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Impedance of Smooth Lines, and Design of Simulating Networks

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Introduction

THE transmission of alternating currents over any transmission line between specified terminal impedances depends only on the propagation constant and the characteristic impedance of the line (at the particular frequency contemplated). In this sense, then, the properties of transmission lines may be classed broadly as propagation characteristics and impedance characteristics. In telephony we are primarily concerned with the dependence of these characteristics on the frequency, over the telephonic frequency range.

Prior to the application of telephone repeaters to telephone lines the propagation characteristics of such lines were more important than their impedance characteristics, because the received energy depended much more on the former than on the latter.

The application of the two-way telephone repeater greatly altered the relative importance of these two characteristics, decreasing the need for high transmitting efficiency of a line but greatly increasing the dependence of the results on the impedance of the line. As well known, this is because the amplification to which a two-way repeater can be set without singing, or even without serious injury to the intelligibility of the transmission, depends strictly on the degree of impedance-balance between the lines or between the lines and their balancing-networks. In the case of the 21-type repeater the two lines must have such impedances as to closely balance each other throughout the telephonic frequency range. In the case of the 22-type repeater, which for long lines requiring more than one repeater is superior to the

21-type, impedance networks are required for closely balancing the impedances of the two lines throughout the telephonic frequency range. Such balancing networks are necessary also in connection with the so-called four-wire repeater circuit.[1]

Smooth lines are fundamental in telephonic transmission; for any telephone line is either a simple smooth line, or a compound smooth line, or a periodically loaded line whose sections are themselves short smooth lines. In any case the characteristic impedances of the constituent smooth lines enter importantly into the impedance of the system. Moreover, the characteristic impedance of the series type of periodically loaded line, at frequencies low relatively to its critical frequency, is closely the same as the characteristic impedance of the corresponding smooth line.

Parts I, II, and III of this paper aim to present in a simple yet comprehensive manner the dependence of the characteristic impedance of the various types of smooth lines on the frequency and on the line constants, by means of description accompanied by equations transformed to the most suitable forms and by graphs of such equations.

Part IV describes the principal networks devised by the writer at various times within about the last ten years, for simulating the impedance of the various types of smooth lines. Of course, the impedance of any line could be simulated, as closely as desired, by means of an artificial model constructed of many short sections each having lumped constants; but such structures would be very expensive and very cumbersome. Compared with them the networks described in this paper are very simple non-periodic structures that are relatively inexpensive and are quite compact; yet the more precise of them have proved to be adequate for simulating with high precision the impedance of most types of smooth lines, while even the least precise (which are the simplest) suffice for a good many applications. The paper includes first approximation design-formulas and outlines a supplementary semi-graphical method for arriving at the best proportioning of the networks. A typical illustrative example is worked out in Appendix E.

It is hoped to devote a succeeding paper to the impedance characteristics of periodically loaded lines, and to various networks devised for simulating and compensating such impedance.

[1]Regarding the broad subject of repeaters and repeater circuits, reference may be made to the paper by Gherardi and Jewett: "Telephone Repeaters," A. I. E. E. Trans., 1919, pp. 1287–1345.

PART I

GENERAL CONSIDERATIONS PERTAINING TO SMOOTH LINES

The exact formula for the characteristic impedance K of any smooth line is usually written in the form

$$K=\sqrt{\frac{R+i\omega L}{G+i\omega C}} \tag{1}$$

R, G, L and C denoting, as usual, the fundamental line constants,[2] namely, the resistance, leakance (leakage conductance), inductance, and capacity, per unit length; ω denoting 2π times the frequency f; and i the imaginary operator $\sqrt{-1}$.

However, this form is neither the simplest nor the most significant. For it involves separately the four quantities R, G, L, and C and is thus a function of not less than four[3] variables, whereas its value evidently depends on only the relative values of these quantities and hence must be expressible as a function of only three independent variables—namely the ratios of any three of them to the fourth.

In deciding just what form or forms of expression to adopt for K we shall here be guided by the following practical considerations:

(A) In telephony we are chiefly interested in the dependence of K on the frequency f; or, stated more generally, in the dependence of some quantity that is approximately proportional to K on some quantity that is approximately proportional to f.

The class of smooth lines is comprised between the following two rather wide extremes, having very different characteristics:

(B) At one extreme are the large gauge open-wire lines, particularly when used at high frequencies. For them R is small relatively to ωL, and G relatively to ωC; and hence K is approximately or at least roughly equal to $\sqrt{L/C}$.

(C) At the other extreme are the small gauge cables, particularly when used at low frequencies. For them R is large relatively to ωL, though G is small relatively to ωC; and hence K is approximately or at least roughly equal to $\sqrt{R/i\omega C}$.

(D) The line constants R, L, C do not change much with frequency over at least the voice frequency range; and hence they, or combinations of them, serve suitably as parameters.

(E) The leakance G, which is nearly always the least important of the four line constants, usually varies greatly with the frequency

[2] Constants as to current and voltage.

[3] Five if ω is regarded separately from L and C.

and hence by itself does not serve very suitably as a parameter. However, in a wide range of applications G is approximately or at least roughly proportional to the frequency; and then a suitable parameter is G/f or preferably $G/\omega C$. This is true of cables except at extremely low frequencies. It is at least roughly true of open-wire lines at very high frequencies, such as carrier frequencies, but usually not at voice frequencies. For most lines the leakance G is usually approximately or at least roughly a linear function of the frequency, namely, $G=G_0+\nu f$, where G_0 is the leakance at $f=0$, and ν is approximately independent of the frequency. For cables, G_0 is small compared with νf except at very low values of f; but for open-wire lines G_0 is usually not negligible except at high values of f.

In the light of these considerations a study of equation (1) suggests the employment of the quantities F, E, k, g, a, b defined by the following six equations. Not all of these substitutions will be employed simultaneously, but it is convenient to set them all down here together.

$$F=\omega L/R, \quad (2) \qquad E=\omega C/R, \quad (3)$$
$$k=\sqrt{L/C}, \quad (4) \qquad g=\sqrt{G/R}, \quad (5)$$
$$a=GL/RC, \quad (6) \qquad b=G/\omega C. \quad (7)$$

Usually F or E will be treated as the independent variable; and k, g, a, b as parameters.

It should perhaps here be emphasized that the approximations mentioned in the foregoing set of five considerations, (A) to (E), are employed merely as guides in the selection of the variables and parameters defined by the above equations (2) to (7), and in the choice of the forms adopted below for the formula for the characteristic impedance. Except where the contrary is definitely indicated, the formulas that will be adopted for the characteristic impedance are rigorously exact; though the variables F and E are never exactly proportional to the frequency, and the parameters k, g, a, b are never exactly independent of the frequency. If the independant variables were exactly proportional to the frequency and the parameters were exactly independent of the frequency, the graphs of the formulas would by a mere change of scales exactly represent the impedance as an explicit function of the frequency.

With particular regard to considerations (B) and (C) it will be found convenient to divide the further treatment of smooth lines into two main parts, pertaining to open-wire lines and to cables respectively; and then, in each of those parts, to present the impedance formulas in the two forms respectively most suitable for the cases where the leakance is approximately proportional to the frequency and approximately independent of the frequency, corresponding to consideration (E).

While the classification of smooth lines into open-wire lines and cables is convenient, there is, of course, no very sharp distinction between the open-wire type of lines and the cable type of lines, since the distinction depends on the line parameters and on the frequency range involved, rather than on the physical form of the line; for, any line at sufficiently high frequencies has the open-wire type of characteristics, and at sufficiently low frequencies the cable type of characteristics. With regard to the relative importance of the fundamental line constants R, G, L, C when the frequency range is that of the voice, it may be said that for the open-wire type of lines L and C are of about equal importance, R of secondary, and G of tertiary importance; while, on the other hand, for the cable type, R and C are of about equal importance, L of secondary, and G of tertiary importance. In illustration of the above remarks it may be noted that smoothly loaded cables (unless loaded very lightly) have the open-wire type of characteristics; as have also periodically loaded cables at low frequencies.

Before proceeding to the separate treatments of open-wire lines and cables, it seems desirable to indicate the general nature of the effect produced on the impedance by leakance.

The General Effect of Leakance

The amount of leakance that is normally allowable as regards its attenuating effects is so small as to produce only very slight effects on the characteristic impedance of either type of line (except at very low frequencies).

In ordinary telephone cables the leakance is so small that, except at very low frequencies, the impedance of such cables is very closely the same as in the limiting case of no leakance; whence that limiting case may be taken as being a good approximation to the actual case. In open-wire lines leakance may be much larger than in cables, yet normally it is small enough so that its effects on the impedance are slight, except at very low frequencies, so that usually the limiting value of zero leakance is still a good approximation when calculating the characteristic impedance. However, during wet weather and in particularly humid climates and locations the leakance in open-wire lines becomes large enough to affect the impedance quite appreciably, even within the voice frequency range, while enormously affecting it at very low frequencies.

The general nature of the effect produced on the characteristic impedance K by any value of the leakance G is readily seen from mere inspection of equation (1), so far as regards the absolute value and angle of the impedance. Thus, increasing G from any initial value decreases the absolute value of the impedance and (algebraically) increases the angle. Starting with $G=0$, the angle is negative and has the value $-\frac{1}{2}\tan^{-1}\frac{R}{\omega L}$; increasing G decreases this negative angle until G has become as large as RC/L, when the angle has become zero and the impedance has become equal to the simple value $\sqrt{L/C}$, and thereby equal also to $\sqrt{R/G}$. Increasing G beyond this transition value RC/L toward infinite values gives to the impedance a positive angle which continually increases toward its limiting value $\frac{1}{2}\tan^{-1}\frac{\omega L}{R}$, while the absolute value of the impedance goes on continually decreasing toward its limiting value of zero.

The statements in the foregoing paragraph hold at all frequencies, though the effects of leakance are usually most pronounced at low frequencies. In fact at zero frequency the characteristic impedance of a line having any finite leakance however small is merely $\sqrt{R/G}$; and at frequencies so low that ωL is small compared with R and ωC small compared with G, the impedance K is, approximately,

$$K=\sqrt{\frac{R}{G}}\left\{1+i\omega\left(\frac{L}{2R}-\frac{C}{2G}\right)\right\}$$

and hence is at least roughly equal to $\sqrt{R/G}$.

