The Theory of Probabilities Applied to Telephone Trunking Problems

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THE Theory of Probabilities lends itself to the solution of many important telephone problems. These problems arise not only in connection with the trunking of calls but also in statistical studies which underlie the making of fundamental plans, in studies carried on in physical research and in the manufacturing of telephone apparatus.

The purpose of the present paper is to discuss certain simple types of trunking problems which can readily be handled to a sufficient degree of approximation by well-known probability methods. It would be quite impossible, within the scope of a single paper to give a complete discussion of trunking problems in general. For years [1] it has been known that light could be shed on these problems by the application of probabilities and many articles [2] have appeared on this subject; however the treatment to be found in the literature is, as yet, by no means comprehensive.

About 1905, the development of machine switching systems arrived at a stage where the relative efficiencies of different sizes of trunk groups became of prime importance.

In designing and engineering machine switching systems, it is necessary to compare the costs of various plans using trunk groups of widely different sizes, in order to choose the cheapest arrangement. Some plans use trunk groups as small as 5 and others groups as large as 90.

Machine switching development, therefore, gave a great impetus to the application of the Theory of Probabilities to telephone engineering and in the Bell System work along this line has been in progress, systematically, for many years. This work has included not only the theoretical solutions of various trunking problems, but has also involved the computation of special probability tables and collection of data by means of which theoretical results have been closely checked.

In the articles which have hitherto appeared, little or no effort has been made to present the mathematical theory of trunking in a

[1] G. T. Blood of the A. T. & T. Co. in 1898 found a close agreement between the terms of a binomial expansion and the results of observations on the distribution of busy calls. The first comprehensive paper was one written by M. C. Rorty in October 1903 and was quite widely circulated within the Bell System.

[2] An excellent bibliography is given by G. F. O'Dell in the P. O. E. E. J. for October 1920.

manner that can be understood by those who are not experts on the subject. It is hoped that this article will assist the reader in understanding both what has been and will be written on the subject. As Poisson [3] has said "a problem relative to games of chance and proposed to an austere Jansenist by a man of the world, was the origin of the calculus of probabilities," and today the reader will find that in the majority of text books the subject is introduced by the solution of games of chance and particularly of dice problems. This established custom will be followed by the present writer who, in the course of this article, will show how various fundamental trunking problems can be transformed into equivalent dice problems. This being done, solutions will be found to be at hand.

Three trunking problems, each one step more complicated than the preceding, will be dealt with. In order to facilitate the transformation to the three equivalent dice problems it is desirable that the basic assumptions made be as simple as possible. The asumptions made in all three problems are:

A—*During the period of time under consideration, the busy hour of the day, each subscriber's line makes one call which is as likely to fall at any one instant as at any other instant during the period.*

Conditions substantially approximating this assumption frequently occur in practice.

B—*If a call when initiated obtains a trunk immediately it retains possession of that trunk for exactly two minutes. In other words, a constant holding time of two minutes duration will be assumed.*

In practice, holding times, of course, vary from a few seconds to many minutes and it may at first sight seem that the assumption of a constant holding time might lead to results deviating too much from practice to be of value. On this point, the theory of probabilities itself sheds some interesting light. As will be pointed out in the following problems, the assumption of a constant holding time is the equivalent of a dice problem in which a single die, or several identical dice are considered. The telephone problem with variable holding times may be reduced to the consideration of many dice, each with a different number of faces. Suppose 600 throws are made with a die having 6 faces so that on the average $\frac{1}{6}$ of 600 or 100 aces would be expected. With Bernoulli's formula it is easy to find the probability that the number of aces which turn up shall lie between 75 and 125, that is to say, within 25 on each side of the average. Now suppose 200 throws are made with a die having 20 faces, 200 with a

[3] Poisson, Recherches Sur La Probabilite Des Jugements, 1837.

10 face die, 100 with a 5 face die and finally 100 with a 2 face die. These 600 throws would also give on the average 100 aces. Using Poisson's generalization of the Bernoulli formula [3] we can calculate the probability that these 600 throws with various kinds of dice shall give a number of aces lying between 75 and 125. This probability will be *greater* than in the case of the 600 throws with the die with a constant number of faces, *i.e.*, the chance that the result will come outside the range 75 to 125 is *less*.

