Wave Propagation Over Parallel Tubular Conductors: The Alternating Current Resistance

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SYNOPSIS: On the basis of Maxwell's laws and the conditions of continuity of electric and magnetic forces at the surfaces of the conductor, the fundamental equations are established for the axial electric force and the tangential magnetic force in a non-magnetic tubular conductor with parallel return. The alternating current resistance per unit length is then derived as the mean dissipation per unit length divided by the mean square current. The general formula is expressed as the product of the alternating current resistance of the conductor with concentric return and a factor, termed the "proximity effect correction factor," which formulates the effect of the proximity of the parallel return conductor. The auxiliary functions which appear in the general formula are each given by the product of the corresponding function for the case of a solid wire and a factor involving the variable inner boundary of the conductor.

In general, the resistance may be calculated from this formula, using tables of Bessel functions. The most important practical cases, however, usually involve only the limiting forms of the Bessel functions. Special formulae of this kind are given for the case of relatively large conductors, with high impressed frequencies, and for thin tubes. A set of curves illustrates the application of the formulae.

I. INTRODUCTION

WHERE circular conductors of relatively large diameter are under consideration, the effect on the alternating current resistance of the tubular as distinguished from the solid cylindrical form becomes of practical importance. Mr. Herbert B. Dwight has worked on a special case of this problem and developed a formula for the ratio of alternating to direct current resistance in a circuit composed of two parallel tubes when the tubes are thin.[1] As infinite sums of infinite series are involved, however, his result is not well adapted to computation.

Mr. John R. Carson has given a complete solution for the alternating current resistance of two parallel solid wires in his paper "Wave Propagation Over Parallel Wires: The Proximity Effect," *Phil. Mag.*, April, 1921. The analysis of that paper may readily be extended to the more general case of propagation over two tubular conductors by a parallel method of development. This is done in the present paper. As the underlying theory is identical in the two problems, familiarity with the former paper will be assumed and the analysis will merely be sketched after the fundamental equations are established.

[1] "Proximity Effect in Wires and Thin Tubes," Trans. *A. I. E. E.*, Vol. XLII (1923), p. 850.

In this paper formulae for the alternating current resistance have been worked out in detail with particular reference to the case of relatively large conductors at high frequencies and to relatively thin tubes. In general the auxiliary functions involved are expressed as the product of the corresponding functions for solid wires by a correction factor which formulates the greater generality due to the variable inner boundary of the conductors. As far as possible the symbols are the same as in the solid wire case but refer now to the system of tubular conductors. Primes are added where the letters denote the corresponding functions for the solid wire case. This will hardly lead to confusion with the primes used in connection with the Bessel functions to denote differentiation.

The general solution is developed in section II. The alternating current resistance of one of the tubular conductors is expressed as the product of the alternating current resistance of the conductor with concentric return and a factor which formulates the effect of the proximity of the parallel return conductor. Section III is a summary of the general formula, special asymptotic forms and forms for thin conductors.

II. Mathematical Analysis and Derivation of Formulae

We require the expression for the axial electric force, E_z, in the conductors. Since the tubular conductor does not extend to $r=O$, the electric force must be expressed by the more general Fourier-Bessel expansion,

$$E_z=\sum_{n=O}^{\infty} A_n\,[J_n(\rho)+\lambda_n K_n(\rho)]\cos n\theta,$$

where

$$\begin{aligned}\rho &= ir\sqrt{4\pi\lambda\mu i\omega}\\ &=\xi = xi\sqrt{i}\text{ when } r=a\\ &=\zeta = yi\sqrt{i}\text{ when } r=\alpha,\end{aligned}$$

a and α being the outer and inner radii, respectively, of the conductors. The additional set of constants $\lambda_o, \lambda_1 \ldots \lambda_n$ is to be determined by the conditions of continuity at the inner boundary of the conductor. It is necessary to satisfy the boundary conditions at the surface of one conductor only, since the symmetry of the system insures that they will then be satisfied at the surface of the other also.

