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Mutual Impedances of Grounded Circuits

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SYNOPSIS: Formulas are derived for the direct-current mutual resistance and inductance between circuits grounded at the surface of the earth. For circuits composed of straight filaments, the mutual inductance is reduced to known Neumann integrals which involve only comparatively simple expressions for the case of horizontal, coplanar conductors above, below or on the surface of the earth. Numerical values for these integrals may be readily obtained from new and accurate graphs for straight filaments which meet at a point or start from a common perpendicular. It is shown that these new results supply a useful first approximation to the actual alternating-current mutual impedance of grounded circuits, when the frequency and extent of the circuits are not larger than occur in many practical applications.

1. INTRODUCTION

THE important discovery of the possibility of using the earth as the return conductor for electric telegraphic communication was announced by Steinheil in the *Comptes Rendus* of September 10, 1838, and throughout the entire development of telegraphy grounded circuits have been extensively employed. Considering the extensive application of such a capital discovery extending over a period of 85 years, it is surprising that so little is known quantitatively about grounded circuits. We have, however, long known that conditions are not of the extreme simplicity pictured under the early view that the earth acts as a reservoir presenting no resistance to the return current and introducing no interference between parallel returns. This view was expressed in 1857 by Bakewell as follows: "There is no mingling of currents, the electric current of each battery being kept as distinct as if separate wires were used both for the transmitted and the return currents. It would indeed be as impossible for the separate currents transmitted from the two batteries to be mingled together as it would be for the written contents of two letters enclosed in the same mail bag to intermix."

Measurements made a few years ago of the mutual impedances between grounded circuits which are restricted to a territory six miles square, at frequencies of 25 to 60 cycles per second, showed that within 10 per cent. the mutual reactance increased in the same ratio as the frequency. It was inferred that the effective inductance under the conditions of these tests was approximately the same as for direct current, or in other words, the incomplete penetration of the alternating currents into the earth was not of controlling importance in tests upon this scale.

This led to my making a theoretical investigation of the mutual inductance between direct-current grounded circuits which did, in fact, show that the calculated numerical results are in reasonable agreement with these actual experimental data. It is the purpose of this paper to describe this work; the mathematical discussion of the theoretical corrections for the incomplete penetration of alternating currents into the earth will form the subject of another paper.

2. Distribution of Current, Potential and Magnetic Force with Direct-Current Earth Return Flow

On the assumption of an infinite earth of uniform resistivity the lines of flow and the equipotential surfaces for a direct current I entering the earth at a point source at A and leaving the earth at a point sink at B, both A and B being on the surface of the earth, assumed flat, and the distance $AB=2b$, are given by the equations,[1]

$$\left.\begin{aligned}\frac{C}{I} &= \tfrac{1}{2}(\cos\theta_1-\cos\theta_2)\\ &= \tfrac{1}{2}\left(\frac{x_1}{r_1}-\frac{x_2}{r_2}\right)\\ &= \sin\tfrac{1}{2}(\theta_1+\theta_2)\sin\tfrac{1}{2}(\theta_2-\theta_1)\\ &= b\,\frac{\sin^2\theta}{r},\ \text{if}\ \frac{b}{r}\ \text{is small,}\end{aligned}\right\}\qquad(1)$$

$$\left.\begin{aligned}\frac{V}{I} &= \frac{\rho}{2\pi}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\\ &= -\frac{b\rho}{\pi}\,\frac{\cos\theta}{r^2},\ \text{if}\ \frac{b}{r}\ \text{is small,}\end{aligned}\right\}\qquad(2)$$

where C is the total current flowing in the earth, from the source at A to the sink at B, outside the current sheet of revolution defined by (1); and V is the potential, with respect to the midplane, upon the equipotential surface of revolution defined by (2).

These equipotential lines and stream lines are identical with the equipotential lines and lines of force for a uniformly magnetized filament. The formulas may be checked by regarding the return flow from A to B as being due to the superposition of two flows, a

[1] The coordinates used in this paper (x_1, y, z), (x_2, y, z), (x, y, z) and (r_1, θ_1, ϕ), (r_2, θ_2, ϕ), (r, θ, ϕ) are rectangular and spherical coordinates with origins at A, B and the midpoint of AB, respectively, the direction AB being the polar axis or positive x-axis in all cases, z being vertical, and ϕ being measured from the earth's surface in the plane perpendicular to AB.

return flow of direct current I from the point A to some infinitely distant point and a second return flow of direct current I from this infinitely distant point to the point B. For these component flows the current diverges from A or converges towards B radially and with equal intensity in all directions in the earth; the total current for one of the component flows flowing through any surface in the earth will thus be equal to $I/2\pi$ times the solid angle subtended at A or B, respectively, by the boundary of the surface, since the entire solid angle filled by the earth at a point on the surface is 2π. The total radial flow from A through the lower half of the circular cone having its axis in AB, the elements of the cone making the angle θ_1 with AB, is $\frac{1}{2}I(1-\cos\theta_1)$; similarly, the total radial flow toward B through the lower half of a cone with the angle $\pi-\theta_2$ will be $\frac{1}{2}I(1+\cos\theta_2)$. For the combined superposed flows the total current flowing through the semicircle in which the cones intersect is the sum of these two values or $\frac{1}{2}I(2-\cos\theta_1+\cos\theta_2)$, from which (1) is immediately obtained, since the total current flowing in the earth from A to B is I. This assumes that the semicircle lies between A and B, but the same formula holds for the entire current sheet of revolution. The lines of flow in the earth are symmetrical about AB and lie in planes through AB, since, in the earth, both component flows are symmetrical about AB.

For the component flows the equipotential surfaces are hemispherical and, since the resistance of a hemispherical shell of radius r, thickness dr, is $\rho dr/2\pi r^2$, the potentials at distances r_1 or r_2 from A or B, referred to the potential at infinity, are $I\rho/2\pi r_1$ or $-I\rho/2\pi r_2$, respectively, from which equation (2) follows by addition.

Fig. 1 accurately reproduces the flow and equipotential lines as given by formulas (1) and (2). At the midpoint of a line of flow its distance from each electrode is $r_1=r_2=bI/C$ and it may be shown that every other point of a line of flow is at a still shorter distance from the nearer electrode. It follows, for example, that less than 1/10 of the total current reaches, in its flow through the earth, any point lying at a distance greater than $5AB$ from the line AB connecting the electrodes.

If a uniform radial flow of current I in the horizon plane converging on the point A is combined with the uniform radial flow in the earth outward from A, we have a closed flow which is symmetrical about the vertical axis through A. Below the horizon plane the magnetic lines of force will be horizontal circles and the magnetic force at any point distant r_1 from A, α being the angle included between r_1 and the nadir, is $H=2I(1-\cos\alpha)/(r_1\sin\alpha)$

$=2I\, r_1^{-1} \tan (\alpha/2)$. Above the horizon plane there is no magnetic field, since any magnetic lines of force are, by symmetry, horizontal circles and the intensity is zero, since there is no current threading any horizontal circle above the surface of the earth.

Superposing this closed flow and a similar closed flow through B from the earth to the horizon plane, we obtain a closed flow from A

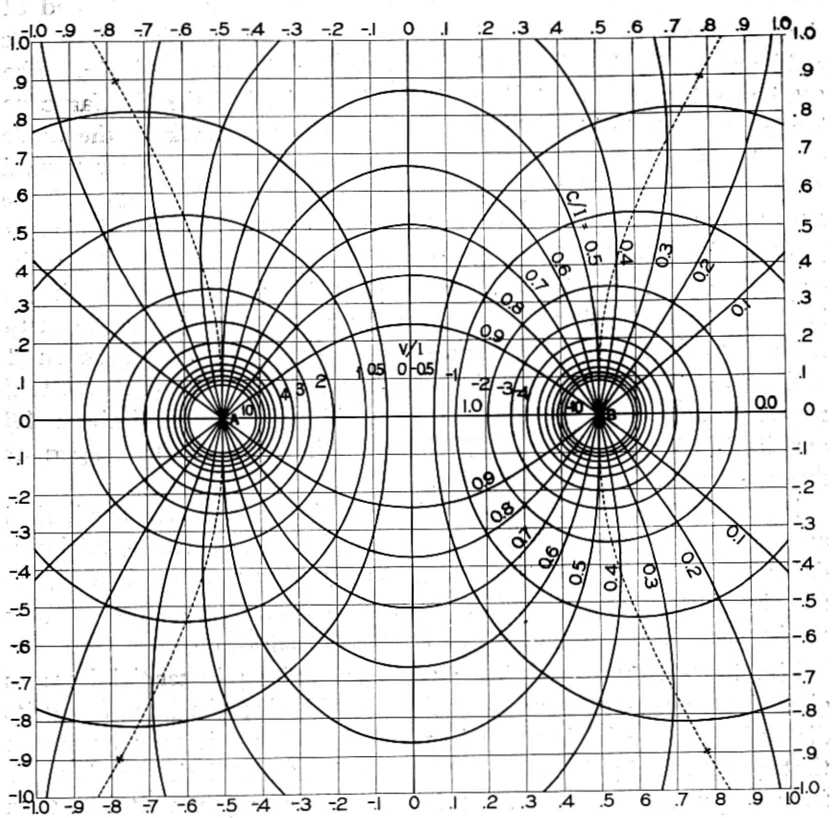

Fig. 1—Flow and equipotential lines on the earth's surface for an earth return flow from A to B. C/I is the fraction of the current flowing in the earth outside the flow surface of revolution; V/I is the resistance to the flow of the portion of the earth lying between the equipotential surface of revolution and the mid or zero potential plane, if the earth's resistivity is $\rho=2\pi$. The flow and equipotential lines through each point on the dotted curve are perpendicular and parallel, respectively, to AB

to B in the earth and back from B to A in the horizon plane. The magnetic field for this closed flow, being the sum of the magnetic fields for the component flows, will be zero above the horizon plane, while below it will consist of lines of force in horizontal planes. This result also applies to any closed flow which does not extend above the horizon plane and may be resolved into any number of component flows, each of which is radially symmetrical about a vertical axis.

