_p + R)} \qquad a_{2(h+k)} = \frac{\frac{1}{2}\left[\frac{\partial\mu}{\partial E_p}\frac{r_p}{2}(2r_p + z_h + z_k) - \mu r_p r_p'\right]}{(r_p + z_h)(r_p + z_k)(r_p + z_{h+k})}$$

$$a_{2(h-k)} = \frac{\frac{1}{2}\left[\frac{\partial\mu}{\partial E_p}\frac{r_p}{2}(2r_p + z_h + \bar{z}_k) - \mu r_p r_p'\right]}{(r_p + z_h)(r_p + z_k)(r_p + z_{h-k})} \tag{51}$$

$$a_{2(Oe)} = \frac{\frac{1}{2}\left[\frac{\partial\mu}{\partial E_g}(r_p + z_k)^2 + \mu\frac{\partial\mu}{\partial E_p}(r_p^2 - z_k)^2 - \mu^2 r_p r_p'\right]}{(r_p + z_k)^2 (r_p + R)}$$

When z is a resistance, R, the expression for i_p reduces to

$$i_p=\frac{(\mu e+E)}{r_p+R}-\frac{\frac{1}{2}r_p r_p'}{(r_p+R)^3}E^2$$
$$+\frac{\frac{1}{2}\left[\frac{\partial\mu}{\partial E_g}(r_p+R)^2+\mu\frac{\partial\mu}{\partial E_p}(r_p^2-R^2)-\mu^2 r_p r_p'\right]}{(r_p+R)^3}e^2 \qquad (52)$$
$$+\frac{\frac{\partial\mu}{\partial E_p}r_p(r_p+R)-\mu r_p r_p'}{(r_p+R)^3}Ee+ \text{ - - - }.$$

If μ is constant, this becomes

$$i_p=\frac{\mu e+E}{r_p+R}-\frac{\frac{1}{2}r_p r_p'}{(r_p+R)^3}(\mu e+E)^2+ \text{ - - - } \qquad (53)$$

which shows that the circuit then acts as though a voltage, $(\mu e+E)$ had been impressed in series with the plate circuit.

Oscillation

The subject of vacuum tube oscillators has been so extensively treated elsewhere that but little new material has thus far been obtained from the general equations now offered. The method of handling the problem is, however, illuminating as it gives an example of what is meant by the statement that no sharply drawn line should be placed between oscillation, detection, amplification, or other uses of the thermionic vacuum tube.

In treating the oscillator problem we consider the amplification term of the general equations; namely

$$i_p=\frac{\mu e}{(r_p+z_n)}\frac{r_g}{\left[r_g+q_n(1-\frac{\mu}{\nu}\frac{z_n}{r_p+z_n}\right]}.$$

The oscillating conditions require that current shall flow without a driving voltage. Hence, as e is zero, i_p can be finite only if one of the factors in the denominator is zero. Thus either

$$r_p+z_n=0 \qquad (54)$$

or

$$r_g+q_n(1-\frac{\mu}{\nu}\frac{z_n}{r_p+z_n}=0 \qquad (55)$$

gives the conditions for oscillation. Fig. 5 and the relations of (45), (46), (47), (48) and (49) are applicable here. The condition of (54) requires a negative value of r_p, and hence is not the usual oscillation condition. The condition of (55) therefore gives the criterion for the oscillation condition. As before, neglecting quantities in $\frac{1}{\nu}$ we may write (55) in the following form

$$r_g+q_n=0$$

or

$$r_g+\frac{z_1(z_g+r_g)}{z_g}=0. \tag{56}$$

When applied to a hypothetical Hartley oscillator, Fig. 9, with the circuit constants

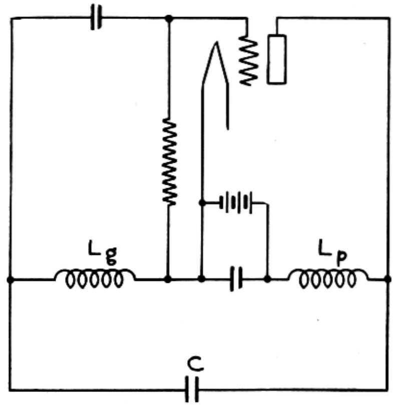

Fig. 9—Hartley oscillator

$$z_2=j\omega L_p \qquad z_3=\frac{1}{j\omega C} \qquad z_1=j\omega L_g,$$

equation (56) gives as the conditions for oscillation

$$\omega^2=\frac{1}{\left[L_p+L_g+\frac{L_pL_g}{Cr_pr_g}\right]}=\frac{1}{(L_p+L_g)\,C}\text{ (nearly)}, \tag{57}$$

$$L_p=\left[\mu-\frac{r_p}{r_gL_p}\right]L_g=\mu L_g \text{ (nearly)}. \tag{58}$$

The relations of (57) and (58) have been given many times, and are included here only in order to illustrate the ease with which simple problems may be solved from fundamental relations.