In the dielectric space inside the tube where $r<\alpha$, the axial electric force may be written

$$E_z=\sum_{n=0}^{\infty} C_n J_n(\rho) \cos n\theta, \tag{1}$$

or replacing the Bessel functions by their values for vanishingly small arguments,

$$E_z=\sum_{n=0}^{\infty} D_n r^n \cos n\theta \tag{2}$$

where D_o, $D_1 \ldots D_n$ are constants determined by the boundary conditions. Applying Maxwell's law relating the normal and tangential magnetic forces H_r and H_θ to the axial electric force, gives

$$\mu i\omega H_\theta=\frac{\rho}{r}\sum_{n=0}^{\infty} A_n\,[J_n'(\rho)+\lambda_n K'_n(\rho)] \cos n\theta, \tag{3}$$

$$\mu i\omega H_r=\frac{1}{r}\sum_{n=0}^{\infty} A_n\,[J_n(\rho)+\lambda_n K_n(\rho)] \sin n\theta, \tag{4}$$

for the space inside the conductor, and

$$i\omega H_\theta=\sum_{n=0}^{\infty} nD_n r^{n-1} \cos n\theta, \tag{5}$$

$$i\omega H_r=\sum_{n=0}^{\infty} nD_n r^{n-1} \sin n\theta, \tag{6}$$

for the inner dielectric ($\mu=1$). Equating the two expressions for the tangential magnetic force H_θ and for the normal magnetic induction μH_r term by term at the surface $r=\alpha$,

$$[\zeta J_n'(\zeta)-\mu n J_n(\zeta)]+\lambda_n[\zeta K_n'(\zeta)-\mu n K_n(\zeta)]=0. \tag{7}$$

Whence, for the practically important case of non-magnetic conductors in which $\mu=1$, we have

$$\lambda_n=-\frac{J_{n+1}(\zeta)}{K_{n+1}(\zeta)} \tag{8}$$

and

$$E_z=\sum_{n=0}^{\infty} A_n\left[J_n(\rho)-\frac{J_{n+1}(\zeta)}{K_{n+1}(\zeta)}K_n(\rho)\right] \cos n\theta. \tag{9}$$

In the subsequent analysis $J_n\ (\xi)$ of the solution for the solid wire case is replaced by

$$J_n(\xi)-\frac{J_{n+1}(\zeta)}{K_{n+1}(\zeta)}K_n(\xi)=M_n(\xi), \tag{10}$$

and $J_n'\ (\xi)$ is replaced by

$$J_n'(\xi)-\frac{J_{n+1}(\zeta)}{K_{n+1}(\zeta)}K_n'(\xi)=M_n'(\xi). \tag{11}$$

Otherwise the formulation of the alternating current resistance of the conductor proceeds exactly as in the solid wire case. For the electric force at the surface $r=a$ in the conductor, we write

$$E_z=A_o[M_o(\xi)+h_1M_1(\xi)\cos\theta+h_2M_2(\xi)\cos 2\theta+\ .\ .\ .] \tag{12}$$

and determine the fundamental coefficient A_o in terms of the current in the conductor. The resistance R of the tubular conductor per unit length is defined as the mean dissipation per unit length divided by the mean square current where the mean dissipation is calculated by Poynting's theorem. Accordingly, we get

$$R=\text{Real}\frac{2\mu i\omega}{\xi}\left\{\frac{M_o\ (\xi)}{M_o'(\xi)}+\frac{1}{2}\sum_{n=1}^{\infty}|h_n|^2\frac{M_n\ (\xi)}{M_o'(\xi)}\text{conj.}\frac{M_n'(\xi)}{M_o'\ (\xi)}\right\}. \tag{13}$$