The thought is at once suggested that for the same total volume of traffic and average holding time, fewer calls would be lost when the holding time is *not* constant.[4] The above theory was tested in practice a few years ago by the engineers of the American Telephone and Telegraph Company, who made pen register records of hundreds of thousands of actual calls as handled by groups of machine switching trunks at Newark, New Jersey. A pen register was made which operated as follows: Each trunk in the group was represented by a pen. These pens were mounted side by side and each was controlled by a magnet in such a manner that when the trunk was busy the pen made a mark on a wide strip of paper driven at constant speed under the pens. There was thus obtained a record showing when each call originated and when it was concluded. An artificial record was now made showing what would have happened if each call had lasted for the average holding time as determined from the original record. Some 100,000 calls were analyzed in this manner and it was found that with a group of trunks of a size to carry the calls of the original record with only a small loss, 30 per cent. more calls would have been lost if the traffic had been as shown by the *artificial* record. It should be borne in mind, however, that a 30 per cent. change in a probability of the order of one in one hundred, considering the values we are dealing with, is practically negligible.

C—*If a trunk is not obtained immediately the calling subscriber waits for two minutes and then withdraws his call. If while waiting a trunk becomes idle he takes it and converses for the interval of time remaining before his two minutes are up.*

This assumption, although artificial, simplifies materially the analysis of the problems. Just what happens in practice to every call

[4] This result is here reached by assuming that each subscriber originates one call per hour. The conclusions are the same, however, even when this is not true, provided the term "holding time" is understood to mean the aggregate of all the talking times of the subscriber in an hour.

It may also be mentioned in passing that for a fixed volume of traffic, the discrepancy decreases as the number of subscribers who originate that traffic increases; that is, it is less when the group is composed of a large number of relatively idle lines than when it is composed of a small number of very busy ones.

which fails to get a trunk immediately is unknown. It is obvious, however, that when the number of trunks is such that the liability of the call failing to get a trunk immediately is very small—for example: of the order of one in one hundred—the reaction of these calls on other calls must be negligible independently of whatever assumption [5] is made in place of C.

Problem I

Referring to Fig. 1 consider a group of 269 subscribers' lines each equipped with a 20-point line switch. When a subscriber removes his receiver his line switch revolves and picks up the first idle trunk which

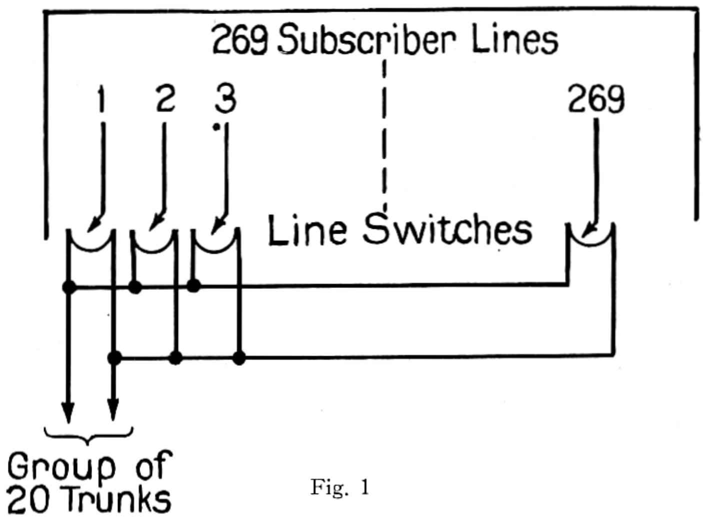

Fig. 1

it comes to. The 20 points of all switches are multipled together so that a single group of 20 trunks must handle the calls originating from these 269 lines.

What is the probability that when a particular subscriber X calls he fails to obtain a trunk immediately?

Referring to Fig. 2 let point P represent the unknown instant within the hour at which X calls. Consider the two minutes immediately preceding the instant P. Evidently, by assumption C, calls falling outside of this particular two-minute interval can not prevent X from obtaining a trunk.

[5] It is well known that the Erlang formula which is based on an assumption diametrically opposed to assumption C, namely that calls which find all trunks busy do not wait for a trunk to become idle, gives essentially the same results (for small probabilities, which are the only ones of interest in practice) as the Poisson furmula which assumes C.

If, however, at least 20 of the remaining 268 subscribers initiate their calls within the particular two minutes under consideration, there will be no trunk immediately available for X. This follows from assumptions B and C.