3. Mutual Resistance of Grounded Circuits

By definition, if e.m.f.'s E and e in grounded conductors AB and ab produce the currents I and $i=0$ in the conductors, the mutual impedance between the two conductors is e/I. In the present case we are dealing with direct current and thus the mutual impedance is a mutual resistance Q, and by (2) its value is [2]

$$\left.\begin{aligned} Q &= \frac{\rho}{2\pi}\left(\frac{1}{Aa}-\frac{1}{Ab}-\frac{1}{Ba}+\frac{1}{Bb}\right) \\ &= \frac{\rho}{2\pi}\int\int\frac{d^2}{dSds}\left(\frac{1}{R}\right)dSds \\ &= \frac{\rho}{2\pi}\int\int\frac{-2dUdu+dVdv+dWdw}{R^3} \\ &= \frac{\rho}{2\pi}\int\left\{\frac{\cos(\theta_1-\epsilon)}{r_1^2}-\frac{\cos(\theta_2-\epsilon)}{r_2^2}\right\}ds. \end{aligned}\right\} \quad (3)$$

The third form of (3) shows that the mutual resistance falls off as the inverse third power of the distance between grounded circuits when this distance has become large compared with the length of these circuits between grounding points.

The first form of (3) shows that the mutual resistance between grounded circuits does not depend upon the location of the conductors but only upon the location of the terminal grounding points A, B, a, b.

The mutual resistance for the case $\rho=2\pi$ is obtained from Fig. 1 by taking the value of V/I at the point corresponding to a reduced by its value at the point corresponding to b; if b is anywhere on the center line, for which $V/I=0$, the diagram gives directly the value of the mutual resistance. Employing ordinary units the diagram gives the mutual resistance directly in ohms if $AB=1$ mile and the earth has a resistivity of about one million ohms per centimeter cube (more exactly 1.011×10^6) which is its actual order of magnitude.

[2] In addition to the earlier notation there are employed in the different expressions for formula (3), and also in formula (5) below, the following: R is the distance between two elements dS and ds of any two paths extending from A to B and a to b; the rectangular projections of these elements along and perpendicular to R are dU, dV, dW and du, dv, dw, the two sets being parallel and with the same positive directions; θ_1, θ_2, ϵ are the angles which r_1, r_2 and ds make with AB, when the path ab lies in a plane with A and B.

4. Neumann Integrals for Return Flows

The required mutual inductances of grounded circuits will be found by means of the Neumann integral

$$N=\int\int\int\int(\cos \epsilon/r)dI\, dS\, di\, ds$$

extended over every current filament in both flows. Since the earth return portions of the two flows are independent of the flows in the arbitrarily located conductors on the earth's surface, it is convenient to divide the Neumann integral into four partial integrals which involve either no return flow, one return flow or both return flows according to the following formula [3]

$$\begin{aligned} N_{(\mathcal{X}-\mathcal{E})(x-e)} &= N_{\mathcal{X}x}-N_{\mathcal{X}e}-N_{\mathcal{E}x}+N_{\mathcal{E}e} \\ &= N_{\mathcal{X}x}-(\tfrac{1}{2}+\tfrac{1}{2}-1)\Delta, \text{ by Table I,} \\ &= N_{\mathcal{X}x}. \end{aligned} \qquad (4)$$

Checking the entries of Table I may be accomplished without performing more than two integrations. It will be convenient to make the integrals somewhat more general than is required in checking the table and find $N_{\mathcal{F}s}$ and $N_{\mathcal{X}'A}$ where $\mathcal{F}$ is any flow in space from A to B, which need not be coplanar points with the terminals a and b of s, and $\mathcal{X}'$ is any flow in a plane parallel to the horizon plane between terminal points A' and B'.

Consider first the part of a space return flow s which is radial from a in connection with an element dS on any filament of current dI of a flow $\mathcal{F}$ from A to B. The component dx of dS along the line x from a to ds is the only component which need be considered, since by symmetry the normal component contributes nothing to the Neumann integral. As the total radial flow is to be taken equal to unity, the amount flowing out through a ring, taken as the volume element, lying between the spheres of radii s and $s+ds$ and between the cones making angles θ and $\theta+d\theta$ with x will be $\frac{1}{2}\sin\theta\, d\theta$. If this ring lies at a distance r from dS the Neumann integral will be

$$N=\int_{Aa}^{Ba}dx\int_0^{\infty}ds\int_0^{\pi}\frac{\cos\theta\sin\theta\, d\theta}{2r},\ r^2=x^2+s^2-2xs\cos\theta,$$

$$=\frac{1}{4}\int_{Aa}^{Ba}\frac{dx}{x^2}\int_0^{\infty}\frac{ds}{s^2}\int_{|x-s|}^{(x+s)}(x^2+s^2-r^2)dr$$

[3] Each term indicates the Neumann integral for the pair of flows designated by the script letter subscripts, as explained in the note accompanying Table I. Both $(\mathcal{X}-\mathcal{E})$ and $(x-e)$ are arbitrary flows on the earth's surface closed by earth return flows from A to B and from a to b, respectively.

$$=\frac{1}{3}\int_{Aa}^{Ba}\frac{dx}{x^2}\left\{\int_0^x s\,ds+x^3\int_x^\infty \frac{ds}{s^2}\right\}$$

$$=\frac{1}{2}\int_{Aa}^{Ba}dx=\tfrac{1}{2}(Ba-Aa).$$

TABLE I

Entries are the value of k in the formula $N=k\Delta$ for the Neumann integral between the specified flows, $\Delta=-Aa+Ab+Ba-Bb$, or $2AB$ if $A=a$, $B=b$, and points A B, a, b are all on the earth's surface.

Flows†	h	s	e	a	n	z	$(h-s)$	$(h-e)$	$(h-a)$	$(h-n)$	$(h-z)$	$(e-a)$
Any surface $=\mathcal{X}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0
Space return $=\mathcal{S}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0
Earth return $=\mathcal{E}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{3}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	-1	1	1
Air return $=\mathcal{A}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	-1	-1
Nadir return $=\mathcal{N}$	0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	*	-1	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	*	1	2
Zenith return $=\mathcal{Z}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	-1	*	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	1	*	-2
Closed $(\mathcal{X}-\mathcal{S})$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0
" $(\mathcal{X}-\mathcal{E})$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	2	0	-1
" $(\mathcal{X}-\mathcal{A})$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	0	1	0	2	1
" $(\mathcal{X}-\mathcal{N})$	1	0	-1	1	*	1	1	2	0	*	0	-2
" $(\mathcal{X}-\mathcal{Z})$	1	0	1	-1	1	*	1	0	2	0	*	2
" $(\mathcal{E}-\mathcal{A})$	0	0	1	-1	2	-2	0	-1	1	-2	2	2

* Not of the assumed form and in general infinite.

† Each type of flow is designated by a script letter. Any flow in the surface of the earth, assumed to be a plane, is designated by $\mathcal{X}$, there being no restriction on the number of filaments of current but each filament must start from a common point, A, and extends to another common point, B. A special case of $\mathcal{X}$ is $\mathcal{H}$ the horizon return flow made up of two superposed uniform radial flows, from A to every point of the horizon and back to B. The space return flow, $\mathcal{S}$, is made up of two superposed uniform radial flows, one outward in all directions from the point A and the other inward from all directions toward the point B. The earth return and air return flows, $\mathcal{E}$ and $\mathcal{A}$, are similar except that the flows are uniformly distributed over all directions in the earth and air, respectively. The nadir return and zenith return flows, $\mathcal{N}$ and $\mathcal{Z}$, consist of two infinite vertical filaments from A and back to B by way of the nadir and zenith, respectively. Small script letters indicate similar types of flow with any independent terminal points, a and b. Differences such as $(\mathcal{X}-\mathcal{S})$ designate closed flows; thus $(\mathcal{X}-\mathcal{E})$ designates any flow on the earth's surface from A to B where it enters the earth and after spreading out uniformly through the earth returns to the terminal A, thus closing the flow.

To this is to be added the corresponding expression $\frac{1}{2}(Ab-Bb)$ for the radial flow converging on b, giving the result $\frac{1}{2}\Delta$. As the path of the line of flow between A and B does not enter into the result, it is immaterial whether the flow is confined to a single filament or is spread out in any way whatsoever in space, provided only all stream lines extend from A to B, as assumed for $\mathcal{F}$. Thus

$$N_{\mathcal{F}_s} = \tfrac{1}{2}\Delta = N_{\mathcal{S}_f}.$$

To find $N_{\mathcal{X}'\lambda}$ let r and s be the distances of any element dS of a line of flow forming a part of $\mathcal{X}'$ from a and from any element of a plane radial flow from a, respectively, the projections of r and s on either of the planes being x and y, which include the angle ϕ; $s^2=y^2+z^2$, $r^2=x^2+z^2$. The component of dS parallel to x will be dx and this is the only component which need be considered, since, on account of the symmetry of the radial flow, the normal component in the plane of flow contributes nothing to the integral.