Of course, with actual lines the whole physically possible range of variation of G from zero to infinite values is never traversed. On the contrary the leakance G even in open-wire lines seldom reaches a value as large as the transition value RC/L and hence the angle seldom becomes positive; while in cables the angle probably always remains negative and indeed is at least roughly equal to its limiting value of $-\frac{1}{2}\tan^{-1}\frac{R}{\omega L}$, except at very low frequencies.

Although, as already indicated, the effects of normal amounts of leakance are usually very small for both cables and open-wire lines, yet the effects in the two cases differ rather markedly in their nature, owing to the difference in the angles of the impedances of these two types of lines; the angle of cables begin almost $-45°$, while that of open-wire lines, though likewise negative, is much smaller (except at very low frequencies).

To formulate analytically the effects of any value of the leakance G, let K_0 denote the value of K when $G=0$, and let ΔK denote the increment $K-K_0$ due to the presence of the leakance G. A suitable measure of the effect of the leakance is then the ratio $\Delta K/K_0$. In order to obtain for the value of this ratio a formula which will be convenient for use with the formula for K (which involves a radical) it is advantageous to write ΔK in the form $\Delta K=(K^2-K_0^2)/(K+K_0)$, and to introduce for brevity the quantity μ defined by the equation

$$\mu=K^2/K_0^2-1. \tag{7.1}$$

This procedure leads readily to the following simple identity for $\Delta K/K_0$, namely

$$\frac{\Delta K}{K_0}=\frac{\mu}{1+\sqrt{1+\mu}}=\sqrt{1+\mu}-1. \tag{7.2}$$

In particular, when this is applied to the formula (1) for K, the value of μ is found to be

$$\mu=\frac{iG/\omega C}{1-iG/\omega C}. \tag{7.3}$$

Equations (7.2) and (7.3) enable the exact value of $\Delta K/K_0$ to be calculated for any value of $G/\omega C$. For small values of $G/\omega C$, the formula for $\Delta K/K_0$ takes the very simple approximate form

$$\Delta K/K_0=iG/2\omega C; \tag{7.4}$$

and this shows that, to the degree of approximation involved, ΔK is proportional to iK_0 through a proportionality factor $(G/2\omega C)$ which is real and positive. Now, for cables, K_0 has an angle of nearly $-45°$; and hence, by (7.4), it is seen that the addition of small leakance increases the resistance component and decreases the negative-reactance component of the impedance by about equal amounts. For open-wire lines, on the other hand, the angle of K_0 is much smaller, though negative, and hence a small increase in the leakance changes the reactance component of the impedance much more than it does the resistance component; evidently, the change in the negative-reactance component is a decrease, but the change in the resistance component may be of either sign, depending on the frequency. This fact regarding the effect of leakance on the resistance component of the impedance is not completely represented by the approximation (7.4)—which indicates the change as being always an increase—but it can be inferred from a study of the exact formulas (7.2) and (7.3). These would have to be employed also if the effects of large leakance were being studied—or, more generally, large $G/\omega C$. If it were not for a need of these exact formulas, the approximate formula (7.4) would have been derived by the mere application of Taylor's theorem to (1).

Part II

Impedance of Open-Wire Lines

It will be recalled that the characteristic impedance of an ordinary open-wire line depends primarily on its inductance and its capacity, only secondarily on its resistance, and far less still on its leakance; and hence that its impedance is at least roughly equal to $\sqrt{L/C}$.

Of the quantities defined by equations (2), . . . (7), the four most suitable for describing open-wire lines are F, k, b, and a. F is suitable as the independent variable, approximately proportional to the frequency. k is suitable as one parameter. For the other parameter, which evidently must involve the leakance, b or a respectively is the most suitable according as the leakance G is approximately proportional to the frequency or is approximately independent of the frequency. The corresponding suitable forms of the equation for the characteristic impedance K are then

$$K=k\sqrt{\frac{1+iF}{(b+i)F}}, \qquad (8) \qquad\qquad K=k\sqrt{\frac{1+iF}{a+iF}}. \qquad (9)$$

The quantity $k=\sqrt{L/C}$ which occurs in (8) and (9) as a mere factor is significant as being the value that the impedance approaches when the frequency is indefinitely increased[4]; it is also the value the impedance would have at all frequencies if, without changing L and C, the line could be rendered non-dissipative. For ordinary open-wire lines at voice frequencies ($R/\omega L$ small or fairly small compared to unity) it is at least a rough approximation to the value of the impedance. This limiting value $k=\sqrt{L/C}$ will be termed the " nominal impedance " or, more fully, the " nominal characteristic impedance." [5]

The amount $K-k$ by which the characteristic impedance K exceeds the nominal characteristic impedance k will be termed the " excess impedance," and hence its two components the " excess resistance " and the " excess reactance "; (or, more fully, for the three: the " excess characteristic impedance," " excess characteristic resistance," and " excess characteristic reactance," respectively). The latter two

[4] Provided that G/f approaches zero.

[5] Strictly speaking, k varies slightly with the frequency, because of the variations of L and even C.

have respectively the values $M-k$ and N, since k is real; M and N denoting the resistance and reactance components of K. The concept "excess impedance" will be found convenient in various connections, particularly in the description of networks for simulating the impedance of smooth lines.

The ratio K/k of the characteristic impedance K to the nominal impedance k will be termed the "relative impedance" and will be denoted by $z=x+iy$; whence $x=M/k$ will be termed the "relative resistance," and $y=N/k$ the "relative reactance." This complex number z is roughly equal to unity over most of the voice frequency range, and approaches unity as a limit when F is indefinitely increased. Its exact value, written in the two forms corresponding to (8) and (9) respectively, is

$$z=\sqrt{\frac{1+iF}{(b+i)F}}, \qquad (10) \qquad\qquad z=\sqrt{\frac{1+iF}{a+iF}}. \qquad (11)$$

Thus z, which is proportional to the characteristic impedance K (except for the fact that the proportionality factor k is not strictly independent of the frequency), depends merely on the two quantities F and b, or F and a, and hence can be readily represented by tables or graphs.

When z has once been tabulated or graphed the value of K in any specific case (R, G, L, C specified) is readily obtained therefrom by entering such tables or graphs of z with the values of the arguments $F=\omega L/R$ and $b=G/\omega C$ of (10) or the arguments $F=\omega L/R$ and $a=GL/RC$ of (11), and then multiplying the value of z there found by $k=\sqrt{L/C}$. (Graphically this would amount merely to a change of scales if the parameters employed were strictly independent of the frequency.) Thus the function

$$z=\sqrt{(1+iF)/(b+i)F}=\sqrt{(1+iF)/(a+iF)}$$

represents simply and comprehensively the properties of the characteristic impedance of all smooth lines, though it is more suitable for representing open-wire lines than cables.

The two components x and y of z are represented as functions of F by the curves in Figs. 1 and 2 with b and a respectively as parameters. (Explicit formulas for x and y are included in Appendix A.)

The effects produced on $z=x+iy$ by the leakance G are exhibited, in Figs. 1 and 2, through the parameters b and a. These effects may be conveniently represented analytically in a manner formally the same as that already outlined in connection with equations (7.1) and

(7.2); with K, K_0, ΔK (there) corresponding to z, z_0, Δz (here). Thus, by applying (7.1) to (10) and (11),

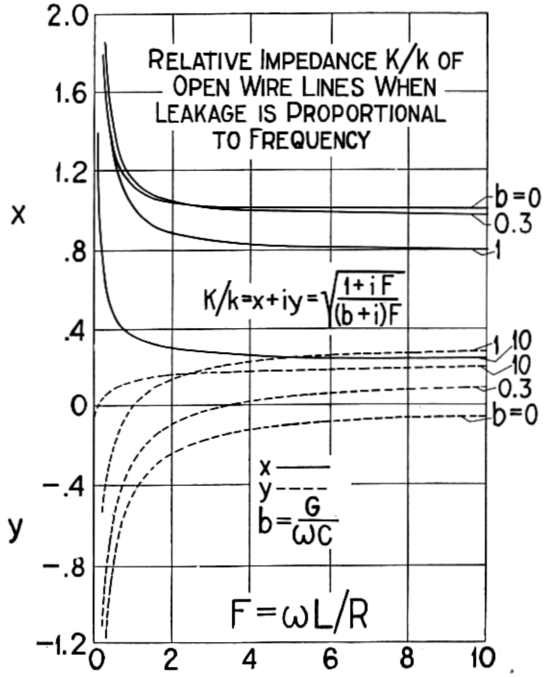

Fig. 1

$$\mu = ib/(1-ib) = ia/(F-ia).$$

Or, approximately, when b and a are small,

$$\mu = ib = ia/F;$$

and thence, approximately, by (7.2),

$$\Delta z/z_0 = ib/2 = ia/2F.$$

This analysis serves to account, approximately, for the nature of the effects of small leakance, as depicted in Figs. 1 and 2 by the curves for small b and small a. To account for the effects of large leakance, as depicted by the curves for large b and large a, recourse to the exact formula for $\Delta z/z_0$ would be necessary; but the curves for large leakance possess hardly more than academic interest, as will be realized from the remarks already made under the heading *The General Effect of Leakance*.