To determine the harmonic coefficients $h_1\ .\ .\ .\ h_n$ or $A_1\ .\ .\ .\ A_n$, the total tangential magnetic force and the total normal magnetic induction at the outer surface of a conductor are expressed in terms of the coordinates of that conductor alone, and the conditions of continuity at the surface are applied. This leads to the set of equations

$$q_n=(-1)^n 2\rho_n k^n-\frac{(-1)^n}{(n-1)!}\rho_n k^n\sum\nolimits_n(q) \qquad n=1,2,3\ .\ .\ .\ \infty \tag{14}$$

where

$$\sum\nolimits_n(q)=\frac{n!}{1!}kq_1-\frac{(n+1)!}{2!}k^2q_2+\ .\ .\ .\ ,$$

$$\sigma_n=(\xi M_n'\ (\xi)-n\mu M_n\ (\xi))/\xi\ M'(\xi),$$

$$\rho_n=(\xi M_n'\ (\xi)-n\mu M_n(\xi))/(M_n'(\xi)+n\mu M_n(\xi)),$$

$$q_n=\sigma_n h_n,$$

$$\frac{a}{c}=k.$$

When the permeability is unity, the solution, to the same order of approximation as in the solid wire case, is

$$|h_n|^2=\frac{u_1^2+v_1^2}{u_{n-1}^2+v_{n-1}^2}\left|\frac{1+\lambda_o K_1(\xi)/J_1(\xi)}{1+\lambda_n K_{n-1}(\xi)/J_{n-1}(\xi)}\right|^2 p_n^2(1+2ngk^2/s^{n-1}) \tag{16}$$

where

$$g=\frac{\sqrt{2}}{x}\,\frac{p[u_1(u_o+v_o)-v_1(u_o-v_o)]-q[u_1(u_o-v_o)+v_1(u_o+v_o)]}{u_o^2+v_o^2}, \tag{17}$$

$$p+iq=\frac{1+\lambda_1 K_1(\xi)/J_1(\xi)}{1+\lambda_1 K_o(\xi)/J_o(\xi)}, \tag{18}$$

$$J_n(\xi)=u_n+iv_n,$$

$$p_n=(-1)^n 2k^n s^n,\ n=1,2\ldots\infty,$$

$$s=2\frac{1-\sqrt{1-(2k)^2}}{(2k)^2}.$$

Since the resistance R_o of an isolated tubular conductor is given by

$$R_o=\text{Real}\,\frac{2\mu ip}{\xi}\,\frac{M_o(\xi)}{M_o'(\xi)} \tag{19}$$

equation (13) becomes equation (I) of the formulae in the next section. This is the general solution for the case of non-magnetic conductors.

In general R may be calculated from this formula and tables of Bessel functions. The ber, bei, ker and kei functions [2] and the recurrence formulae are sufficient to evaluate the Bessel functions but the process is long. In the most important practical cases, the conductors are rather large and the applied frequencies fairly high. When this is true as well as when the tubes are very thin the formulae usually involve only the limiting forms of the Bessel functions. These special results are given in the next section.

III. Alternating Current Resistance Formulae for Non-Magnetic Conductors

The symbols used are:

a = outer radius of conductor in centimeters,

α = inner radius of conductor in centimeters,

c = interaxial separation between conductors in centimeters,

$k=a/c$

λ = conductivity of conductor in electromagnetic c.g.s. units,

[2] A convenient table of these functions for arguments from 0 to 10 at intervals of 0.1 is incorporated in Mr. Dwight's paper "A Precise Method of Calculation of Skin Effect in Isolated Tubes," *J. A. I. E. E.*, Aug., 1923.