$$\begin{aligned}
N &= \int_{Aa}^{Ba} dx \int_0^\infty \int_0^{2\pi} \frac{x-y\cos\phi}{x^2+y^2-2xy\cos\phi}\,\frac{y\,dy\,d\phi}{2\pi s} \\
&= \frac{1}{2\pi}\int_{Aa}^{Ba}\frac{dx}{x}\int_z^\infty ds \int_0^\pi \left(1+\frac{x^2-y^2}{x^2+y^2-2xy\cos\phi}\right)d\phi \\
&= \frac{1}{2\pi}\int_{Aa}^{Ba}\frac{dx}{x}\int_z^\infty ds\left[\phi-\sin^{-1}\frac{(x^2+y^2)\cos\phi-2xy}{x^2+y^2-2xy\cos\phi}\right]_0^\pi \\
&= \int_{Aa}^{Ba}\frac{dx}{x}\int_z^r ds,
\end{aligned}$$

since inspection shows that the two values of the definite integral 2π and 0 are to be used for $s \lessgtr r$ respectively, and therefore

$$\begin{aligned}
N &= \int_{Aa}^{Ba}\frac{r-z}{x}\,dx \\
&= \int_{A'a}^{B'a}\frac{r\,dr}{r+z} \\
&= \Big[r - z\log(r+z)\Big]_{A'a}^{B'a} \\
&= (B'a - A'a) - z\log\frac{B'a+z}{A'a+z}.
\end{aligned}$$

This is for the radial flow from a. Adding the corresponding expression for the radial flow towards b, we have finally for the complete integral

$$N\mathscr{X}'_{h}=(-A'a+A'b+B'a-B'b)-z\log\frac{(A'b+z)\,(B'a+z)}{(A'a+z)\,(B'b+z)},\quad \textbf{(4a)}$$

which becomes, if $z=0$,

$$N\mathscr{X}_{h}=\Delta=N\mathscr{H}_{x}.$$

The first line of Table I can now be filled in at once since the integrations have shown that the first two values of k are 1 and $\frac{1}{2}$; the next two entries are also $\frac{1}{2}$ since by symmetry $N\mathscr{X}_{s}=N\mathscr{X}_{e}=N\mathscr{X}_{a}$; $N\mathscr{X}_{n}=N\mathscr{X}_{z}=0$ since the nadir and zenith flows are perpendicular to the $\mathscr{X}$ flow in the horizon plane. The remaining six entries in the first row are for closed flows which are expressed as differences between the flows already considered, and the corresponding k's are the differences of the k's for the component flows. The first column of k's may also be filled in, the table being symmetrical, since interchanging capital and small script letters leaves N unchanged and h is a special case of x.

The second row of the table involves only special cases of $N\mathscr{S}_{f}$ and only the values $\frac{1}{2}$ and 0 occur.

From the flows included in the table nine pairs of closed flows may be formed having zero mutual inductances, because one of the closed flows of each pair has no magnetic field below the surface of the earth and the other closed flow includes no current above the surface of the earth, and thus there is no interlinkage of induction between the two closed flows. The portion of Table I referred to is repeated in Table II, where the flows at the top are those for which any difference such as $(h-a)$ has no magnetic field below the earth's surface, just as $(\mathscr{H}-\mathscr{E})$ has no magnetic field above the earth's surface, as was proved above, while no flow at the side penetrates above the earth's surface.

TABLE II

	h	a	z
$\mathscr{X}$	1	$\frac{1}{2}$	0
$\mathscr{E}$	$\frac{1}{2}$	0	$-\frac{1}{2}$
$\mathscr{N}$	0	$-\frac{1}{2}$	-1

The top row of Table II includes only values of k already found, and the remainder of the first column follows from symmetry and the fact that $\mathscr{h}$ is a special case of the general flow $\mathscr{x}$. Any other entry in Table II is now found in terms of three of the entries in this border, thus

$$0 = N(\mathscr{X}-\mathscr{E})(\mathscr{h}-\mathscr{a}) = N\mathscr{X}\mathscr{h} - N\mathscr{X}\mathscr{a} - N\mathscr{E}\mathscr{h} + N\mathscr{E}\mathscr{a} = (1-\tfrac{1}{2}-\tfrac{1}{2})\Delta + N\mathscr{E}\mathscr{a},$$

and we have the interesting result $N\mathscr{E}\mathscr{a}=0$; the remainder of the table follows.

The result $N\mathscr{Y}\mathscr{x} = -\Delta$ may be readily checked directly since it involves only the mutual Neumann integral between straight parallel filaments, and by using the expanded form of the integral for equal filaments beginning at a common perpendicular with opposite positive directions[4] the result can be written down at once.

The important difference Δ which is utilized in Table I may be expressed in the following useful forms:

$$\left.\begin{aligned} \Delta &= (-Aa+Ab+Ba-Bb) \\ &= -\int\int \frac{d^2R}{dSds}\,dSds \\ &= \int\int \frac{dVdv+dWdw}{R} \\ &= 2\int \sin\tfrac{1}{2}(\theta_2-\theta_1)\sin\left[\tfrac{1}{2}(\theta_1+\theta_2)-\epsilon\right]ds, \end{aligned}\right\} \qquad (5)$$

where the notation is the same as for formula (3) above. The third form of (5) shows that when the separation R is great the mutual inductance varies inversely as the first power of the separation.

5. Mutual Inductance Between Grounded Circuits Lying on the Surface of the Earth

It has now been shown that *for direct currents the mutual inductance between grounded circuits consisting of conductors lying on the surface of the earth and grounded at their terminals is equal to the mutual Neumann integral between the conductors alone*, since in the complete Neumann integral for the closed flows the total contribution of those parts which involve the ground returns is zero. For low frequencies the effective inductance can differ but little from the direct-current inductance, and it is therefore of practical importance to investigate

[4] Mutual Inductances of Circuits Composed of Straight Wires. *Physical Review*, 5, pp. 452–458, June, 1915, formula (6).

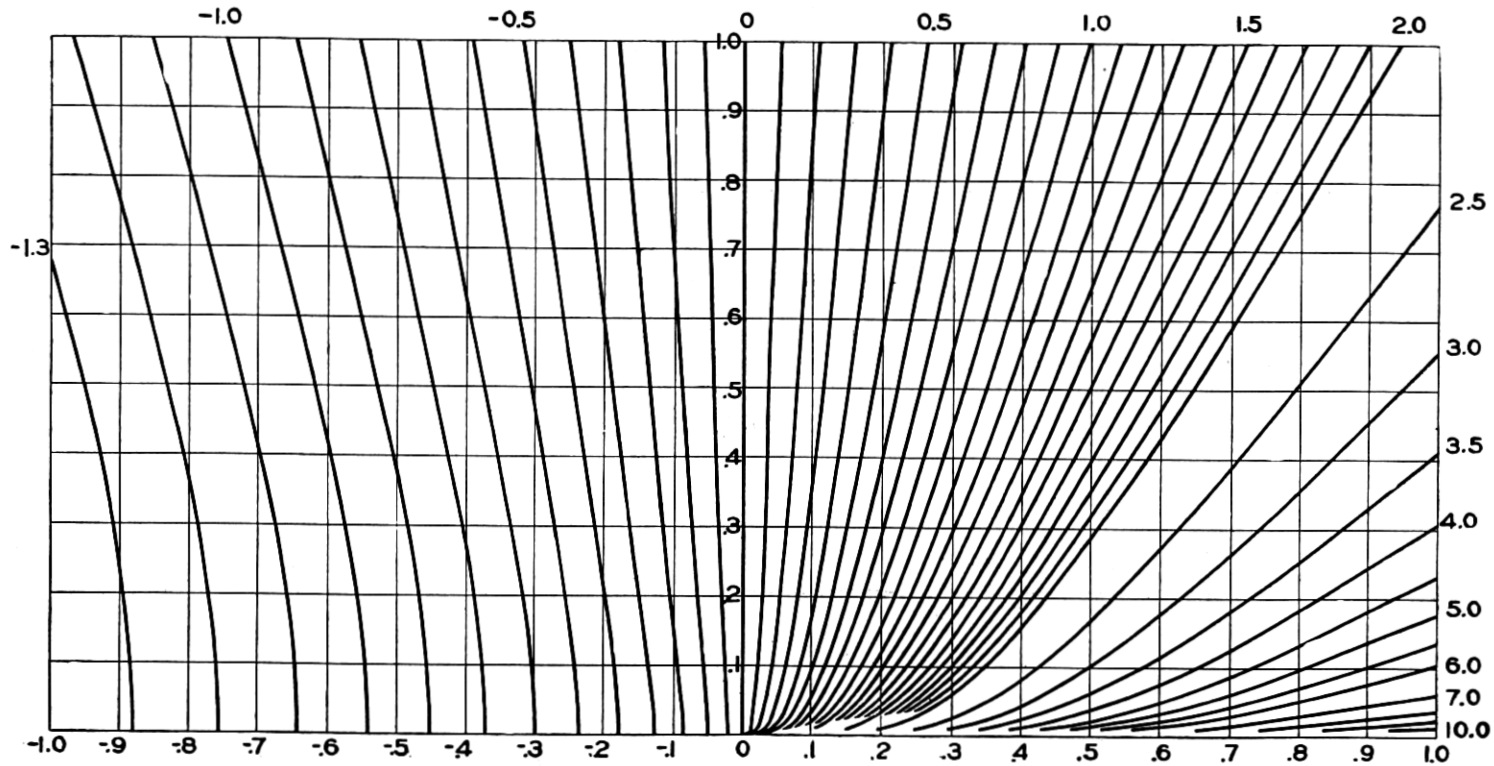

Fig. 2—Contour lines of the mutual Neumann integral between two straight filaments meeting at a point, one filament being of unit length