When there is no leakance ($G=0$, and hence $b=0$ and $a=0$) equations (10) and (11) reduce to the same form, namely

$$z=\sqrt{1-i/F}. \tag{12}$$

This limiting form of the equation for the relative impedance z is rather important because it is comparatively simple and yet is a close approximation for the impedance of most actual lines except at very low frequencies (since the effects of normal amounts of leakance are

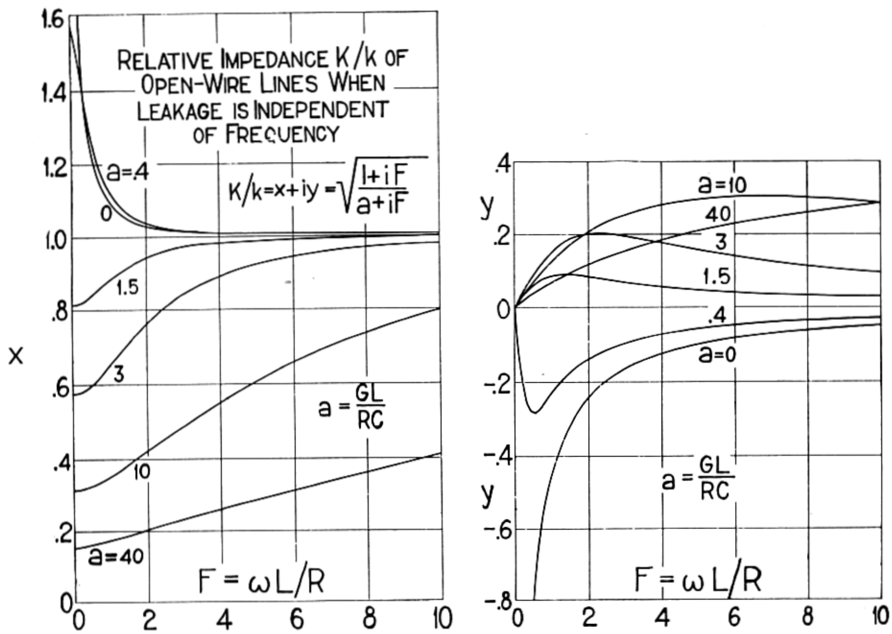

Fig. 2

very small except at very low frequencies). It will therefore now be discussed with some fullness:

For the case of no leakance the formulas for x and y are given under equation (12) in Appendix A; and are graphed in Fig. 1, ($b=0$), and in Fig. 2, ($a=0$). If the wires were devoid of resistance ($R=0$), x would be equal to unity and y would be zero. Thus the effect of wire resistance (in a non-leaky line) is to make x greater than its limiting value unity by the amount $x-1$ (the "relative excess resistance"), and to introduce a negative value of y (the "relative excess reactance," which is equal to the "relative reactance"). Both $x-1$ and $-y$ increase with decreasing F; the increase being slow at large values of F, but more and more rapid as F is decreased. $x-1$

is always smaller than $-y$; and is much smaller except at low values of F, where the two approach equality as F approaches zero. The statements regarding x and y hold also for the effect of wire resistance on the characteristic resistance and the characteristic reactance, since these are (approximately) proportional to x and y respectively, the proportionality factor being the nominal impedance $\sqrt{L/C}$.

Before leaving equation (12) attention will be directed to certain approximate and exact forms of this equation that have been found very useful in devising and proportioning networks for simulating the characteristic impedance of smooth lines, as will appear more fully in the latter part of this paper. At large values of F equation (12) yields immediately the approximation

$$z=1+\frac{1}{8F^2}-i\frac{1}{2F}, \tag{13}$$

whence $x-1$ and y have approximately the values

$$x-1=\frac{1}{8F^2}, \tag{14}$$ $$y=-\frac{1}{2F}. \tag{15}$$

From equation (12) of Appendix A the exact values of $x-1$ and y are known to be

$$x-1=\frac{1}{8F^2}\,\frac{2}{(x+1)x^2}, \tag{16}$$ $$y=-\frac{1}{2Fx}. \tag{17}$$

Thus it is seen that each of the approximations (14) and (15) is always somewhat larger than the exact value, since x is always greater than unity. However, these two approximations are fairly good for values of F as small even as unity, since there x does not exceed 1.1; and they rapidly approach exactness when F is increased, since x rapidly approaches unity. The exact equation for z will now be set down for purposes of reference; by (16) and (17) it is

$$z=1+\frac{1}{8F^2}\,\frac{2}{(x+1)x^2}-i\frac{1}{2Fx}. \tag{18}$$

At small values of F formula (12) shows that z is approximately equal to $z''=x''+iy''$, defined by the equation $z''=1/\sqrt{iF}$. The exact value of the fractional departure $(z-z'')/z''$ is

$$\frac{z-z''}{z''}=\frac{iF}{1+\sqrt{1+iF}}, \tag{18.1}$$

which, at small F, is approximately equal to $iF/2$ merely. Thus, at small F, z exceeds its approximate value z'' by an amount which is proportional to iz'', through a proportionality factor $(F/2)$ which is real and positive; since the angle of z'' is $-45°$ it follows that x is greater than x'' by about the same amount that $-y$ is less than $-y''$. This analysis serves to account for the shape of the curves of x and y at small values of F and no leakance (the curves $b=0$ in Fig. 1, and $a=0$ in Fig. 2). The shape of the curves at any value of F can be accounted for by means of the exact formula (18.1), or a suitable approximation thereof. In fact formula (18.1) shows immediately that

$$x-x''>(-y'')-(-y)$$

and that this inequality increases with F.

Part III

Impedance of Cables

It will be recalled that the impedance of an ordinary cable depends chiefly on its capacity and resistance, relatively little on its inductance, and far less still on its leakance; and hence that its impedance is at least roughly equal to $\sqrt{R/i\omega C}=(1-i)\sqrt{R/2\omega C}$.

Of the quantities defined by equations (2), . . . (7), the four most suitable for describing cables are E, k, b and g. E is suitable as the independent variable, approximately proportional to the frequency. k is suitable as one parameter. For the other parameter, which evidently must involve the leakance, b or g respectively is the most suitable according as the leakance G is approximately proportional to or approximately independent of the frequency. The corresponding suitable forms of the equation for the impedance are then

$$K=\sqrt{\frac{1+ik^2E}{(b+i)E}}, \qquad (19) \qquad\qquad K=\sqrt{\frac{1+ik^2E}{g^2+iE}}. \qquad (20)$$

These two formulas (19) and (20) for cables are less simple than the corresponding formulas (8) and (9) for open-wire lines, because in (19) and (20) neither of the two parameters enters as a mere factor, and hence the number of effective parameters cannot be reduced to less than two. For purposes of mere specific computations this is not much of a complication; but in graphical representation it is enough to prevent the desired simplicity and compactness, if the representation is required to be exact and comprehensive. (Explicit formulas for the two components M and N of K are included in Appendix A.)

The effects of the leakance G, as represented through the medium of the parameters b and g, may if desired be conveniently formulated and analyzed in a manner formally the same as that already outlined under the heading *The General Effect of Leakance*.

In cables, particularly, the effect of leakance is usually extremely small except at very low frequencies. Hence in the graphical representation of formulas (19) and (20) it will suffice very well to confine ourselves to the limiting case of no leakance ($G=0$, and hence $b=0$ and $g=0$), when these two equations reduce to the same form, namely

$$K=\sqrt{k^2-i/E}. \tag{21}$$

The curves in Fig. 3 represent the resistance and reactance components M and N of K as functions of E with k as parameter.

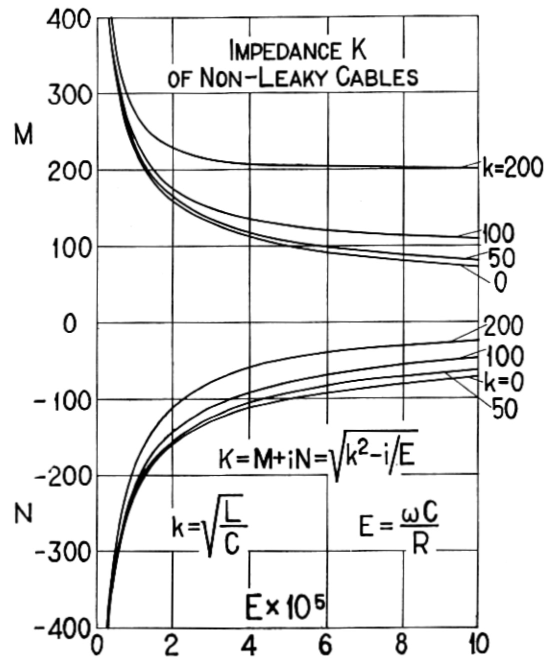

Fig. 3

The effects produced on $K=M+iN$ by the inductance L are exhibited, in Fig. 3, through the parameter k. These effects may be conveniently represented analytically in a manner formally the same as that already employed for the effects of leakance, under the heading *The General Effect of Leakance*. Thus, if K' denotes the value of K, expressed by (21), when $k=0$, then K' here will correspond to K_0 there; hence $\mu=ik^2E$, and thence, when k^2E is small compared to unity,

$$\Delta K/K' = ik^2E/2.$$

This analysis serves to account approximately for the nature of the effects of small inductance as depicted in Fig. 3. When the leakance is not zero but is small, the effects of inductance are still about the same. The general nature of the effect of the inductance L on the characteristic impedance of any smooth line, so far as regards the absolute value and the angle of the impedance, can be readily determined by mere inspection of equation (1), in a manner similar to that already outlined regarding the effect of leakance under the heading *The General Effect of Leakance.*

An alternative mode of representing the characteristic impedance of cables is suggested by the fact, already mentioned, that the impedance of a cable is at least roughly equal to $\sqrt{R/i\omega C}$, whence its absolute value is at least roughly equal to $\sqrt{R/\omega C}$. This suggests that we study a relative impedance consisting of the ratio of K to $\sqrt{R/\omega_1 C}$, where ω_1 denotes any fixed value of ω; and that we adopt the ratio ω/ω_1 as the independent variable. In this mode of treatment it will be convenient to employ the quantities w, r, F_1, b, b_1 defined by the equations

$$w = \frac{K}{\sqrt{R/\omega_1 C}} = \frac{K}{|K_1|}, \tag{22}$$

$$r = \omega/\omega_1 = f/f_1, \quad (23) \qquad F_1 = \omega_1 L/R, \quad (24)$$

$$b = G/\omega C, \quad (25) \qquad b_1 = G/\omega_1 C. \quad (26)$$

Thus, w denotes the relative impedance to be studied; its real and imaginary components will be denoted by u and v, so that $w=u+iv$. r denotes the relative frequency—relative to any fixed frequency f_1. F_1 is one parameter. The other parameter is, respectively, b or b_1 according as the leakance G is approximately proportional to or approximately independent of the frequency.[6] The corresponding forms of the equation for the relative impedance w are

$$w = \sqrt{\frac{1+iF_1r}{(b+i)r}}, \quad (27) \qquad w = \sqrt{\frac{1+iF_1r}{b_1+ir}}. \quad (28)$$

These are seen to be of the same functional forms as (19) and (20) respectively; with w corresponding to K, r to E, F_1 to k^2, and b_1 to g^2.