μ = permeability of conductor in electromagnetic c.g.s. units,
$\omega = 2\pi$ times frequency in cycles per second,
$i = \sqrt{-1}$
$x = a\sqrt{4\pi\lambda\omega}$
$y = \alpha\sqrt{4\pi\lambda\omega}$
$\xi = xi\sqrt{i}$
$\zeta = yi\sqrt{i}$
$\lambda_n = -J_{n+1}(\zeta)/K_{n+1}(\zeta)$
$J_n(\xi) = u_n + iv_n$
= Bessel function of first kind of order n and argument $xi\sqrt{i}$,

$$J_n'(\xi) = \frac{dJ_n(\xi)}{d\xi}$$

$$u_n' + iv_n' = \frac{dJ_n(\xi)}{dx}$$

$K\ (\xi)$ = Bessel function of second kind of order n and argument $xi\sqrt{i}$,

$$K_n'(\xi) = \frac{dK_n(\xi)}{d\xi}$$

R = resistance per unit length of tubular conductor with parallel return,
R_o = resistance per unit length of tubular conductor with concentric return in electromagnetic c.g.s. units,
C = proximity effect correction factor,

$$R = C\,R_o. \tag{I}$$

The auxiliary functions involved are:

$$^{3}\,R_o = R_o'm\left(1 - \frac{n}{m}\,\frac{u_o u_o' + v_o v_o'}{u_o v_o' - u_o' v_o}\right) \tag{20}$$

where

$$R_o' = \frac{1}{a}\sqrt{\frac{\omega}{\pi\lambda}}\,\frac{u_o v_o' - u_o' v_o}{u_1^2 + v_1^2} \tag{21}$$

= resistance of solid wire with concentric return,

$$m + in = \frac{1 + \lambda_o K_o(\xi)/J_o(\xi)}{1 + \lambda_o K_o'(\xi)/J_o'(\xi)}, \tag{22}$$

$$g = g'p\left\{1 - \frac{q\,[u_1(u_o - v_o) + v_1(u_o + v_o)]}{p\,[u_1(u_o + v_o) - v_1(u_o - v_o)]}\right\}, \tag{23}$$

[3] The ratio R_o/R_o' oscillates about unity which it approaches more and more closely as the frequency increases. It is due to the fact that the phase of the current in the inner portion of the solid conductor may be such as to oppose the current in the outer portion, that the resistance of the solid conductor may be greater than that of the tube even though the heating effect in the latter is the greater.

where

$$g' = \frac{\sqrt{2}}{x}\ \frac{u_1(u_o+v_o)-u_1(u_o-v_o)}{u_o^2+v_o^2}, \tag{24}$$

$$p+iq = \frac{1+\lambda_1 K_1(\xi)/J_1(\xi)}{1+\lambda_1 K_o(\xi)/J_o(\xi)}, \tag{25}$$

$$w_n = w_n' \frac{a_n}{|1+\lambda_n K_{n-1}(\xi)/J_{n-1}(\xi)|^2}\left(1-\frac{b_n}{a_n}\ \frac{u_n u_n' + v_n v_n'}{u_n v_n' - u_n' v_n}\right), \tag{26}$$

where

$$w_n' = \frac{u_n v_n' - u_n' v_n}{u_{n-1}^2 + v_{n-1}^2}, \tag{27}$$

$$a_n + ib_n = \left(1+\lambda_n \frac{K_n(\xi)}{J_n(\xi)}\right) \text{conj.} \left(1+\lambda_n \frac{K_n'(\xi)}{J_n'(\xi)}\right), \tag{28}$$

$$s = 2\frac{1-\sqrt{1-(2k)^2}}{(2k)^2}. \tag{29}$$

The formula for the correction factor C is then

$$C = 1 + \frac{2}{aR_o}\sqrt{\frac{\omega}{\pi\lambda}}(S_1 + 2gk^2 S_2) \tag{II}$$

where

$$S_1 = \sum_{n=1}^{\infty} w_n k^{2n}\ s^{2n}, \tag{30}$$

$$S_2 = \sum_{n=1}^{\infty} n w_n k^{2n} s^{n+1}. \tag{31}$$

For large values of the argument

$$R_o = R_o'\left[m - n\left(1-\frac{1}{\sqrt{2}x}\right)\right] \tag{32}$$

and the correction factor is

$$C = 1 + 2\frac{\sqrt{2}-1/x}{m-n(1-1/\sqrt{2}x)}\left(S_1 - \frac{2\sqrt{2}}{x}\left[p+q\left(1-\frac{1}{\sqrt{2}x}\right)\right]k^2 S_2\right) \tag{III}$$