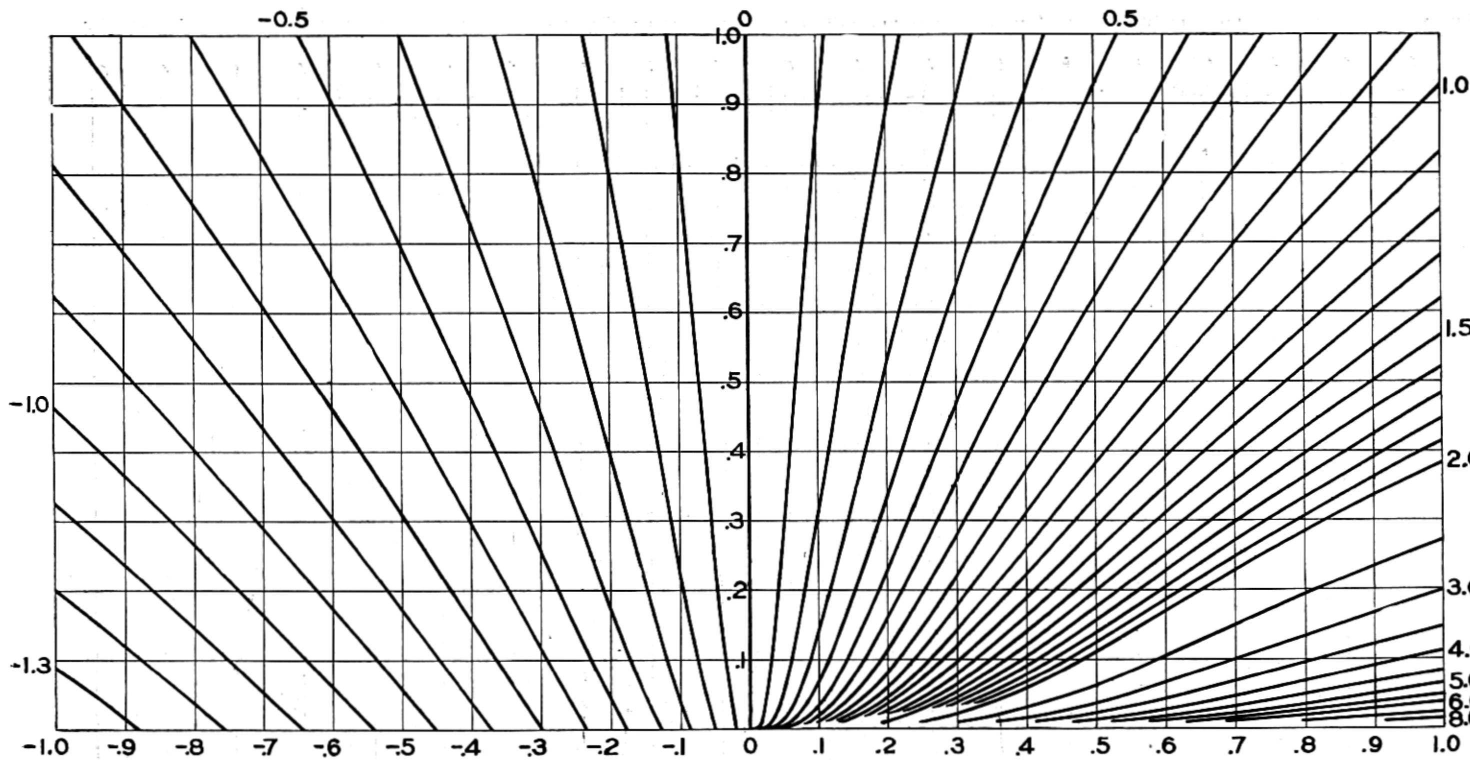

Fig. 3—Contour lines of the mutual Neumann integral between two parallel filaments starting at a common perpendicular, one filament being of unit length

the numerical magnitude of these Neumann integrals between grounded circuits. In order to visualize the magnitudes involved and supply means by which they may be readily calculated, a number of diagrams have been prepared for the important case of straight conductors.[5]

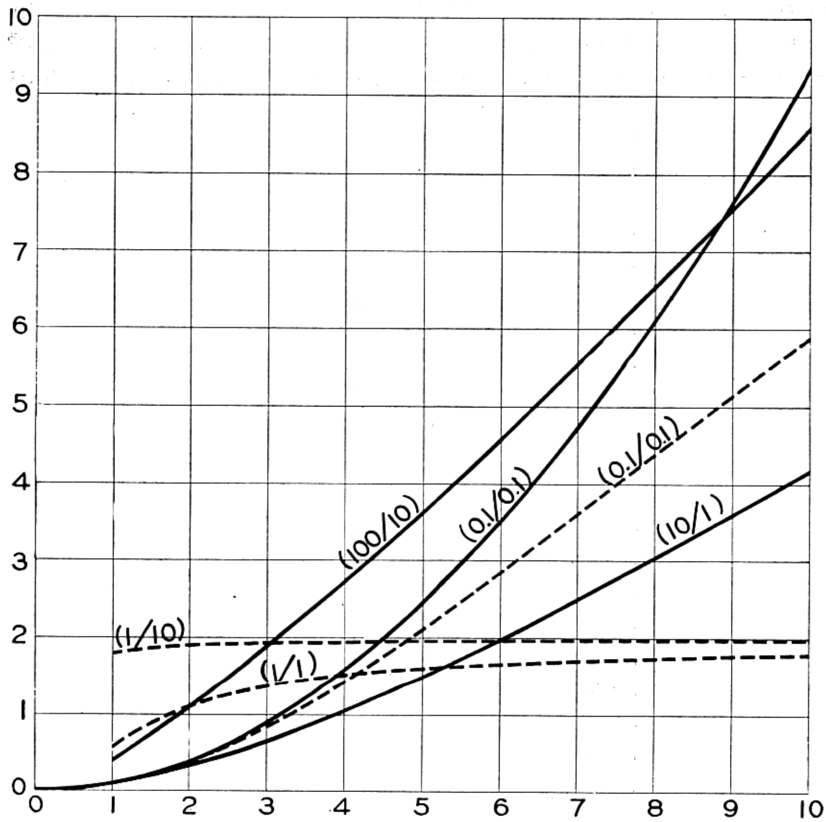

Fig. 4—Mutual Neumann integral between filaments forming opposite sides of a rectangle with unit separation. The dashed curves show the mutual resistance between the filaments if the circuits are grounded through the earth and its resistivity is $\rho = 2\pi$. The numerator and denominator of the bracketed fraction on each section of the curve show the factors by which the vertical and horizontal scales must be multiplied for use on this section

If the two straight conductors OA, Oa start from a common point O, the mutual Neumann integral is shown by Fig. 2; the curves give the locus of terminal a for constant values of the integral when the other conductor OA is the unit base. The Neumann integral between

[5] The necessary formulas are given in the paper loc. cit. Additional transformations of these formulas are given in the appendix to the present paper.

any two straight filaments $\mathscr{R}$ and r having terminals A, B and a, b may be expressed as

$$N_{\mathscr{R}r} = N_{(Aa)} - N_{(Ab)} - N_{(Ba)} + N_{(Bb)}, \tag{6}$$

where $N_{(Aa)}$ stands for the Neumann integral between the two straight filaments, OA and Oa, beginning at O, the point of intersection of $\mathscr{R}$ and r, extended if necessary, and ending at the terminals A and a; thus

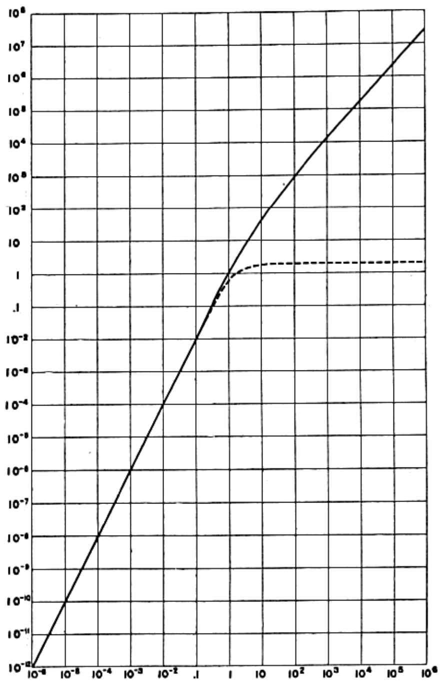

Fig. 5—Mutual Neumann integral between filaments forming opposite sides of a rectangle with unit separation. The dashed curves show the mutual resistance between the filaments if the circuits are grounded through the earth and its resistivity is $\rho = 2\pi$. This is Fig. 4, but with logarithmic scales

the integral between any two filaments which would intersect within a finite distance may be readily found after reading four values from Fig. 2.

In the special case of parallel filaments Fig. 2 fails; the corresponding curves for this case are presented by Fig. 3, which assumes a unit base filament and a parallel filament starting at a point on the left hand perpendicular to the base. Differences will give the general case of parallel filaments which do not start at a common perpendicular, which may thus be derived from Fig. 3.

The mutual Neumann integral between any two parallel filaments may also be obtained by means of the formula

$$N_{\mathscr{R}r} = \tfrac{1}{2}[-N_{(Aa)} + N_{(Ab)} + N_{(Ba)} - N_{(Bb)}], \tag{7}$$

where $N_{(Aa)}$ now stands for the Neumann integral between the projections of Aa on $\mathscr{R}$ and r, extended if necessary, the projections

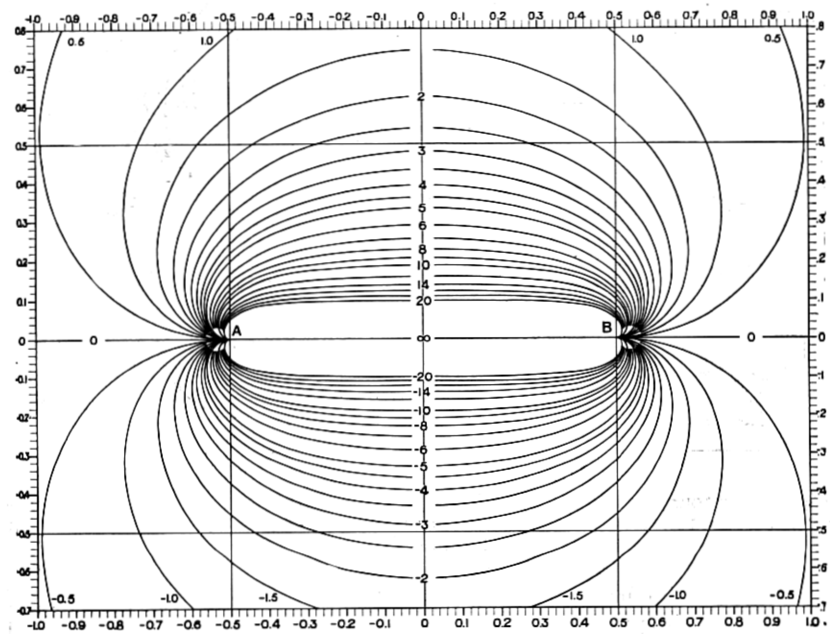

Fig. 6—Contour lines of the mutual Neumann integral between a counter-clockwise small loop on the surface of the earth, per unit area, and a straight grounded filament AB of unit length

having the same or opposite positive directions in agreement with $\mathscr{R}$ and r. This formula for the mutual Neumann integral presents the advantage of requiring only a single entry diagram, which is supplied by Fig. 4 and on a logarithmic scale by Fig. 5.