[6] It will be noted that b is the same as already defined by (7); b_1 is related to b; and F_1 is related to F, which has already been defined by (2).

In other respects, however, there are marked differences: K is an impedance, while w is a pure number, being the ratio of K to $\sqrt{R/\omega_1 C}$; E, though approximately proportional to the frequency, is not a pure number (for it is not dimensionless), while r is a pure number, being the ratio of the general frequency to any fixed frequency; of the parameters, k^2 is very different from F_1, and b_1 is very different from g^2. The fact that (27) and (28) are of the same functional forms as (19) and (20) respectively renders formally applicable the material pertaining to equations (19) and (20) given in Appendix A.

As already remarked, the effect of leakance in cables is usually extremely small except at very low frequencies. Hence in the graphi-

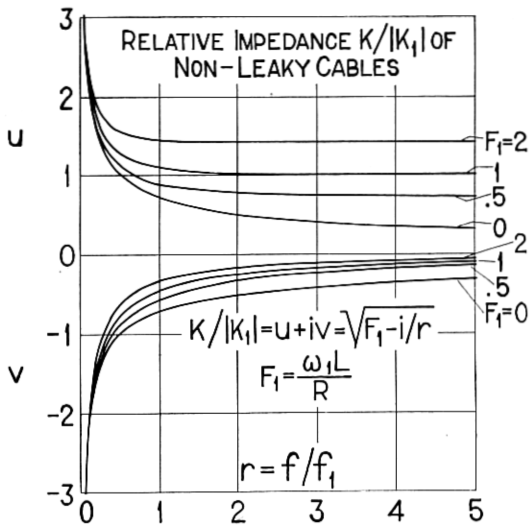

Fig. 4

cal representation of formulas (27) and (28) it will suffice for most purposes to confine ourselves to the limiting case of no leakance ($G=0$, and hence $b=0$ and $b_1=0$), when these two equations reduce to the same form, namely,

$$w=\sqrt{F_1-i/r}. \tag{29}$$

This has the same functional form as (21), with w corresponding to K, r to E, and F_1 to k^2; a circumstance rendering formally applicable the material pertaining to equation (21) given in Appendix A. The curves in Fig. 4 represent the two components u and v of w as functions of r with F_1 as parameter.

PART IV

NETWORKS FOR SIMULATING THE IMPEDANCE OF SMOOTH LINES

Under this heading will be described the various networks devised by the writer, for simulating the characteristic impedance of smooth lines, as mentioned in the latter part of the INTRODUCTION. Before proceeding to the systematic description of these networks, some of their practical uses will be mentioned. Foremost of these is their employment for balancing purposes in connection with 22-type repeaters, already spoken of in the INTRODUCTION. Another application is for properly terminating an actual telephone line in the field or an artificial line in the laboratory, usually for electrical testing purposes or electrical measurements on the lines. In making certain tests on apparatus normally associated with a telephone line, such line may be conveniently represented for impedance purposes by the appropriate simulating network.

Some of the networks to be shown are potentially equivalent in impedance; but may differ somewhat in cost, space occupied, etc. For the purpose of this paper any two networks will be called " potentially equivalent " if, when the elements of either network are assigned any arbitrary values, the other network can be so proportioned as to have at all frequencies identically the same impedance as the first network. Evidently the mathematical condition for such equivalence is that the expressions for the impedances of the two networks have the same functional forms when the frequency is regarded as the independent variable. The two networks will then have the same number of independent parameters, or degrees of freedom for adjustment; and this number is the same as the minimum number of elements requisite for the construction of a network to have identically the impedance of the given network.

For most of the networks described, there are included design-formulas for the values of the network elements (resistances and capacities). But in any applications requiring the highest simulative precision attainable with such networks, these formulas should be regarded merely as first approximations serving to reduce the requisite detailed design-work down to a relatively small amount but not permitting it to be dispensed with entirely; for the best values of the network elements depend somewhat on the particular frequency-range involved, and on the preassigned weighting of the desired simulative precision with respect to the frequency. Moreover, these formulas completely ignore leakance; while actually leakance may not always be quite negligible. even in the voice frequency-range.

A supplementary semi-graphical method as an aid to finally arriving at the best proportioning of the networks will be found outlined in Appendix C.

The Basic Resistance and the Excess Simulator

The first approximation to a network for simulating the characteristic impedance K of a smooth line is evidently a mere resistance R_1

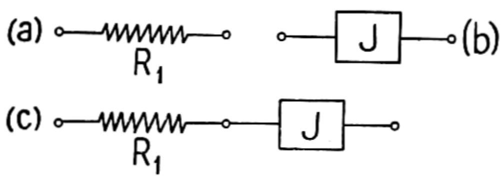

Fig. 5—Synthesis of the General Form of Complete Network. (a). Basic Resistance Element R_1 for Simulating Nominal Impedance. (b). Excess-Simulator J (Abstractly Symbolized) for Simulating Excess Impedance. (c). Complete Network for Simulating Line Impedance

(Fig. 5a) approximately equal to the nominal impedance k of the line, that is,

$$R_1=\sqrt{L/C}, \tag{30}$$

and this is a very close approximation, for instance, in the case of open-wire lines at the frequencies of carrier current transmission.

Over the voice frequency range, however, a mere resistance does not suffice; since there the excess characteristic impedance $K-k$ is not negligible, particularly at the lower frequencies. But the resistance R_1 equal to the nominal impedance may be retained as the natural basis of a network if it is supplemented by an element or elements such as to approximately simulate the excess characteristic impedance. Such a supplementary network is here termed an "excess-simulator"[7], and is symbolized abstractly by Fig. 5b; while Fig. 5c represents the corresponding complete network consisting of the basic resistance R_1 in series with the excess-simulator, whose impedance is denoted by J. The requisite excess-simulator is obviously less simple in structure and proportioning than the mere basic resistance; whence most of the remainder of this paper will be concerned with various specific types of excess-simulators.

[7] But in practice the term "low frequency corrector" has become rather firmly established. It was suggested by the fact that the excess impedance to be simulated is largest at relatively low frequencies.

The Simplest Excess-Simulator, and Complete Network

The simplest type of excess-simulator is a mere capacity C_1 (Fig. 6b). This is adequate only for those lines whose excess characteristic resistance is negligible; as, for instance, large gauge open-wire lines, and even then not at very low frequencies. The capacity C_1 is cap-

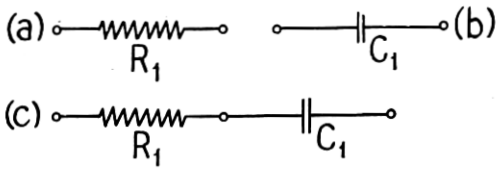

Fig. 6—Synthesis of the Simplest Type of Complete Network. (a). Basic Resistance. (b). Excess-Simulator. (c). Complete Network

able of simulating the reactance N of such a line rather closely, and its proper value for that purpose is approximately

$$C_1 = \frac{2\sqrt{LC}}{R} = C\frac{2\sqrt{L/C}}{R}, \tag{31}$$

although the most suitable value depends somewhat on the specific frequency-range involved. The complete network (Fig. 6c) thus consists merely of a resistance R_1 and a capacity C_1 in series with each other, having approximately the values expressed by (30) and (31).[8]

The simple network in Fig. 6c was devised a good many years ago.[9] The majority of present-day applications require such high simulative precision that the excess characteristic resistance of the line is not negligible, and also a mere capacity does not in all cases simulate the excess characteristic reactance quite as closely as desirable. To meet these needs there have been devised the much more precise, yet fairly simple, excess-simulators and complete networks described under several of the following headings.

Two Precise Types of Excess-Simulators, and Their Limiting Forms

Fig. 7 represents two potentially equivalent[10] excess-simulators that in most cases admit of such proportioning as to simulate with

[8] See Appendix B for the derivation of formula (31) for C_1, and incidentally formula (30) for R_1; and for a discussion of the simulative precision of this network; also for the values R_1' and C_1' requisite for exact simulation at any preassigned single frequency.

[9] In 1913. U. S. Patent No. 1,167,694 of January 11, 1916.

[10] In comparing networks as to equivalence I have found very useful the general theorems on equivalence given by O. J. Zobel in his paper on electric wave-filters in the January Number of this JOURNAL, pages 45–46.

the requisite high precision the excess characteristic impedance of the line; the complete network then consisting of either of these excess-simulators in series with the basic resistance element R_1 of Fig. 5a.