Application of the Theory

The illustrations will serve to give a sufficiently comprehensive view of the methods of applying the general equations to special cases.

Inasmuch as the derivation of the equations requires no assumptions other than that the static curves of grid current-grid potential, and plate current-plate potential of the tube remain constant, the accuracy with which a given problem may be calculated depends only upon the ability to determine the effective differential coefficients required by the Taylor's series expansions, and the number of terms of the series included. Practically, the component of current of a given frequency resulting from any higher order term is entirely negligible with respect to the component of the same frequency resulting from lower order terms. For precise results in a general case the calculations are necessarily tedious, since the physical processes are quite complex. However, in any given special case one of the respective approximations indicated is usually allowable, which greatly simplifies matters. In the event that any question arises concerning the proper phase angles for the complex impedances, the correct result may always be arrived at by writing the voltages in full complex form, as illustrated in the mathematical digression. The impedances will then take care of themselves.

While it is difficult to show mathematically the convergence of the series of (31), experience has shown that the convergence is so rapid that higher order terms may be neglected, unless new frequencies developed by them are under investigation. In these cases, the conditions of the problem are often such that simplifying assumptions may be made at the outset. If familiarity with the complex impedances has been attained, it will, in many cases, be sufficient to derive all equations on the basis of resistance only, and then introduce the complex impedances in the manner indicated by the analogy between these and the general equations.

The higher order coefficients are given below for the special case where resistances, only, are considered, and where the voltage, e_g, is known. It is found more convenient to use the P's, equation (4), in their derivative form than to attempt to express them in terms of μ and r_p, so referring to the expansion

$$i_p = a_1 e_g + a_2 e_g^2 + a_3 e_g^3 + a_4 e_g^4 + a_5 e_g^5 + \cdots,$$

we have

$$a_1=\frac{P_1}{1+P_2Z}$$

$$a_2=\frac{\frac{1}{\underline{|2}}}{1+P_2Z}\left[P_3-2P_4\frac{P_1Z}{1+P_2Z}+P_5\frac{Z^2P_1^2}{(1+P_2Z)^2}\right],$$

$$a_3=\frac{1}{1+P_2Z}\left(\frac{1}{\underline{|2}}\left[-2P_4a_2Z+2P_5a_1a_2Z_1Z_2\right]\right.$$

$$\left.+\frac{1}{\underline{|3}}\left[P_6-3P_7a_1Z+3P_8a_1^2Z^2-P_9a_1^3Z^3\right]\right),$$

$$a_4=\frac{1}{1+P_2Z}\left(\frac{1}{\underline{|2}}\left[-2P_4a_3Z+P_5(a_2^2Z^2+2a_1a_3Z_1Z_3\right]\right.$$

$$+\frac{1}{\underline{|3}}\left[-3P_7a_2Z+6P_8a_1a_2Z^2-3P_9a_1^2a_2Z^3\right]$$

$$\left.+\frac{1}{\underline{|4}}\left[P_{10}-4\mathrm{P}_{11}a_1Z+6P_{12}a_1^2Z^2-4P_{13}a_1^3Z^3+P_{14}a_1^4Z^4\right]\right)$$

$$a_5=\frac{1}{1+P_2Z}\left(\frac{1}{\underline{|2}}\left[-2P_4a_4Z+2P_5(a_1a_4Z^2+a_2a_3Z^2)\right]\right.$$

$$+\frac{1}{\underline{|3}}\left[-3P_7a_3Z+3P_8(a_2^2Z^2+2a_1a_3Z^2)\right.$$

$$\left.-P_9(a_1Za\overline{{}_2^2Z^2+2a_2a_3Z^2}+2a_1a_2^2Z^3+a_1^2a_3Z^3)\right]$$