The mutual inductance may be required between a small, closed loop lying upon, but insulated from, the surface of the earth and a straight grounded conductor. The value depends upon the location, area and assumed positive direction around the loop, but is independent of the shape of the loop. Contour curves for the mutual inductance per unit area of the loop are given by Figs. 6 and 7; the positive direction around the loop is counter-clockwise; the straight grounded conductor AB of Fig. 6 is of unit length while in Fig. 7 grounded terminal B alone appears, terminal A being at an infinite

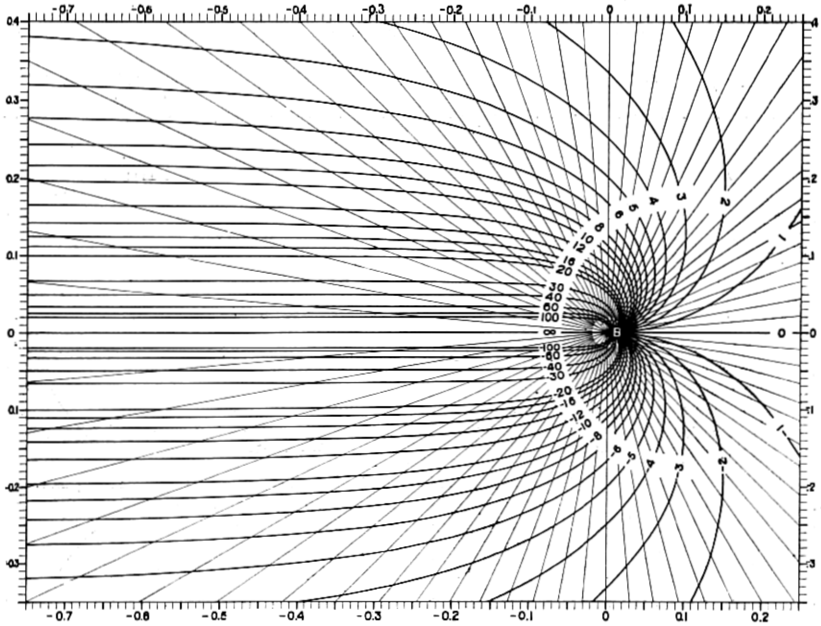

Fig. 7—Contour lines of the mutual Neumann integral between a counter-clockwise small loop on the surface of the earth, per unit area, and a straight grounded filament of infinite length

distance to the left. These curves show the vertical component of the magnetic field due to unit current in AB. The formulas employed for $AB=1$ and infinity, respectively, are [6]

$$-\frac{d^2N}{dxdy}=\frac{2y(r_1+r_2)}{r_1r_2[(r_1+r_2)^2-1]}=\frac{r_1+r_2}{r_1r_2}\sqrt{\frac{1-(r_1-r_2)^2}{(r_1+r_2)^2-1}}, \tag{8}$$

$$-\frac{d^2N}{dxdy}=\frac{y}{r_2(x_2+r_2)}=\frac{1}{r_2}\tan\tfrac{1}{2}\theta_2. \tag{9}$$

Large loop mutual inductances may be calculated either by integrating the value of $-d^2N/dxdy$ over the loop or by integrating the value of dN/ds around the boundary. If the boundary may be approximated to by a broken straight line, the curves of Figs. 2 and 3 may be employed.

[6] These formulas may be derived by differentiating (8) of the paper loc. cit., $dN/dx=2\tanh^{-1}[AB/(As+sB)]$, with respect to y.

6. Mutual Impedances for Conductors Lying on the Surface of the Earth

In order to arrive as directly as possible at a concrete numerical idea of the magnitudes and angles occurring in the mutual impedances encountered in engineering work, we may advantageously start with the following specific constants:

Base Length (AB, OA or Aa)................1 Mile,

Frequency of the Alternating Current..... 1 Kilocycle,

Resistivity of the Earth per Centimeter Cube.1 Megohm,

which are of the right order of magnitude and make the factors

$$\rho/(2\pi\ AB) = 10^6/(2\pi\ 0.1609 \times 10^6) = 0.989,$$

$$2\pi\ f \times AB \times 10^{-9} = 2\pi\ 0.1609 = 1.011,$$

which are both equal to unity within about 1 per cent., so that the approximate resistance and reactance components of mutual impedances may be read directly from Figs. 1–8 without applying multipliers. On the other hand, when mutual impedances are required for other lengths, frequencies and specific resistances, the correcting factors are readily applied. The tangent of the angle of the mutual impedance is proportional to the frequency, to the square of the linear dimensions of the circuits and to the reciprocal of the earth's resistivity.

Grounded circuits separated by a distance large compared with the dimensions of the circuits have a mutual impedance with negligible resistance component since ultimately this component varies inversely as the cube of the distance by (3), whereas ultimately the mutual reactance varies inversely with the distance.

For two parallel grounded conductors separated by one mile and forming opposite sides of a rectangle, the two components of the mutual impedance are shown by Fig. 5 for the assumed constants (approximately a kilocycle and a megohm). The resistance component of the mutual impedance is then always less than the reactance component; when the rectangle becomes a square, the mutual impedance angle is $\tan^{-1}1.595 = 57.9°$. Reducing the frequency to 0.627 kilocycles or reducing the side of the square to 0.792 miles or increasing the resistivity to 1.595 megohms would reduce this angle to 45°.

Consider two straight grounded conductors AB and ab, the latter distance being small compared with the other dimensions of the system and assume that while ground a is fixed, ground b is rotated about a in a circle of fixed radius ab. This will vary the mutual impedance between the two grounded conductors. The reactance component will vary as the cosine of the angle between ab and AB. The resistance component will also vary as the cosine of an angle, measured, in general, from a direction other than AB. The maximum resistance component will also differ, in general, from the maximum reactance component. The locus of the mutual impedance obtained for all positions of ab will be an ellipse. The ellipse becomes a straight line when ground a lies on the bisecting normal of AB, for then the direction giving the maximum resistance component is parallel to AB and thus the same as for the maximum reactance component. The straight line limit is also obtained when ground a lies on the prolongation of AB in either direction, for the resistance component then has its maximum value in the direction opposite to AB. The maximum resistance component will be perpendicular to AB at the points on Fig. 1 where the C/I contour, if drawn, would be vertical. The locus of these points is

$$y^2=(x^2-b^2)^{2/3}[(x+b)^{2/3}+(x-b)^{2/3}]. \tag{10}$$

At remote points on this locus $y=\pm\sqrt{2}\,x$ and the maximum resistance is negligible compared with the maximum reactance since the circuits are widely separated, while in the neighborhood of A and B it is the maximum reactance which is negligible compared with the maximum resistance. At some intermediate point the two maxima are equal, and, since they occur for directions differing by 90°, the elliptical impedance locus becomes a circle; and the mutual impedance between the two grounded circuits does not change in magnitude as ground b is rotated about ground a. For the assumed constants (1 mile, 1 kilocycle and 1 megohm) this point lies at distances of 1.562 and 0.939 miles from the two terminals A and B, and its four possible locations are shown by the four small crosses on Fig. 1.

If a is rotated counter-clockwise about AB, the direction giving the maximum resistance component also rotates counter-clockwise, making two complete revolutions, while the ground a makes one revolution about AB.

7. Equivalent Ground Plane

To a first approximation the direct-current mutual inductance between two straight conductors AB and ab, forming opposite sides of a rectangle on the surface of the earth, at a separation Aa which is small compared with the length AB is, by (25) of the appendix, neglecting the first and higher powers of $1/s$,[7]

$$N = 2AB \log \frac{\frac{2}{e}AB}{Aa} = 2AB \log \frac{0.736\,AB}{Aa}. \tag{11}$$

This expression has the form ($2\,l \log s/r$) of the commonly employed mutual inductance formula for two long parallel conductors, each of length l, separated by distance r, the common return being a perfectly conducting earth in which the image of each conductor is at the distance s from the other physical conductor. For our direct-current case, therefore, the effective distance to the images is about $\frac{3}{4}$ of the length of either grounded conductor. Since this distance is by assumption large compared with the distance between the conductors, the images are approximately at this same depth below the actual surface of the earth, and the hypothetical perfectly conducting earth would be at one-half this depth, or $\frac{3}{8}AB$. The effective image distance is necessarily directly proportional to the dimensions of the grounded circuits and independent of the earth's resistivity because the shape and relative distribution of the lines of flow are independent of the resistivity and of the length of the grounded circuits. Inspection of Fig. 1 shows that somewhat over $\frac{1}{2}$ of the return flow attains a distance $\frac{3}{4}AB$ from AB, while the remainder of the current remains closer to the grounded conductor.