In any specific application the most suitable values for the elements constituting either of the two excess-simulators in Fig. 7 depend

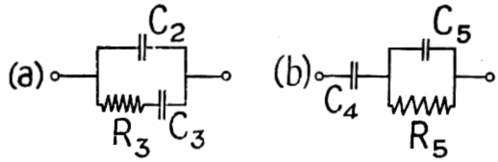

Fig. 7—Two Potentially Equivalent 3-Element Excess-Simulators Possessing High Simulative Precision for Most Applications, Except at very Low Frequencies

somewhat on the particular frequency-range involved, and also on the weighting of the desired simulative-precision with respect to the frequency. As might be expected, therefore, the work of determining closely the best combination of values for the elements of the excess-simulator can hardly avoid a certain amount of tentative detailed design-work; but usually this can be reduced to a relatively small amount by a semi-graphical method such as outlined in the latter part of Appendix C. Moreover, first-approximation values that will usually prove to be rather close, can be quickly found by means of the following approximate design-formulas (32), . . . (37), which are explicit except for containing the single undetermined parameter D. These formulas are such that the excess-simulator will possess high simulative-precision at large and even fairly large values of F, for all physcially admissible values of D $(0 \leqq D \leqq 1)$; and at the lower values of F will have a considerable range of adjustment by means of D, whose optimum value can be readily determined from inspection of Figs. 8 and 9, as described below. The above-mentioned approximate design-formulas for the elements of the two excess-simulators in Fig. 7 are[11]:

$$C_2 = \frac{2\sqrt{LC}}{R}, \tag{32}$$

$$C_3 = \frac{D}{1-D} \frac{2\sqrt{LC}}{R}, \tag{33}$$

$$R_3 = 2\sqrt{\frac{L}{C}}. \tag{34}$$

[11] The first part of Appendix C gives the derivation of these formulas, and also the equation of the curves in Fig. 8.

$$C_4 = \frac{1}{1-D} \frac{2\sqrt{LC}}{R}, \tag{35}$$

$$C_5 = \frac{1}{D} \frac{2\sqrt{LC}}{R}, \tag{36}$$

$$R_5 = D^2 2\sqrt{\frac{L}{C}}. \tag{37}$$

When the excess-simulator is proportioned in accordance with these design-formulas the corresponding complete network consisting of such excess-simulator J in series with the basic resistance $R_1 = \sqrt{L/C}$ will possess the simulative precision represented by the set of graphs in Fig. 8, which shows the percentage impedance-departure δ of the

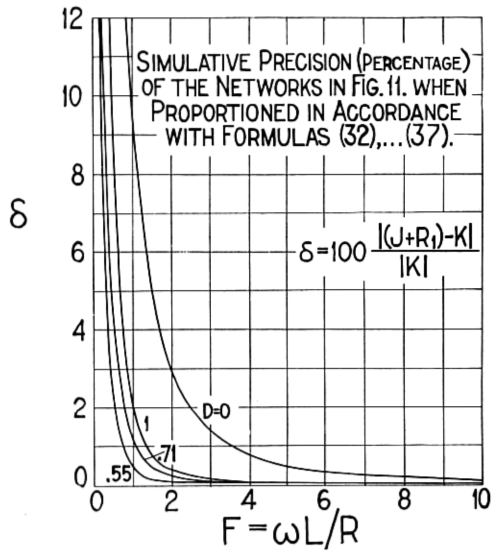

Fig. 8

complete network $R_1 + J$ from the line-impedance K, as function of F with D as parameter. In any specific case, where, of course, the F-range would be known, inspection of these graphs (Fig. 8) enables the best value of D to be readily determined, and the corresponding resulting precision δ to be seen as function of F. The curves show that the best value of D is determined by the lowest value of F contemplated, since the departure δ is largest at small values of F and rapidly decreases toward the larger values of F. It will be noted that the curves for the limiting values $D=0$ and $D=1$ have been included in Fig. 8; the corresponding limiting forms of the excess-simulators are considered a little further on.

Fig. 9, derived from Fig. 8, represents the optimum value of D as function of F; and shows also the corresponding minimum departure δ_m of the complete network. If D is chosen to be the optimum

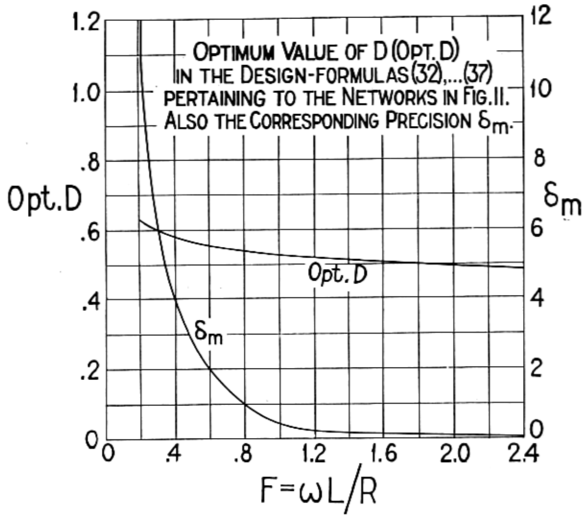

Fig. 9

value at any fixed F, the resulting network will have at that F exactly the departure shown on Fig. 9, but at all other values of F will, of course, have departures larger than those on Fig. 9.

It should be noted that these statements regarding the departures pertain to the network when the excess-simulator is proportioned in accordance with formulas (32), . . . (37). As those are only first-approximation formulas, the ultimate precision attainable will usually be better, and may be adjusted to possess a somewhat different distribution over the frequency-range.

Although the two excess-simulators in Fig. 7 are potentially equivalent as regards impedance there is a slight choice between them from the viewpoints of cost and space occupied. For it is readily seen by mere inspection of the networks at zero frequency that when they have equal impedances the total capacity C_2+C_3 of the excess-simulator in Fig. 7a is equal to merely the capacity C_4 of the excess-simulator in Fig. 7b, thus leaving C_5 in excess. As regards the relative magnitudes of their various elements the two excess-simulators can be readily compared by means of the following equations (38) derived from (32), . . . (37):

$$\frac{C_4+C_5}{C_2+C_3}=\frac{1}{D}=\frac{C_5}{C_2}=\frac{C_4}{C_3}=\sqrt{\frac{R_3}{R_5}}. \tag{38}$$

Fig. 10 represents the two limiting forms of the excess-simulators in Fig. 7, corresponding to the limiting values 0 and 1 of the parameter D occurring in the design-equations (32), . . . (37). For $D=0$ the limiting form is that in Fig. 10a, and will be recognized as

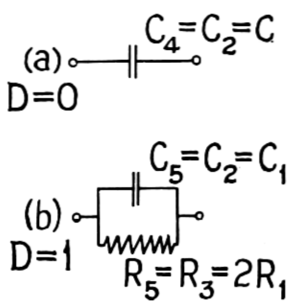

Fig. 10—The Two Limiting Forms of the Excess-Simulators in Fig. 7, Corresponding to the Limits 0 and 1 of the Parameter D

the simple excess-simulator already shown in Fig. 6b consisting of a mere capacity C_1 having the value expressed by (31); while for $D=1$ the limiting form is that in Fig. 10b, and is thus of the same form as one mentioned below, under the heading *Modifications for Very Low Frequencies*, as being capable of furnishing approximate simulation extending down to zero frequency. The departure-curves for these two limiting forms ($D=0$ and $D=1$) are included in Fig. 8, as already mentioned; and from them it is seen that the form in Fig. 10b ($D=1$) possesses much higher simulative precision than the form in Fig. 10a ($D=0$)—as would be expected.

Four Precise Types of Complete Networks, and Their Limiting Forms

Figs. 11a and 11b represent the two potentially equivalent complete networks that can be constructed from the basic resistance R_1 of Fig. 5a, and the excess-simulators in Figs. 7a and 7b respectively; and hence having for their elements approximately the values expressed by equations (30), (32), . . . (37).

Figs. 11c and 11d represent two other complete networks that also are potentially equivalent to those in Figs. 11a and 11b[12].

Appendix D gives the three sets of formulas expressing the values of the elements constituting the networks in Figs. 11b, 11c, 11d re-

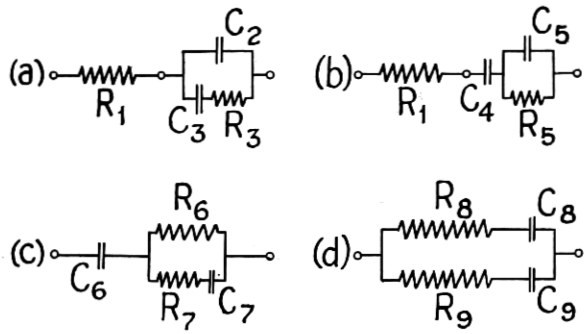

Fig. 11—Four Potentially Equivalent 4-Element Complete Networks Possessing High Simulative Precision for Most Applications, Except at Very Low Frequencies

spectively, in terms of the elements constituting the network in Fig. 11a, when those four networks have equal impedances.

Although the four complete networks in Fig. 11 are potentially equivalent as regards impedance there is some choice among them from the viewpoint of cost and space occupied. For it is readily seen by mere inspection of the networks at zero frequency that, when they have equal impedances,

$$C_2+C_3=C_8+C_9=C_4=C_6. \tag{39}$$

Thus the networks in Figs. 11a and 11d have the same total capacity; and this is less than the total capacity of the network in Fig. 11b by the amount C_5, and is less than the total capacity of that in Fig. 11c by the amount C_7. Similarly by mere inspection of the networks at infinite frequency it is seen that

$$G_6+G_7=G_8+G_9=G_1, \tag{40}$$

the G's being the reciprocals of the R's and thus being the corresponding conductances.

Before leaving Fig. 11 it may be noted that the network in Fig. 11d has the same form as though obtained by connecting in parallel two networks having the same form as Fig. 6c but with elements R_1', C_1' and R_1'', C_1'', say. Now it is known that, in most applications, the

[12] In connection with Figs. 11b and 11c the network shown in U. S. Patent No. 1,240,213 of September 18, 1917 may be of some interest.

network in Fig. 11d has much higher simulative precision than that in Fig. 6c. These considerations suggest the possibility of attaining still higher precision by connecting in parallel several such networks, having constants R_1', C_1'; R_1'', C_1''; R_1''', C_1''', ...

Fig. 12 represents the two limiting forms of the network in Fig. 11, corresponding to the limiting values 0 and 1 of the parameter D occurring in the design-equations (32), . . . (37). For $D=0$ the limiting form is that represented in Fig. 12a, and this will be recognized

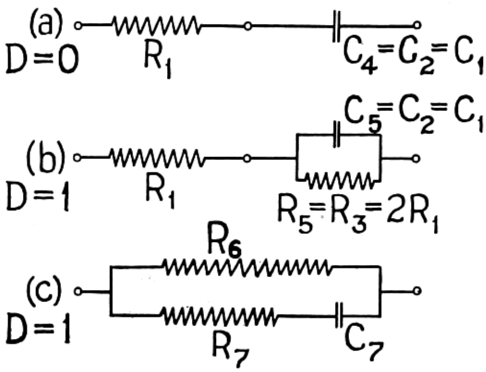

Fig. 12—The Two Limiting Forms of the Networks in Fig. 11, Corresponding to the Limits 0 and 1 of the Parameter D. Networks (b) and (c) are Potentially Equivalent

as the simple 2-element network already shown in Fig. 6c; while for $D=1$ the limiting forms are the two potentially equivalent 3-element networks represented in Figs. 12b and 12c, and are thus of the same forms as two mentioned below, under the heading *Modifications for Very Low Frequencies*, as being capable of furnishing approximate simulation extending down to zero frequency. The values of the elements of the network in Fig. 12c in terms of the elements of the network in Fig. 12b, for equivalence of these two networks as regards impedance, are

$$R_6 = R_1 + R_3 = 3R_1, \tag{41}$$

$$R_7 = R_1(1 + R_1/R_3) = 3R_1/2, \tag{42}$$

$$C_7 = \frac{C_2}{(1 + R_1/R_3)^2} = \frac{4C_2}{9}. \tag{43}$$

Thus the network in Fig. 12c requires only four-ninths as much capacity as the network in Fig. 12b.