When x and y are both large quantities, the auxiliary functions are as follows, provided terms of the second order in $1/x$ and $1/y$ are negligible, n in d and h below being equal to the number of terms in which S_1 and S_2 converge to a required order of approximation.

With the notation

$$\cos = \cos\sqrt{2}(x-y),$$
$$\sin = \sin\sqrt{2}(x-y),$$
$$\exp = \exp\,[-\sqrt{2}(x-y)],$$

$$R_o = R_o' \frac{1+[(1+a)\sin-(1-a)\cos]\exp - a\exp^2}{1-[(1-b)\sin+(1+b)\cos]\exp + b\exp^2} \tag{33}$$

where

$$a=1-\frac{1}{2\sqrt{2x}}-\frac{3}{2\sqrt{2y}},$$

$$b=1+\frac{3}{2\sqrt{2x}}-\frac{3}{2\sqrt{2y}},$$

$$\frac{1}{aR_o'}\sqrt{\frac{\omega}{\pi\lambda}}=\sqrt{2}-\frac{1}{x}, \tag{34}$$

$$g=g'\frac{1+[(1-c)\cos-(1+c)\sin]\exp-c\exp^2}{1-[(1+c)\cos+(1-c)\sin]\exp+c\exp^2}, \tag{35}$$

where

$$c=1-\frac{1}{2\sqrt{2x}}-\frac{15}{2\sqrt{2y}},$$

$$g'=-\sqrt{2}/x, \tag{36}$$

$$w_n=w_n'\frac{1-[(1-d)\cos-(1+d)\sin]\exp-d\exp^2}{1-[(1+h)\cos+(1-h)\sin]\exp+h\exp^2}, \tag{37}$$

where

$$d=1+\frac{4n^2-1}{2\sqrt{2x}}-\frac{4(n+1)^2-1}{2\sqrt{2y}},$$

$$h=1+\frac{4(n-1)^2-1}{2\sqrt{2x}}-\frac{4(n+1)^2-1}{2\sqrt{2y}},$$

$$w_n'=\frac{1}{\sqrt{2}}-\frac{2n-1}{2x}. \tag{38}$$

At frequencies sufficiently high to afford practically skin conduction, the following formulae indicate the way in which the resistance of the tubular conductor approaches its limit, the resistance of the solid wire.

$$R_o=R_o'\frac{1+2\sin\exp}{1-2\cos\exp}, \tag{39}$$

$$\frac{1}{aR_o'}\sqrt{\frac{\omega}{\pi\lambda}}=\sqrt{2}-\frac{1}{x},$$

$$C=C_m(1-A/x), \tag{IV}$$

$$C_m=\frac{1+k^2s^2}{1-k^2s^2}, \tag{40}$$

$$A=2\sqrt{2}\,\frac{k^2s^2}{1-k^4s^4}\left\{1+2k^2\frac{(1-k^2s^2)^2}{(1-k^2s)^2}\,\frac{1-2\sin\exp}{1-2\cos\exp}\right\}. \tag{41}$$