It may be inquired what would be the effect of confining both return currents to a thin uniform conducting layer on the earth's surface, so that they become horizon return flows. For the closed flows ($\mathscr{X}-\mathscr{H}$) and ($\mathscr{x}-\mathscr{h}$) in general and the particular flows ($\mathscr{R}-\mathscr{H}$) and ($\mathscr{r}-\mathscr{h}$), where $\mathscr{R}$ and $\mathscr{r}$ are close parallel straight conductors,

$$\begin{aligned} N_{(\mathscr{X}-\mathscr{H})(\mathscr{x}-\mathscr{h})} &= N_{\mathscr{X}\mathscr{x}} - N_{\mathscr{X}\mathscr{h}} - N_{\mathscr{H}\mathscr{x}} + N_{\mathscr{H}\mathscr{h}} \\ &= N_{\mathscr{X}\mathscr{x}} - \Delta; \\ N_{(\mathscr{R}-\mathscr{H})(\mathscr{r}-\mathscr{h})} &= N_{\mathscr{R}\mathscr{r}} - 2Ab + 2Aa \\ &= 2AB \log \frac{\frac{2}{e^2}AB}{Aa} = 2AB \log \frac{0.271AB}{Aa}. \end{aligned} \tag{12}$$

[7] If the term $1/s = Aa/AB$ of the expansion is retained the equivalent ground plane has the depth $(AB+Aa)/e$ and thus becomes deeper as ab is moved away from AB. But the equivalent ground plane may be kept fixed at the distance AB/e from AB provided it is tipped at the angle $\sin^{-1}2/e = 47°$ so that ab moves away from the ground plane as it moves away from AB. If it were worth while, still closer approximations might be secured by using a perfectly conducting cylindrical earth of suitable cross-section.

Thus, the assumption that the return current is confined to the earth's surface does not change the order of magnitude of the effective image distance, but reduces it from about $\frac{3}{4}$ to about $\frac{1}{4}$ of the length of the exposure. For space returns the effective image distance is

$$2AB/e^{3/2}=0.446\,AB.$$

Now take another practical case by assuming that the conductor ab is of negligible length compared with its separation r from the parallel conductor AB and the separation is, in turn, negligible compared with the length AB. The formula for dN/dx given in footnote 6 shows that the required inductance depends only on the ratio $AB/(r_1+r_2)$ and is thus constant upon an ellipse. Equivalent expressions in logarithmic form are

$$N=2\,ab\log\left(2\cos\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\frac{\sqrt{r_1r_2}}{y}\right) \qquad (13)$$

$$=2\,ab\log\left(\frac{\cos\frac{1}{2}\theta_1}{\cos\frac{1}{2}\theta_2}\sqrt{\frac{r_1}{r_2}}\right), \qquad (14)$$

or approximately

$$N=2\,ab\log\frac{AB}{r},\ \text{if } ab \text{ is opposite the midpoint of } AB, \qquad (15)$$

$$N=\frac{1}{2}\left[2\,ab\log\frac{AB}{r_2}\right],\ \text{if } ab \text{ is at distance } r_2 \text{ beyond } B.\text{[8]} \qquad (16)$$

Thus, from (13) the effective image distance is never greater than twice the geometrical mean distance from ab to A and B. Its maximum value is approximately AB and occurs when ab is opposite the midpoint of AB. Its minimum value is approximately $\sqrt{r_2AB}$ and occurs when ab is at the distance r_2 from A or B in the prolongation of AB. This makes the inductance one-half of what it is at the symmetrical position. Thus, wherever ab is placed, its mutual inductance with AB lies somewhere between 50 per cent. and 100 per cent. of the mutual inductance, due to an effective image distance AB, ab remaining always parallel to AB and the locus of ab being a rectangle with semicircular ends of radius r_2 and centers A and B.

8. Mutual Impedances of Grounded Circuits Which Depart from the Surface of the Earth

Consider a system of conductors following any paths in space and insulated from the earth except at two grounding points on the sur-

[8] The shortest distance from ab to AB might have been designated by a single letter in place of using y, r and r_2 in formulas (13), (15) and (16).

face of the earth. Any flow of current through this system of conductors may be divided into elementary filaments each of which is made up of segments beginning and ending at the earth's surface and not crossing the earth's surface between these terminals. Ground each segment at both ends. Let $\mathcal{W}$ and $\mathcal{U}$ designate segments having terminals A and B, $\mathcal{W}$ (for example, an open wire circuit) never going below the surface of the earth and $\mathcal{U}$ (for example, an underground cable circuit) never going above the surface of the earth. Add the underground flow $\mathcal{U}$ to Table II at the foot of the left-hand column and, proceeding as before,

$$0 = N(\mathcal{U}-\mathcal{X})(z-a) = N\mathcal{U}_z - N\mathcal{U}_a + \tfrac{1}{2}\Delta = N\mathcal{U}_z + N\mathcal{U}_o - \tfrac{1}{2}\Delta,$$

$$0 = N(\mathcal{U}-\mathcal{X})(h-a) = N\mathcal{U}_h - N\mathcal{U}_a - \tfrac{1}{2}\Delta = N\mathcal{U}_h + N\mathcal{U}_o - \tfrac{3}{2}\Delta,$$

$$\text{or } N\mathcal{U}_o = \tfrac{1}{2}\Delta - N\mathcal{U}_z = \tfrac{3}{2}\Delta - N\mathcal{U}_h,$$

and similarly for the flow $\mathcal{W}$ above ground,

$$N\mathcal{W}_o = \tfrac{1}{2}\Delta + N\mathcal{W}_n = N\mathcal{W}_h - \tfrac{1}{2}\Delta.$$

Hence the three cases which may occur give

$$\left.\begin{aligned} N(\mathcal{W}-\mathcal{E})(w-o) &= N\mathcal{W}_w - N\mathcal{W}_n - N\mathcal{N}_w \\ &= N\mathcal{W}_w - N\mathcal{W}_h - N\mathcal{H}_w + 2\Delta, \end{aligned}\right\} \quad (17)$$

$$\left.\begin{aligned} N(\mathcal{U}-\mathcal{E})(u-o) &= N\mathcal{U}_u + N\mathcal{U}_z + N\mathcal{Z}_u \\ &= N\mathcal{U}_u + N\mathcal{U}_h + N\mathcal{H}_u - 2\Delta, \end{aligned}\right\} \quad (18)$$

$$\left.\begin{aligned} N(\mathcal{W}-\mathcal{E})(u-o) &= N\mathcal{W}_u - N\mathcal{W}_n + N\mathcal{Z}_u \\ &= N\mathcal{W}_u - N\mathcal{W}_h + N\mathcal{H}_u. \end{aligned}\right\} \quad (19)$$

The importance of these equations lies in the fact that the earth return flows are replaced by the simpler nadir, zenith and horizon return flows. *If the conductors comprise only broken straight filaments, making any angles with each other and the earth, the required Neumann integrals, if we use the expressions involving the nadir and zenith returns, are the known expressions between straight filaments.* If the conductors lie in horizontal planes with vertical ground connections, it is convenient to employ the expressions involving the horizon return flows, since the required integral is (4a) derived above. Formulas (33)–(41) of the appendix are the resulting formulas for the three general cases and for a number of important special cases.

In general we may say that the effect of changing the height of one or both conductors by an amount which is small compared with the length of the conductors will be relatively small, since the effective image distance has been shown above to be of the order of the length of the conductors. To illustrate this fact, in Fig. 8 four dotted curves have been added to the curve of Fig. 5 showing the mutual inductances of the two parallel grounded conductors when they are in the

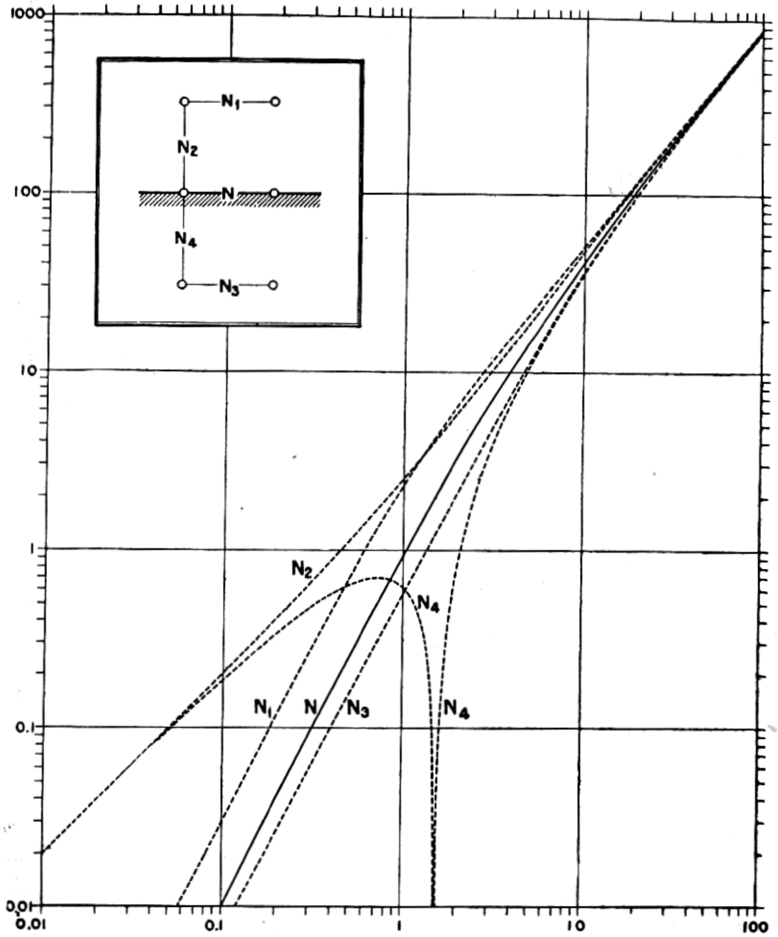

Fig. 8—Mutual Neumann integral between grounded filaments forming opposite sides of a rectangle with unit separation, on the surface of the earth (from Fig. 5) and between the same filaments when one or both of the filaments is raised above or depressed below the surface of the earth as shown by the insert

four positions indicated by N_1, N_2, N_3 and N_4 on the insert of Fig. 8, calculated by formulas (28)–(30) of the appendix, which are special cases of (35), (36), (39) and (40). When the conductors are long, the relative change in the mutual inductance is small. Depressing the