Modifications for Very Low Frequencies

Thus far the present paper has dealt with the characteristic impedance of smooth lines as distinguushed from their sending-end impedance, strictly speaking. The two are closely equal when the lines are electrically long, which is usually the case for the telephonic frequency range; but at very low frequencies the sending-end impedance of even a rather long line may depend very greatly on the distant terminating impedance and hence depart widely from the characteristic impedance. In case the terminating impedance is conductive to direct current the sending-end impedance of even a strictly non-leaky line would have a finite value at zero frequency; its resistance component evidently being equal to the total line-wire resistance plus the terminating resistance, while its reactance component would, of course, be zero. Actually, on account of line leakance, the resistance component would be somewhat less; and in case the distant terminating impedance permits no passage of direct current the sending-end impedance of the line at zero frequency would depend largely on the line leakance.

Most of the simulating networks thus far described were devised primarily with regard to the voice range of frequencies, without reference to frequencies very far below that range. At very low frequencies these networks become unsuitable because their impedance is not only much too large but also has not even approximately the proper angle. There have not been many occasions for modifying the networks so as to extend their range of simulation down toward zero frequency; but it seems likely that in most cases the requisite modification in the network impedance could be attained, at least roughly, by shunting the excess-simulator (Fig. 5b) with a mere resistance S' approximately equal to the zero-frequency sending-end resistance of the line diminished by the resistance R_1 of the basic resistance element. Clearly this modification will give the network the desired impedance at zero-frequency, without affecting its impedance at infinite frequencies; since the impedance of the unshunted excess-simulator is infinite [13] at zero-frequency and is zero at infinite frequencies. At the intermediate frequencies the resulting modification would doubtless be slight except toward the lower frequencies, where it would increase more and more rapidly as zero-frequency is approached. Of course, the addition of the modifying element S' would usually entail some alterations in the proportioning of the

[13] Except for the limiting form in Fig. 10b.

original network, as indicated by Fig. 13a, where the altered values of J and R_1 are denoted by J' and R_1' respectively.

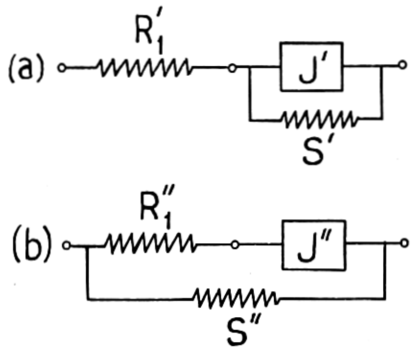

Fig. 13—Two Potentially Equivalent Modifications for Extending Range of Simulation Down to Zero Frequency. (a). Modification by Shunting the Excess-Simulator J'. (b). Modification by Shunting the Complete Network $R_1''+J''$

Fig. 13b represents an alternative but potentially equivalent form of modification, obtained by shunting the original form of network (Fig. 5c) with a resistance S''; and the conditions for equivalence are

$$S''=S'+R_1', \tag{44}$$

$$R_1''=R_1'(1+R_1'/S'), \tag{45}$$

$$J''=J'(1+R_1'/S')^2. \tag{46}$$

Since the shunts S' and S'' are potentially equivalent in their effects their simultaneous application would be potentially equivalent to the application of either alone.

Thus far the suggested modifications have been stated only with reference to the excess-simulator regarded abstractly. When the specific structure of the excess-simulator is regarded, the modifications can take several different forms which, for any one excess-simulator, are equivalent as regards impedance. Certain of these are noted in the following paragraphs:

Among the modified excess-simulators will evidently be found one having the limiting form already depicted in Fig. 10b.

Fig. 14 represents by (a) and (b), respectively, the 3-element excess-simulators in Figs. 7a and 7b modified by the shunt resistance S', and thereby converted to 4-element excess-simulators. Figs. 14c and 14d represent two other 4-element excess-simulators that are potentially equivalent to those in Figs. 14a and 14b as regards impedance.

Before leaving Fig. 14 it may be noted that the excess-simulator in Fig. 14d has the same form as though obtained by connecting in series two of the simple form of modified excess-simulator in Fig. 10b having elements C_2', R_3' and C_2'', R_3'', say. This observation suggests

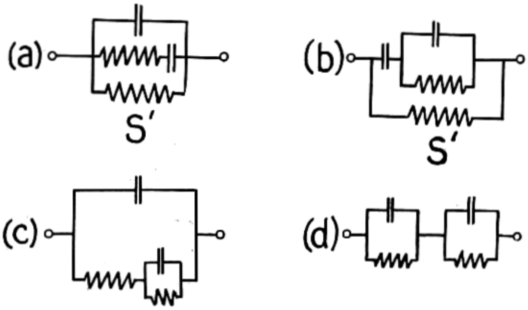

Fig. 14—Four Potentially Equivalent 4-Element Excess-Simulators Embodying Shunt-Resistance Modifiers for Extending the Range of Simulation down to Zero Frequency

the possibility of attaining still higher simulative precision for a modified excess-simulator by connecting in series several such simple modified excess-simulators.

Among the modified complete networks will evidently be found two having respectively the forms already depicted in Figs. 12b and 12c.

Fig. 15 represents four potentially equivalent complete networks derived from Fig. 11d by application of a shunt resistance S''. The

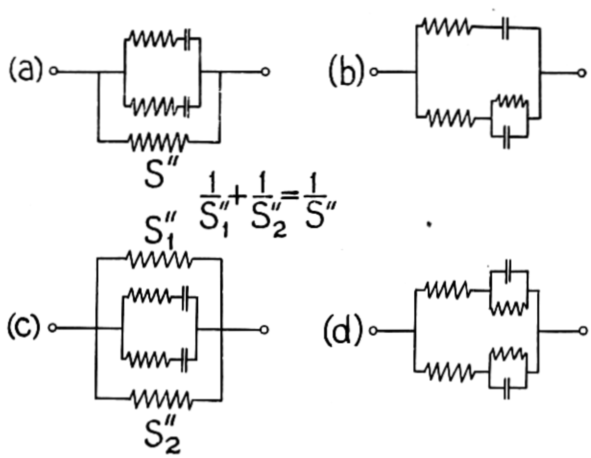

Fig. 15—Four Potentially Equivalent Complete Networks Embodying Shunt-Resistance Modifiers for Extending the Range of Simulation Down to Zero Frequency

forms in Figs. 15c and 15d, though each containing a superfluous element, are of interest because they have the same forms as though obtained by connecting in parallel two networks of the forms already depicted in Figs. 12c and 12b respectively.

Modifications For Leaky Lines

For lines whose leakance is not quite negligible a study of the formulas and graphs of the line impedance indicates that the effects of such leakance can be sufficiently taken into account by a mere slight reproportioning of the network without the addition of any further element, except that a small series inductance might be a slight improvement in those cases where the leakance increases rapidly with the frequency.

Appendix A

Equations of the Components of the Line Impedance

This Appendix contains the equations for the rectangular components and the equation for the angle of the relative impedance z and of the impedance K, corresponding to most of the various forms of the equations employed in this paper for expressing z and K. It thereby includes the equations for all the graphs employed in representing z and K. It contains also the equations for the network of curves of z and K in the complex plane for certain of the more important limiting cases involving not more than one parameter.

With regard to the notation it will be recalled that $z=x+iy$ and $K=M+iN$. The angles of z and K will be denoted by $ag\,z$ and $ag\,K$, respectively, "*ag*" being an abbreviation for "angle of".

Equation (10):

$$z=\sqrt{\frac{1+iF}{(b+i)F}};$$

$$x=\sqrt{\frac{b+F+\sqrt{(1+b^2)\,(1+F^2)}}{2\,(1+b^2)\,F}},$$

$$y=-\frac{1-bF}{2(1+b^2)Fx},$$

$$ag\,z=-\tfrac{1}{2}\tan^{-1}\frac{1-bF}{b+F}.$$

Equation (11):

$$z=\sqrt{\frac{1+iF}{a+iF}};$$

$$x=\sqrt{\frac{a+F^2+\sqrt{(1+F^2)(a^2+F^2)}}{2(a^2+F^2)}},$$

$$y=-\frac{(1-a)F}{2(a^2+F^2)x},$$

$$ag\,z=-\tfrac{1}{2}\tan^{-1}\frac{(1-a)F}{a+F^2}.$$

Equation (12):

$$z=\sqrt{1-i/F};$$

$$x=\frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1+1/F^2}},$$

$$y=-1/2Fx=-\sqrt{x^2-1},$$

$$x-1=\frac{1}{8F^2}\,\frac{2}{(x+1)x^2},$$

$$ag\,z=-\tfrac{1}{2}\cot^{-1}F.$$

The relation $y=-\sqrt{x^2-1}$ can be written in the form

$$\frac{x-1}{-y}=\sqrt{\frac{x-1}{x+1}},$$

which shows that $x-1$ is always smaller than $-y$; and is very much smaller except at small values of F, where the two approach equality as F approaches zero.

The locus of z in the xy-plane is the hyperbola $x^2-y^2=1$. For any preassigned value of F the corresponding values of x and y on this locus can be accurately calculated by means of the above equations for x and y. For any pair of values of x and y situated on this locus the corresponding value of F is given by $F=-1/2xy$.