When the conductors are very thin tubes, i.e., thin as compared to the radius, $(a-\alpha)/a$ is necessarily small and, in general, $x-y$ is small. Of course, when the frequency is high enough, $x-y$ becomes large in any case. When this is true with respect to thin tubes, however, x and y will usually be large enough to make the asymptotic formulae applicable; but, if $x-y$ is small, the approximations

$$J_n(\zeta)=J_n(\xi)-(\xi-\zeta)J_n'(\xi)+\frac{(\xi-\zeta)^2}{2!}J_n''(\xi),$$

$$K_n(\zeta)=K_n(\xi)-(\xi-\zeta)K_n'(\xi),+\frac{(\xi-\zeta)^2}{2!}K_n''(\xi),$$

reduce the correction factor to

$$C=1+2\,\beta^2 f\left\{\sum_{n=1}^{\infty}k^{2n}s^{2n}\frac{d_n}{D_n}-2k^2\frac{\frac{4d_1^2}{x^4}-\beta^3c_1}{D_1}\sum_{n=1}^{\infty}k^{2n}s^{n+1}n\frac{d_n}{D_n}\right\} \qquad \text{(V)}$$

where $\beta=\dfrac{a-\alpha}{a}$,

$$f=\frac{(1+\beta/2)^2}{1+\beta+\beta^2}=\frac{c_o^2}{d_o},$$

$$D_n=\beta^2c_n^2+\frac{4n^2}{x^4}d_n^2,$$

$$c_n=1+\frac{2n+1}{2}\beta,$$

$$d_n=1+(n+1)\beta+\frac{(n+1)\,(n+2)}{2}\beta^2.$$

and the resistance with concentric return to

$$R_o=\frac{1}{2\pi\lambda a(a-\alpha)}\,\frac{1+\beta+\beta^2}{1+\beta/2}. \qquad (42)$$

$1/2\pi\lambda a(a-\alpha)$ is, of course, the direct current resistance of a very thin conductor.

If $(a-\alpha)/a$ is very small and negligible compared with $2n/x^2$, where n is the number of terms in which the series of (V) converge to a required order of approximation,

$$C=1+\frac{x^4}{2}\left(\frac{a-\alpha}{a}\right)^2\left\{\begin{array}{l}\left(1-\dfrac{a-\alpha}{a}\right)\left\{\displaystyle\sum_{n=1}^{\infty}\frac{k^{2n}s^{2n}}{n^2}+2k^2s\log(1-k^2s)\right\}\\[2ex] +\dfrac{a-\alpha}{a}\left\{\log(1-k^2s^2)+2\dfrac{k^4s^2}{1-k^2s}\right\}\end{array}\right\} \qquad \text{(VI)}$$

As a check on formulae (V) and (VI), the limiting cases may be arrived at directly as follows. If the conductors are thin tubes, the harmonic coefficients are given by

$$h_n=(-1)^{n+1}2k^n\frac{\xi-\zeta}{\dfrac{2n}{\xi}-(\xi-\zeta)\left(1-\dfrac{2n(n+1)}{\xi^2}\right)}$$

$$-(-1)^{n+1}\frac{\xi-\zeta}{\dfrac{2n}{\xi}-(\xi-\zeta)\left(1-\dfrac{2n(n+1)}{\xi^2}\right)}k^n\left[nkh_1-\frac{n(n+1)}{2!}k^2h_2+\ldots\right]. \quad (43)$$

When ξ is very large

$$h_n=(-1)^n2k^n\left[1-\frac{1}{2}\left\{nkh_1-\frac{n(n+1)}{2!}k^2h_2+\ \ldots\right\}\right]$$

$$=(-1)^n2k^ns^n, \quad (44)$$

and

$$\frac{M_n}{M_o}=\frac{M_n'}{M_o'}=1 \quad (45)$$

so that

$$C=\text{Real}\left[1+\frac{1}{2}\sum_{n=1}^{\infty}|h_n|^2\frac{M_n}{M_o}\text{conj.}\frac{M_n'}{M_o'}\right]$$

$$=\frac{1+k^2s^2}{1-k^2s^2}, \quad (46)$$

the same result as for the corresponding limiting case of a solid conductor.