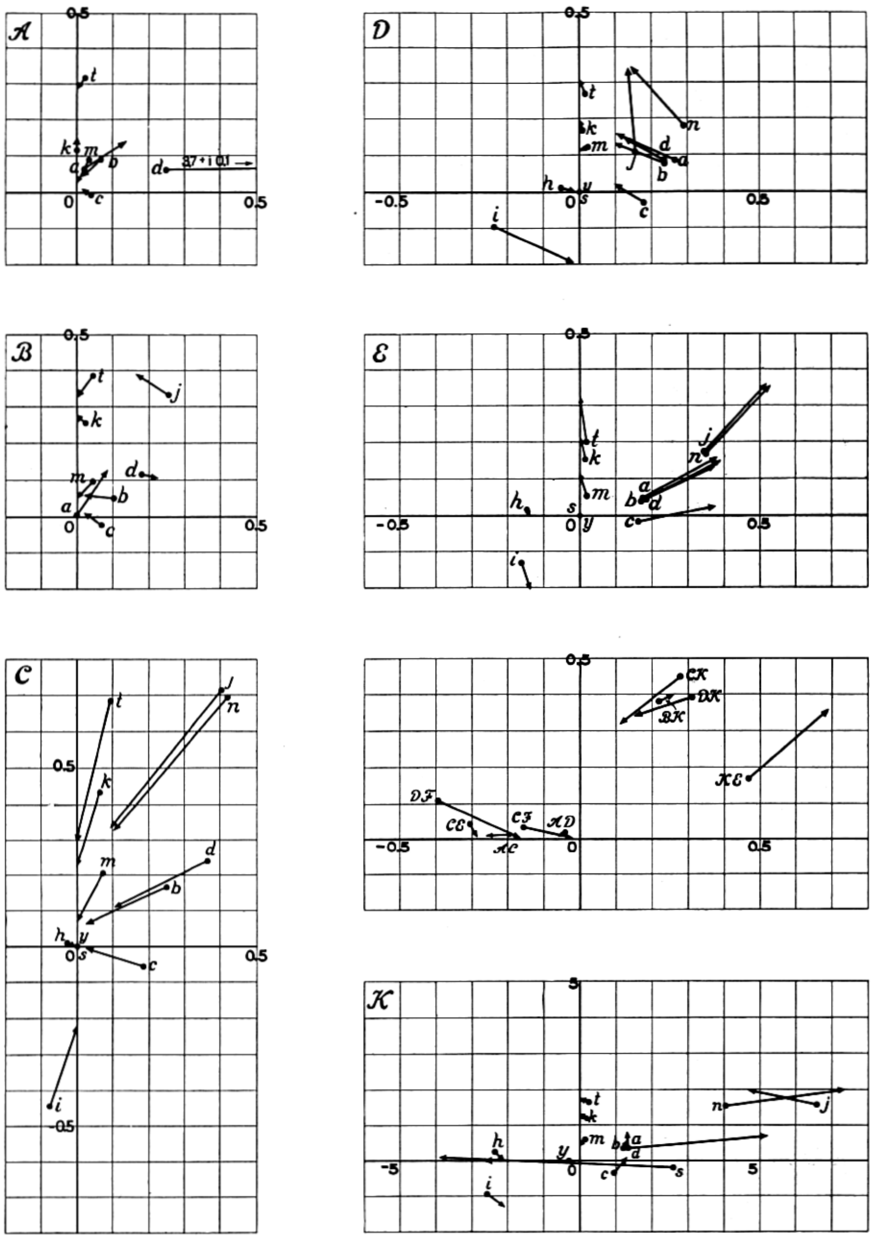

Fig. 9—Comparison of measured and calculated mutual impedances between the indicated circuit pairs. Each arrow extends from the measured impedance located at its tail to the calculated impedance at its head. The location of and positive directions for these circuits are shown by Fig. 10

wires reduces the inductance; the curves show that in the case of N_4 the inductance passes through zero and is reversed in sign when $s=1.560$.

When the departure of the circuits from the earth's surface may be neglected, all terms, but the first, on the right-hand side of formulas (17)–(19) drop out, and each reduces to the simple, fundamental grounded circuit formula (4).

9. Comparison of Theoretical Results with Measurements at 25 and 60 Cycles

Fig. 9 shows, by means of arrows, the impedances which must be added to each of a large group of measured 25-cycle mutual impedances to obtain the results calculated by means of the preceding formulas, on the assumption that the earth has a uniform resistivity

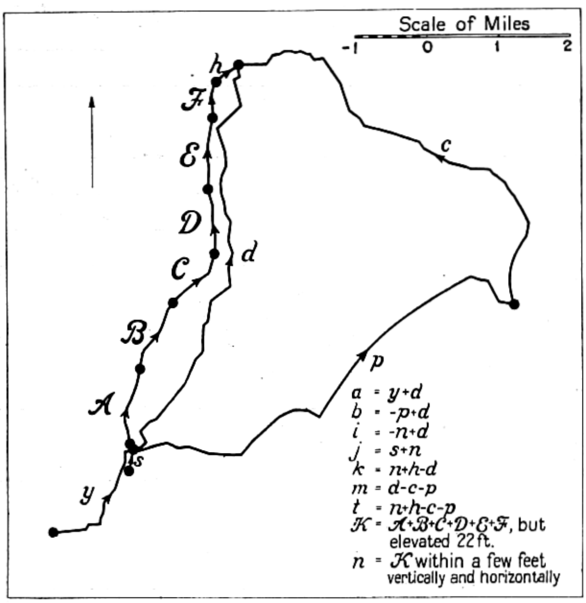

Fig. 10—Location of test circuits, the arrow-heads showing the positive directions in the conductors. When circuits are combined in series, with the removal of intermediate grounds, the new circuit designation is shown by the equations. The test circuits k, m and t are large metallic loops from which all grounds have been removed. The horizontal and vertical displacements of the conductor by a few feet, which render the indicated equalities for $\mathcal{K}$ and n only approximate, were allowed for in determining the calculated results for Fig. 9. The grounds of the capital letter circuits were isolated sections of single track about one mile in length; the midpoint of the section is shown as the effective ground but it may have actually been displaced and have varied with the moisture of the road-bed

of 0.5 megohm per centimeter cube. The measurements were not made for the purpose of this comparison, for which they are not well adapted, but they do give both components of the mutual impedance, which is the absolutely essential requirement. The geometrical irregularity of these circuits is shown by Fig. 10. This was completely allowed for by making detailed computations after substituting an approximately equivalent broken line for each circuit. The variability in the earth's resistivity with location, depth and changing moisture content on different days could not be allowed for. The effect of buried gas and water pipes and of other grounded conductors was also necessarily neglected.

The direct-current theory leaves but one arbitrary constant at our disposal after the frequency and the geometry of the circuits have been fixed. This constant is the earth's resistivity. By trial it was found that 0.5 megohm gave a good average agreement between the calculated resistance components and the entire set of measurements, only a part of which is included in Fig. 9. The individual discrepancies are large but are not so large as to be disconcerting, considering the variations in effective earth resistivity from place to place and from day to day during the progress of the tests.

The calculated reactance component of the mutual impedance based on the direct-current mutual inductance is independent of the earth's resistivity and is uniquely determined by the frequency and geometrical relations. Even a general agreement between the calculated and the measured reactances is significant and Fig. 9 shows not only this, but also a great many good individual agreements. The outstanding discrepancies for circuits $\mathcal{C}$ and $\mathcal{E}$ are systematic, and are apparently to be explained by the effective grounding of these circuits at some other points than the midpoints of the track sections. On the basis of this comparison, it appears that the direct-current theory proves itself adequate to give an approximation to the actual mutual reactances, provided the linear scale and the frequency involved do not greatly exceed those of these tests.

Measurements were also made at 60 cycles. The resistance component remained roughly the same as for 25 cycles; the reactance component doubled as shown by Fig. 11; each component therefore agreeing approximately with the results which would obtain if the direct-current distribution is maintained.

Other comparisons have been made with the same conclusion, but tests should be made, throughout a range of frequencies, at a locality where it is known that the conductivity of the upper layer of the earth's surface is reasonably uniform so that the effect of the lower layers may be determined. Small scale models, having the proper propagation constant, could advantageously be used in determining the alternating-current impedances for uniform earth resistivity or any assigned distribution of resistivity.

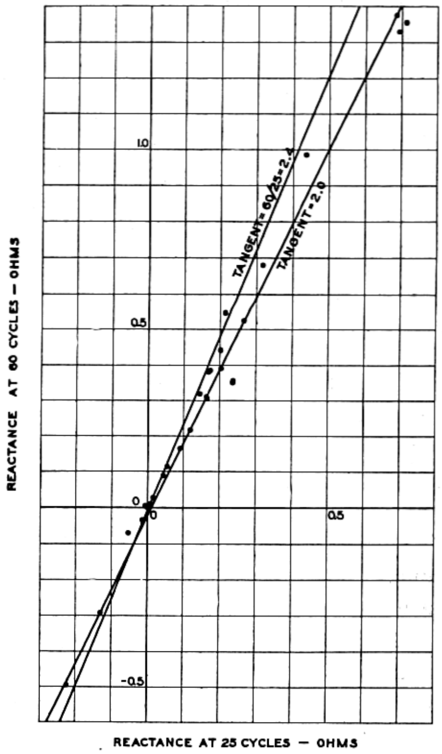

Fig. 11—Comparison of measured mutual reactances at 25 and 60 cycles. A right line with the slope 2.0 fits the point somewhat better than a line with the slope 2.4 which corresponds to the ratio of the frequencies

10. Summary

Formulas for the direct-current mutual resistances and mutual inductances for grounded circuits on the assumption of uniform earth resistivity have been derived and useful diagrams prepared. The applicability of these results as a first approximation to many practical alternating-current cases has been shown.