Equation (19):

$$K=\sqrt{\frac{1+ik^2E}{(b+i)E}};$$

$$M=\sqrt{\frac{b+k^2E+\sqrt{(1+b^2)\,(1+k^4E^2)}}{2(1+b^2)E}},$$

$$N=-\frac{1-bk^2E}{2(1+b^2)EM},$$

$$ag\,K=-\tfrac{1}{2}\tan^{-1}\frac{1-bk^2E}{b+k^2E}.$$

Equation (20)

$$K=\sqrt{\frac{1+ik^2E}{g^2+iE}};$$

$$M=\sqrt{\frac{g^2+k^2E^2+\sqrt{(1+k^4E^2)\,(g^4+E^2)}}{2\,(g^4+E^2)}},$$

$$N=-\frac{(1-g^2k^2)E}{2(g^4+E^2)M},$$

$$ag\,K=-\tfrac{1}{2}\tan^{-1}\frac{(1-g^2k^2)E}{g^2+k^2E^2}.$$

Equation (21):

$$K=\sqrt{k^2-i/E};$$

$$M=\frac{1}{\sqrt{2}}\sqrt{k^2+\sqrt{k^4+1/E^2}},$$

$$N=-1/2EM=-\sqrt{M^2-k^2},$$

$$M-k=\frac{1}{8E^2}\,\frac{2}{(M+k)M^2},$$

$$ag\,K=-\tfrac{1}{2}\cot^{-1}\,k^2E.$$

The relation $N=-\sqrt{M^2-k^2}$ can be written in the form

$$\frac{M-k}{-N}=\sqrt{\frac{M-k}{M+k}},$$

which shows that $M-k$ is always smaller than $-N$, though the two approach equality when E approaches zero.

The network of curves of K in the MN-plane are the equi-k curves consisting of the family of hyperbolas $M^2-N^2=k^2$, and the equi-E curves consisting of the family of hyperbolas $MN=-1/2E$.

Appendix B

On the Simple Type of Complete Network (Fig. 6)

The network in Fig. 6c consisting of a resistance R_1 and capacity C_1 in series with each other and having the values expressed by equations (30) and (31) was originally arrived at by working with values of F large or at least fairly large compared with unity; for then, by equation (12), the characteristic impedance K has approximately the value.

$$K=k-ik/2F. \qquad (1\text{-B})$$

This represents K as having a resistance component k that is independent of frequency, and a reactance component $-k/2F$ that is negative and inversely proportional to the frequency f (since $F=\omega L/R$) and thus leads exactly to the values of R_1 and C_1 expressed by (30) and (31), whence the impedance of this network is exactly equal to the approximate value of the line impedance expressed by (1-B).

To obtain more precise and comprehensive knowledge regarding the simulative precision of this network its exact impedance $k-ik/2F$ will here be compared with the exact value of the line impedance (when leakance is neglected). For this purpose it is convenient to employ the line impedance in the form

$$K=xk-ik/2xF, \tag{2-B}$$

obtained by means of the relation $y=-1/2Fx$ found under equation (12) in Appendix A. The equation (2-B) shows that to exactly simulate the line impedance by a resistance R_1' and capacity C_1' in series with each other these would have to possess the values

$$R_1'=x\sqrt{L/C}, \tag{3-B}$$

$$C_1'=\frac{2x\sqrt{LC}}{R}, \tag{4-B}$$

which differ only by the factor x from the values of R_1 and C_1 expressed by (30) and (31). Thus the ideal resistance R_1' and capacity C_1' for exactly simulating the line impedance would vary with F in precisely the same way as x varies with F. Moreover the ratio of these ideal values to the fixed values of R_1 and C_1 expressed by (30) and (31) is merely x. By reference to Fig. 1 (with $b=0$) it will be seen that, except at small values of F, the factor x is nearly independent of F and is only slightly greater than unity. Thus the values of R_1 and C_1 determined by means of equations (30) and (31) are slightly too small at all frequencies; while the values determined by means of equations (3-B) and (4-B), for any specified frequency (by inserting the appropriate value of x), are slightly too small at lower frequencies and slightly too large at higher frequencies.

Since (3-B) can, by (30), be written in the form

$$R_1'=R_1+(x-1)\sqrt{L/C}, \tag{5-B}$$

and since x is always greater than unity, it is seen that the simulation can be somewhat improved by supplementing the excess-simulator with a small series resistance element R_{11}, the ideal value of which would be

$$R_{11}=(x-1)\sqrt{L/C}. \tag{6-B}$$

Actually, since x varies with frequency, R_{11} is limited to some compromise value. In practice R_{11} would usually be combined with the basic resistance R_1, though the functions of the two are distinctly different. (If the requisite value of R_{11} were negative, R_1 would merely be decreased by that amount.)

Appendix C

On the Precise Types of Excess-Simulators (Fig. 7)

The two sets of formulas (32), (33), (34) and (35), (36), (37), representing first-approximations to the proper values of the elements constituting the excess-simulators in Figs. 7a and 7b respectively, were originally obtained by working with values of F large or at least fairly large compared with unity; for then, by (13), the excess characteristic impedance $K-k$ has approximately the value

$$K-k=\frac{k}{8F^2}-i\frac{k}{2F}, \tag{1-C}$$

while, at large or fairly large values of T, the impedance $J=P+iQ$ of each excess-simulator in Fig. 7 can be expressed approximately by the equation

$$J=\frac{P_0}{T^2}-i\frac{(1+t)P_0}{T}, \tag{2-C}$$

derived from the exact equation (16-C) below, in which t, P_0, and T have the values defined by the following two sets of equations (3-C), (4-C), (5-C) and (6-C), (7-C), (8-C) for the excess-simulators in Figs. 7a and 7b respectively:

$$t=C_2/C_3, \tag{3-C}$$

$$P_0=\frac{R_3}{(1+t)^2}, \tag{4-C}$$

$$T=\frac{\omega C_2R_3}{1+t}, \tag{5-C}$$

$$t=C_5/C_4, \tag{6-C}$$

$$P_0=R_5, \tag{7-C}$$

$$T=\omega C_5R_5, \tag{8-C}$$

P_0 thus being the value of P at $\omega=0$. Comparison of the approximate equations (1-C) and (2-C) gives immediately

$$P_0/T^2=k/8F^2, \tag{9-C}$$

$$(1+t)P_0/T=k/2F, \tag{10-C}$$

as the two conditions that are necessary and sufficient for (approximate) equality of J and $K-k$ at large values of F and T. This pair of equations is equivalent to the more convenient equations (11-C),

$$\frac{T}{F}=\frac{4}{1+t}=\sqrt{\frac{8P_0}{k}}. \qquad \text{(11-C)}$$

Thus the ratio of T to F is fixed as soon as either t or P_0/k is fixed. It will be convenient to adopt $\sqrt{P_0/2k}$ as the arbitrary quantity and to denote it by D, so that

$$D=\sqrt{P_0/2k}, \qquad \text{(12-C)}$$

whence

$$P_0=2D^2k, \qquad \text{(13-C)}$$

and

$$t=\frac{1}{D}-1, \qquad \text{(14-C)}$$

and

$$T=4DF. \qquad \text{(15-C)}$$

Since only positive values of t and D are physically admissible, equation (14-C) shows that the admissible range of D is 0 to 1.

From (13-C), (14-C), (15-C) and the defining equation $F=\omega L/R$ the two sets of equations (32), (33), (34) and (35), (36), (37) follow readily from the two sets of defining equations (3-C), (4-C), (5-C) and (6-C), (7-C), (8-C), respectively.

The formula for plotting the curves in Fig. 8 depends on the exact equation for J/P_0 which is

$$\frac{J}{P_0}=\frac{1}{1+T^2}-i\frac{t+(1+t)T^2}{T(1+T^2)}. \qquad \text{(16-C)}$$

By substituting herein the values of P_0, t, and T expressed by (13-C), (14-C), (15-C) the equation for J/k becomes

$$\frac{J}{k}=\frac{2D^2}{1+16D^2F^2}-i\frac{1+16D^2F^2-D}{2F(1+16D^2F^2)}, \qquad \text{(17-C)}$$

which is thus the exact formula for the relative impedance J/k of each of the excess-simulators in Fig. 7 when these are proportioned in accordance with the formulas (32), . . . (37).

A semi-graphical method will now be outlined in the remainder of this Appendix. In this method the ratio T/f is of frequent occurrence and will be denoted by d. Then, recalling that $P+iQ=J$, it will be seen from equation (16-C) that P/P_0 depends only on f and d; while Q/P_0 depends on f, d, and t. These observations are the basis for the method now to be described for evaluating the three parameters P_0, d, and t which implicitly determine the elements of the excess-simulators in Fig. 7.

In the first step of this method the two parameters d and P_0 are so chosen that the resistance component P of the excess-simulator will be approximately equal to the excess resistance $M-k$ of the line-impedance K, over the specific f-range contemplated, or else will differ therefrom by a nearly constant amount, which can be approximately simulated by a mere series resistance element. In the second step of the method the remaining parameter, t, is so chosen that the reactance component Q of the excess-simulator will be approximately equal to the reactance N of the line impedance, when d and P_0 have the pair of values already chosen in the first step. The technical procedure in these two steps may now be formulated explicitly as follows:

First, over the contemplated f-range, plot a set of curves representing P/P_0 as function of f with d as parameter; and on the same sheet a set of curves representing $(M-k)/P_0$ as function of f with P_0 as parameter. To evaluate d and P_0 choose (by interpolation, if necessary) such P/P_0-curve and $(M-k)/P_0$-curve as most closely coincide. A preliminary idea regarding the useful ranges of d and P_0 can be readily obtained from the approximate formulas (15-C) and (13-C), together with Fig. 8.

Second, on another sheet plot as function of f that particular N/P_0-curve having as parameter the value of P_0 already found in the first step. With this value of P_0 and the corresponding value of d, as found in the first step, plot also a sufficient set of Q/P_0-curves as function of f with t as parameter to find the one that coincides most closely with the single N/P_0-curve already plotted. To abridge this step tentative values for t can be readily obtained from the approximate formula (14-C), together with Fig. 8. But the useful range of t can be demarcated more closely by solving for t the equation obtained by equating the expressions for Q and N; the value for t thus found is [14]

$$t=-\frac{dfN}{P_0}-\frac{d^2f^2}{1+d^2f^2}.$$

This may even be plotted, as function of f, to see whether the requisite value of t varies much in the contemplated f-range.