$$+\frac{1}{\underline{|4}}\left[-4P_{11}a_2Z+12P_{12}a_1a_2Z^2-12P_{13}a_1^2a_2Z^3+4P_{14}a_1^3a_2Z^4\right]$$

$$+\frac{1}{\underline{|5}}\left[P_{15}-5P_{16}a_1Z+10P_{17}a_1^2Z^2-10\,P_{18}a_1^3Z^3\right.$$

$$\left.\left.+5P_{19}a_1^4Z^4-P_{20}a_1^5Z^5\right)\right],$$

APPENDIX I

To Show that with Negative Grid Potentials the Relation:

$$\mu\frac{\partial\mu}{\partial E_p}=\frac{\partial\mu}{\partial E_g}$$

Holds With Fair Precision

We have the fundamental expression:

$$I_p=I_p(E_g,\ E_g) \tag{1}$$

Suppose that E_g and E_p are allowed to vary under the restriction that I_p is maintained constant. Then:

$$d\ I_p=0 \tag{2}$$

Hence:

$$\frac{dI_p}{dE_g}=0=\frac{\partial I_p}{\partial E_g}+\frac{\partial I_p}{\partial E_p}\frac{dE_p}{dE_g} \tag{3}$$

Whence:

$$\left.\frac{dE_p}{dE_g}\right]_{I_p}=-\mu \tag{4}$$

Also:

$$d^2I_p=0 \tag{5}$$

Hence:

$$\frac{d^2I_p}{dE_g^2}=0=\frac{\partial^2I_p}{\partial E_g^2}+2\frac{\partial^2I_p}{\partial E_g\partial E_p}\frac{dE_p}{dE_g}+\frac{\partial^2I_p}{\partial E_g^2}\left(\frac{dE_p}{dE_g}\right)^2+\frac{\partial I_p}{\partial E_p}\frac{d^2E_p}{dE_g^2} \tag{6}$$

Then with the aid of (4), above, and (6) in the body of the paper, we get

$$\frac{\partial\mu}{\partial E_g}-\mu\frac{\partial\mu}{\partial E_p}+\frac{d^2E_p}{dE_g^2}=0 \tag{7}$$

Equation (7) shows that:

$$\frac{\partial\mu}{\partial E_g}=\mu\frac{\partial\mu}{\partial E_p} \tag{8}$$

provided that:

$$\frac{d^2E_p}{dE_g^2}=0 \tag{9}$$

when I_p is constant.

Experimental curves showing the relation between E_p and E_g required to maintain I_p constant are straight lines, to a very close approximation, in the region where the grid potential is negative with respect to the filament as shown in Fig. 10. Hence, in this region (9) is satisfied for all practical purposes, and, therefore, the proof of (8) follows directly.

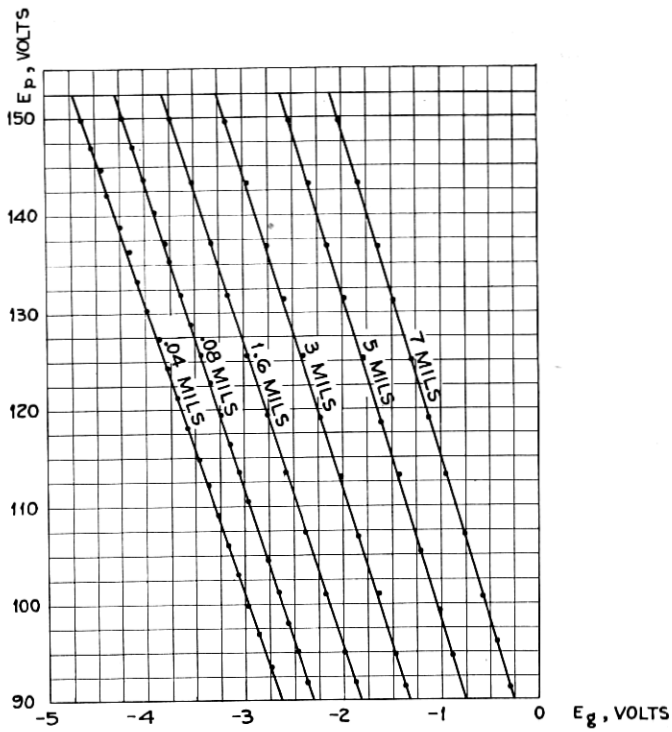

Fig. 10—Relation between E and E for constant plate current