If, as I hope, this paper is free from ambiguities and errors, it is due to a thorough revision by Mr. R. M. Foster; and I am indebted

to Miss Frances Thorndike for the accuracy attained in the numerous curves of Figs. 1–5 and 8, which should make them of practical value in numerical calculation.

Mathematical Appendix

Additional mathematical results which have been employed in connection with the figures and discussion of this paper are brought together below for convenient reference.

Formulas for Fig. 2

$$2s=\rho+d+1, \text{ if } OA=1,\ Oa=\rho,$$

$$d^2=1+\rho^2-2\rho\cos\theta,$$

$$\sin\theta=d\sin(\theta+\phi),$$

$$\sin\phi=\rho\sin(\theta+\phi),$$

$$\cos\theta=(2s-1)-2s(s-1)/\rho,$$

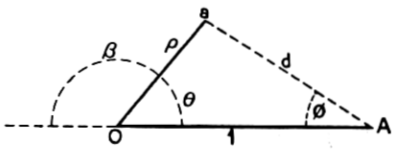

Fig. 12

$$\frac{N\mathcal{R}_r}{AB}=f(\rho,\theta)=\rho f(\rho^{-1},\theta)$$

$$=\cos\theta\left(\log\frac{s}{s-\rho}+\rho\log\frac{s}{s-1}\right) \tag{20}$$

$$=\cos\theta\left\{\log\frac{\tan\frac{1}{2}(\theta+\phi)}{\tan\frac{1}{2}\theta}+\rho\log\frac{1}{\tan\frac{1}{2}\theta\tan\frac{1}{2}\phi}\right\} \tag{21}$$

$$=2\cos\theta\log\frac{2+d}{d}=2\cos\theta\log(1+1/\sin\tfrac{1}{2}\theta), \text{ if } \rho=1,$$

$$=\tfrac{1}{2}\rho\log\frac{2+\rho}{2-\rho}+\tfrac{1}{2}\rho^2\log\frac{2+\rho}{\rho}, \text{ if } d=1,$$

$$=\cos\theta\left[\rho\log\frac{4}{\rho\theta^2}-(1-\rho)\log(1-\rho)\right]$$
$$+\frac{\rho(1+2\rho)}{12(1-\rho)}\theta^2+\ldots, \text{ for } \rho<1,$$

$$=\cos\beta[\rho\log\rho-(1+\rho)\log(1+\rho)]-\frac{\rho}{4(1+\rho)}\beta^2$$
$$+\frac{\rho(11+19\rho+11\rho^2)}{96(1+\rho)^3}\beta^4+\ldots,$$

$$\theta=\frac{2}{\sqrt{\rho}}e^{-[N+(1-\rho)\log(1-\rho)]/2\rho}+\ldots, \tag{22}$$

$$R=\frac{2\rho^2 \log\,[(1+\rho)/\rho]}{2\log(1+\rho)-\rho/(1+\rho)} = \text{(radius of curvature when } \theta=\pi) \tag{23}$$

$=0,\ 1.564,\ \infty$ at $\rho=0,\ 1,\ \infty$, which checks the sharp curvature of the curves which cross the axis just to the left of the origin.

Formulas for Figs. 4 and 5

$$s=AB/Aa,$$

$$\frac{N\mathcal{R}_r}{Aa}=2[s \log\,(s+\sqrt{s^2+1})+1-\sqrt{s^2+1}] \tag{24}$$

$$\left.\begin{aligned}&=2s\left[\log 2s-1+\frac{1}{s}-\frac{1}{4s^2}+\frac{1}{32s^4}-\frac{1}{96s^6}+\ldots\right]\\&=2s\log\left[\frac{2s}{e}\left(1+\frac{1}{s}+\frac{1}{4s^2}-\frac{1}{12s^3}-\frac{1}{48s^4}+\ldots\right)\right]\end{aligned}\right\} \tag{25}$$

$$=s^2\left[1\quad\frac{s^2}{12}+\frac{s^4}{40}-\frac{5s^6}{448}+\ldots\right], \tag{26}$$

$$Q\mathcal{R}_r(Aa)\left(\frac{2\pi}{\rho}\right)=2-\frac{2}{\sqrt{s^2+1}}, \tag{27}$$

$$\frac{N_1}{Aa}=\frac{N\mathcal{R}_r}{Aa}+2\log\,(s^2+1), \tag{28}$$

$$\frac{N_2\ (\text{or } N_4)}{Aa}=\frac{N\mathcal{R}_r}{Aa}\pm 2[\log \tfrac{1}{2}(1+\sqrt{s^2+1})-\sqrt{s^2+1}+s+1], \tag{29}$$

$$\frac{N_3}{Aa}=\frac{N\mathcal{R}_r}{Aa}+2\left[\log\frac{(s^2+1)\ (\sqrt{2}+1)^4}{(1+\sqrt{s^2+2}\ )^4}+4(\sqrt{s^2+2}-\sqrt{s^2+1}-\sqrt{2}+1)\right]. \tag{30}$$

Formulas for the Mutual Inductance Between Any Flows in Two Horizontal Planes Grounded by Vertical Filaments

Let the arbitrary flows be $\mathcal{X}'$ and x' between points A', B' and a', b' in the two horizontal planes, grounded by vertical filaments connecting these four terminals with the points A, B, a, b on the surface of the earth. In order to indicate briefly which of these eight points are involved in each term of the result, we imagine a vertical line which cuts the horizontal planes in the points P', p', P, p, where P and p are the same point on the surface of the earth, since the nonprimed points are all in this plane, and we agree that A' or A occurs in a term, according as P' or P is found in the subscript of the symbol Δ or Γ used to designate the term, where

$$\Delta_{P'p'} = -A'a' + A'b' + B'a' - B'b', \tag{31}$$

$$\Gamma_{P'p'} = \log \frac{(A'b'+P'p')(B'a'+P'p')}{(A'a'+P'p')(B'b'+P'p')}. \tag{32}$$

In these expressions every distance between points, such as $A'b'$, $P'p'$, is a positive quantity. The formulas below are perfectly general, but require the assignment of the capital letters $\mathcal{X}'$, A', B', P' to the upper plane when both flows are above the earth and to the lower plane when both flows are below the earth. They are most readily checked by employing formulas (4a) and (24) in formulas (17), (18) and (19). The results show that the mutual inductance is equal to the Neumann integral between $\mathcal{X}'$ and x' augmented by terms which depend only upon the arithmetical distances between the eight points A', B', a', b', A, B, a, b.

$$N(\mathcal{W}-\mathcal{E})(w-e) = N\mathcal{X}'x' + P'p'\Gamma_{P'p'} + 2Pp'\Gamma_{Pp} - \Delta_{P'p'} + \Delta_{Pp}, \quad \text{where } P'p \geq Pp', \tag{33}$$

$$= N\mathcal{X}'x' + 2Z \log \frac{(Ab)(Ba)}{(Aa)(Bb)}, \text{ if } P' \text{ and } p' \text{ are both at height } Z, \tag{34}$$

$$= N\mathcal{X}'x' + 2Z \log (1+s^2), \text{ if } A'B'b'a' \text{ is a horizontal rectangle and } AB = s(Aa), \tag{35}$$

$$= N\mathcal{X}'x' + 2Z[\log \tfrac{1}{2}(1+\sqrt{1+s^2}) + 1 - \sqrt{1+s^2} + s], \text{ if } A'B'b'a' \text{ is a vertical rectangle with one side on the earth and } AB = s(A'a) = sZ, \tag{36}$$

$$N(\mathcal{U}-\mathcal{E})(u-e) = N\mathcal{X}'x' + P'p'\Gamma_{P'p'} - 2P'p\Gamma_{P'p} - 2Pp'(\Gamma_{Pp'} - \Gamma_{Pp}) - \Delta_{P'p'} + 2\Delta_{P'p} + 2\Delta_{Pp'} - 3\Delta_{Pp}, \quad \text{where } P'p \geq Pp', \tag{37}$$

$$= N\mathcal{X}'x' - 2Z \log \frac{(Aa)(Bb)(Ab'+Z)^2(Ba'+Z)^2}{(Ab)(Ba)(Aa'+Z)^2(Bb'+Z)^2} + 4(\Delta_{P'p} - \Delta_{Pp}), \text{ if } P' \text{ and } p' \text{ are both at distance } Z \text{ below the earth,} \tag{38}$$

$$= N\mathcal{X}'_{x'} + 2(Aa)t \log \frac{(1+s^2)(t+\sqrt{1+t^2})^4}{(t+\sqrt{1+s^2+t^2})^4}$$

$$+8\,(Aa)\,(\sqrt{1+s^2+t^2}+1-\sqrt{1+s^2}$$

$$-\sqrt{1+t^2}),$$ if $A'B'b'a'$ is a horizontal rectangle at the distance $Z=t(Aa)$ below the earth and $AB=s(Aa)$, (39)

$$= N\mathcal{X}'_{x'} - 2Z\,[\log \tfrac{1}{2}(1+\sqrt{1+s^2}) - \sqrt{1+s^2}+1+s],$$ if $A'B'b'a'$ is a vertical rectangle with one edge at the surface of the earth and $AB=ab=s(A'a)=sZ$, (40)

$$N(\mathcal{W}-\mathcal{E})_{(u-o)} = N\mathcal{X}'_{x'} + P'p'\Gamma_{P'p'} - 2Pp'\Gamma_{Pp'} - \Delta_{P'p'} + 2\Delta_{Pp'} - \Delta_{Pp}. \quad (41)$$