If the best compromise value of t found in the second step is unsatisfactory as regards simulation of N by Q, it will be necessary to revert to the first step, choose some other pair of values for d and

[14] It will be recalled that the line-reactance N is practically always negative.

P_0, and with these repeat the second step. In this connection it should be noted that, in the first step, it is not necessary to choose the P/P_0-curve and $(M-k)/P_0$-curve which most closely coincide; on the contrary it suffices to choose two curves that are closely parallel (that is, have closely equal slopes at each f). For, corresponding to the nearly constant distance between such two curves, it will only be necessary to supplement the excess-simulator with a series resistance element R_{11}—which will thus in the complete network be also in series relation to the basic resistance R_1 and hence can be merged therewith (even when the requisite R_{11} is negative, provided it is less than R_1 in absolute value).

After the parameters t, P_0, and $d=T/f$ have been evaluated, the values for the elements of the excess-simulators in Figs. 7a and 7b can be readily obtained from the two sets of equations (3-C), (4-C), (5-C) and (6-C), (7-C), (8-C), respectively; it thus being found that

$$C_2=d/2\pi(1+t)P_0,$$
$$C_3=d/2\pi(1+t)tP_0,$$
$$R_3=P_o(1+t)^2,$$

$$C_4=d/2\pi tP_0,$$
$$C_5=d/2\pi P_0,$$
$$R_5=P_0.$$

The requisite value for the supplementary series resistance element R_{11} is evidently

$$R_{11}=P_0\left(\frac{M-k}{P_0}-\frac{P}{P_0}\right).$$

which will be approximately independent of f if the curves of P/P_0 and $(M-k)/P_0$ chosen in the first step are approximately parallel. If the requisite value of R_{11} is negative, the basic resistance R_1 will merely be decreased by that amount.

For the limiting form of excess-simulator in Fig. 10b the design-procedure is considerably simpler, because the parameter t is fixed $(t=0)$. The two remaining parameters d and P_0 can be evaluated by inspection of two sheets of curves plotted as functions of f: One sheet containing a set of curves of P/P_0 with d as parameter, and curves of $(M-k)/P_0$ with P_0 as parameter; and the other sheet, curves of Q/P_0 with d as parameter, and curves of N/P_0 with P_0 as parameter.

Instead of f as the independent variable it may be more convenient to employ some quantity proportional to f (for instance, F or E); likewise, instead of P, P_0, Q, M, N, some quantities proportional to them (for instance, their ratios to k).

Appendix D

Relative Values of the Elements in the Four Precise Types of Complete Networks (Fig. 11)

The following three sets of formulas express the values of the elements constituting the networks in Figs. 11b, 11c, 11d, respectively, in terms of the elements constituting the network in Fig. 11a when those four networks have equal impedances. These formulas involve the two ratios ξ and ζ pertaining to the network in Fig. 11a and defined by the equations

$$\xi = C_3/C_2, \quad \zeta = R_1/R_3.$$

For Fig. 11b,

$$\frac{R_5}{R_3} = \left(\frac{\xi}{1+\xi}\right)^2, \qquad \frac{C_4}{C_3} = \frac{1+\xi}{\xi}, \qquad \frac{C_5}{C_3} = \frac{1+\xi}{\xi^2}.$$

For Fig. 11c,

$$\frac{R_6}{R_3} = \zeta + \left(\frac{\xi}{1+\xi}\right)^2, \qquad \frac{C_6}{C_3} = \frac{1+\xi}{\xi},$$

$$\frac{R_7}{R_3} = \zeta + \left(\zeta\frac{1+\xi}{\xi}\right)^2, \qquad \frac{C_7}{C_3} = \frac{1}{(1+\xi)\left(\frac{\xi}{1+\xi} + \zeta\frac{1+\xi}{\xi}\right)}$$

For Fig. 11d,

$$\frac{R_8}{R_3} = \frac{2\zeta\tau}{(\eta+\tau) - 2(1+\xi)},$$

$$\frac{R_9}{R_3} = \frac{2\zeta\tau}{2(1+\xi) - (\eta-\tau)},$$

$$\frac{C_8}{C_3} = \frac{2\xi - \zeta(1+\xi)(\eta-\tau)}{2\xi\zeta\tau},$$

$$\frac{C_9}{C_3} = \frac{\zeta(1+\xi)(\eta+\tau) - 2\xi}{2\xi\zeta\tau},$$

where $\eta = 1+\xi+\frac{\xi}{\zeta}$, $\quad \tau = \sqrt{\left(1+\xi+\frac{\xi}{\zeta}\right)^2 - 4\frac{\xi}{\zeta}}$.

Appendix E

Illustrative Example

The example contained in this Appendix serves two purposes. First, it illustrates the use of two types of the general line-impedance graphs contained in Parts II and III of the paper; second, it illustrates the first-approximation design of a simulating network by means of the method in Part IV.

The specific example chosen pertains to a well-insulated open-wire line consisting of two horizontal parallel copper wires, of No. 12 N. B. S. gauge, having per loop-mile the constants

$$R=10.4 \text{ ohms}, \qquad L=.00367 \text{ henry}, \qquad C=.00835\text{x}10^{-6} \text{ farad},$$

the leakance G being regarded as negligible. This particular type and gauge of line was chosen because it is rather extensively employed in practice, and also because its excess impedance is far from being negligible even as regards its resistance component.

For the illustratve purposes contemplated, it will be supposed that it is desired to evaluate the resistance and reactance components M and N of the characteristic impedance K of this line over the frequency-range from 200 to 2500 cycles per second; and also to design a network for approximately simulating this impedance over that frequency-range, and to determine the simulative precision of such network.

The procedure and results are indicated by the following table together with the supplementary description coming thereafter.

f	F	x	$-y$	M	$-N$	r	u	$-v$	δ_0	δ
200	.444	1.32	.86	876	570	.222	1.87	1.22	——	3.0
300	.666	1.19	.64	790	425	.333	1.69	.91	16.6	1.3
500	1.110	1.08	.42	717	279	.555	1.53	.60	8.1	.3
800	1.776	1.04	.27	691	179	.888	1.48	.38	3.6	.1
1200	2.664	1.02	.19	676	126	1.332	1.45	.27	1.7	.1
1600	3.552	1.01	.14	670	93	1.776	1.43	.20	1.0	.1
2000	4.440	1.01	.12	670	80	2.220	1.43	.17	.6	.1
2300	5.106	1.01	.10	670	66	2.553	1.43	.14	.5	.1
2500	5.550	1.01	.10	670	66	2.775	1.43	.14	.4	.1
∞	∞	1	0	663	0	∞	$\sqrt{2}$	0	0	0

The first column in the table is a set of values of the frequency f distributed over the specified range 200 to 2500.

The columns headed $F, x, -y, M, -N$, show the successive steps in evaluating the characteristic impedance $K=M+iN$ by means of

Fig. 2 [15], with $a=0$ (since the leakance G is neglected). The F-column was obtained from the f-column by means of the equation $F=\omega L/R=.00222f$. Next the values of x and y were read from the curves $a=0$ in Fig. 2. Finally M and N were obtained from $M=kx$ and $N=ky$, with $k=\sqrt{L/C}=663$ ohms (whence M and N are in ohms). From the table it will be noted that at $f=200$ the excess resistance is about one-third as large as the nominal impedance, and the reactance is about nine-tenths as large as the nominal impedance.

The columns r, u, $-v$ show the steps in evaluating M and N by means of Fig. 4. After choosing F_1 (in Fig. 4) equal to 2 [16], the r-column was obtained from the f-column by means of the equation $r=f/f_1=2\pi Lf/RF_1=.00111f$. Next the values of u and v were read off from the curves $F_1=2$ in Fig. 4. Finally the values of M and N were obtained from $M=|K_1|u$ and $N=|K_1|v$, with $|K_1|=\sqrt{R/\omega_1 C}=\sqrt{L/CF_1}=469$.

The two last columns (δ_0, δ) of the table show the percentage precision of certain simulating networks designed in accordance with the first-approximation methods of this paper, and having for their elements the values given in the last paragraph of this Appendix.

δ_0 pertains to the simple 2-element network in Fig. 6c when proportioned in accordance with equations (30) and (31). The values of δ_0 were read from the curve $D=0$ in Fig. 8. The precision at the lower frequencies could be considerably improved by the addition of a small resistance R_{11} (as already noted in connection with equation (6-B) of Appendix B), but with a corresponding sacrifice in the rest of the range.

δ pertains to the 4-element networks in Figs. 11a and 11b when proportioned in accordance with (30), (32), (33), (34) and (30), (35), (36), (37) respectively; and δ pertains also to the networks in Figs. 11c and 11d when these are proportioned, by Appendix D, so as to be equivalent to the network in Fig. 11a. The values of δ were read from the curve $D=0.55$ in Fig. 8, this value of D being known from Fig. 9 to be about the best.

The values of the elements of the networks to which δ_0 and δ pertain will now be set down, each preceded by a reference to the corresponding diagram and design-formulas:

Fig. 6c—Formulas (30), (31); Precision δ_0.

$R_1=663$ ohms, $C_1=1{,}063\text{x}10^{-6}$ farad.

[15] Or Fig. 1, with $b=0$; but Fig. 2 is plotted to a larger scale.

[16] $F_1=1$ would be somewhat preferable for reading off values; but the principles are more clearly exhibited by choosing for F_1 some other value than unity.

Fig. 11a—Formulas (30), (32), (33), (34); Precision δ.
$R_1 = 663$ ohms, $C_2 = 1.063 \times 10^{-6}$ farad, $C_3 = 1.300 \times 10^{-6}$ farad, $R_3 = 1326$ ohms.

Fig. 11b—Formulas (30), (35), (36), (37); Precision δ.
$R_1 = 663$ ohms, $C_4 = 2.362 \times 10^{-6}$ farad, $C_5 = 1.932 \times 10^{-6}$ farad, $R_5 = 401$ ohms.

NOTE:—To furnish sufficiently high precision for most engineering applications the curves and the cross-section lines in the line-impedance charts would evidently have to be drawn at much closer intervals than has been done in the present paper, where the purpose of the charts is mainly qualitative, or only roughly quantitative, to exhibit the general nature of the functions involved.