On the other hand, if ξ is not large and $\xi-\zeta$ is very small,

$$h_n=(-1)^{n+1}\frac{k^n}{n}\xi(\xi-\zeta), \quad (47)$$

$$\frac{M_n}{M_o}=1, \quad (48)$$

$$\frac{M_n'}{M_o'}=-\frac{in}{x(x-y)}, \quad (49)$$

so that

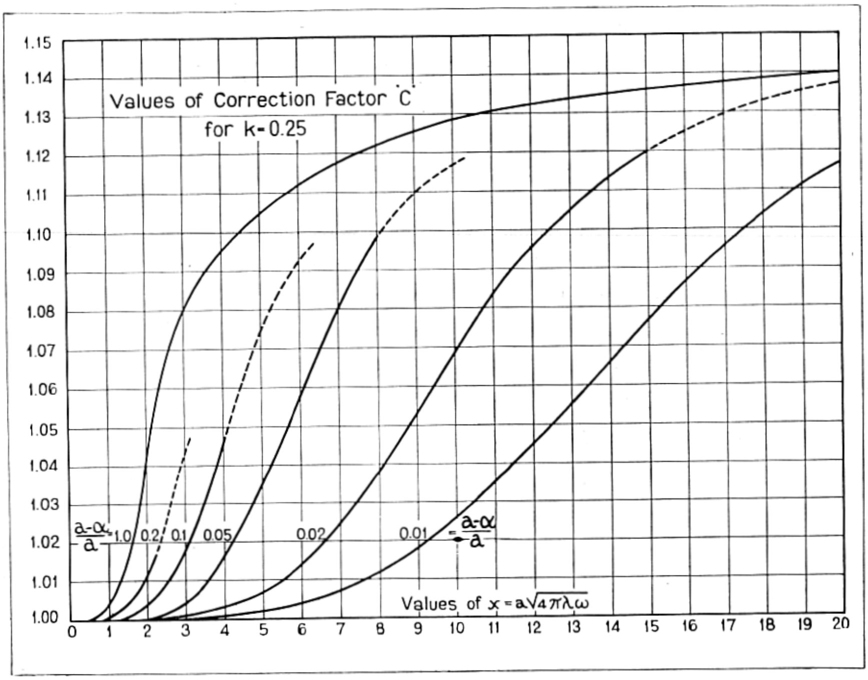
$$C=1, \quad (50)$$

and

$$R=R_o=R_{d.c.}, \quad (51)$$

where $R_{d.c.}$ is the direct current resistance of the thin tubular conductor. Eqs. (46) and (50) agree with the corresponding limits of formulae V and VI respectively.

The curves of the accompanying figure do not pretend to represent the proximity effect correction factor with precision. They are, however, accurate for thin tubes, and indicate the order of magnitude of the factor for various values of the thickness of the tubular conductor and show the nature of its variation with respect to the applied



frequency. They are computed from formula (V) which is valid for quite high frequencies when the tubes are thin. When the thickness of the tubes is greater, however, the range of validity with respect to frequency is smaller, the dotted portions indicating a doubtful degree of precision. It was previously pointed out in connection with formula (IV) and is immediately deducible from physical considerations, that all of the curves eventually coincide with the curve for the solid wire which approaches the value 1.155 asymptotically.

As a simple application, suppose the resistance is required of a tubular conductor with an outer radius of 0.4125 cm. (that of No. O gauge A.W.G. copper wire) whose resistivity is 1696.5 electromagnetic units per cm., where there is an equal parallel return so situated that $k=0.25$ and a frequency of 5,000 cycles per second is applied to the circuit. Then $m=\sqrt{4\pi\lambda\omega}=15.26$ and $x=ma=15.26\times 0.4125=6.30$. When the ratio of the thickness of the conductor to the radius is greater than about 0.01 the proximity effect correction factor C is appreciable. If the ratio is 0.05, reading C from the curves, gives $C=1.064$. From formula (42), $R_o=5.24$ ohms per mi. which makes the resistance $R=5.53$ ohms per mi.