Consider some one of these 268 other subscribers, for example Y. The probability that Y calls in the two minutes under consideration is by assumption A, the ratio of 2 minutes to 60 minutes, or 1/30, which is exactly the same as the probability that he would throw an ace if he were to make a single throw with a 30-face die. Likewise the probability that still another subscriber calls in the two minutes under consideration is exactly the same as the probability that this

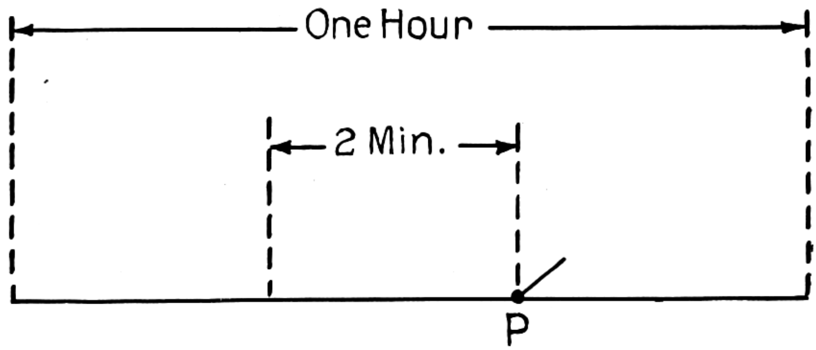

Fig. 2

other subscriber should throw the ace in a single throw with a 30-face die.

It is evident then, that the probability that X fails to get a trunk immediately is the same as the probability of throwing *at least* 20 aces if 268 throws are made with a 30-face die. To facilitate the determination of this probability and the solution of similar problems, probability tables of a type shown in Table I have been computed.[6] In the table, the average number of times an event may be expected is represented by a. The probability that the event occurs at least a greater number of times $c = a + d$ is represented by P. In the problem under consideration, the average number of aces expected is $8.96 = \frac{269}{30}$. Likewise in the present problem $c = 20$. Turning to the table, we find that corresponding to $c = 20$ and $a = 8.96$, the value of the probability P is .001. In the particular telephone problem under consideration this means that once in a thousand times

[6] Table I is to be found in the Appendix and its origin is there explained.

when X calls, at least 20 of the other subscribers will have called in the two minutes immediately preceding, and therefore X fails to get a trunk immediately. In other words, we may consider that on the average one in every thousand calls is lost.

In the problem just considered, a known number of subscribers' lines have had a known number of trunks assigned to them and we have inquired the probability that any subscriber would fail to find an idle trunk. It is frequently desirable to change the statement of

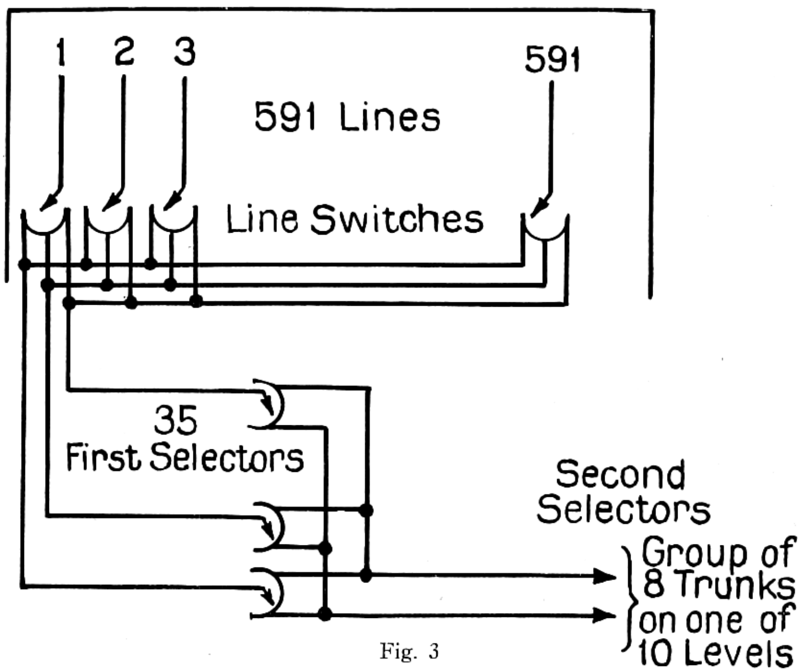

Fig. 3

the problem slightly. For instance: given a known number of subscribers' lines and having decided upon a desirable value of the probability P, we may inquire the number of trunks which must be assigned. It is evident that in this problem we would enter the table knowing the value 8.96 and .001, and would find corresponding to these, the number 20 as representing the size of the required group of trunks.

Problem II

Referring to Fig. 3 consider a group of 591 subscribers' lines each equipped with a 35 point line switch giving access to first selectors. We will suppose for illustration that each first selector has 10 levels or choices. The reader unfamiliar with automatic systems may consider a 10 level selector as one from which calls may be sent in 10 different directions. Assume that each level is equipped with 8 trunks to second selectors. The 591 line switches are multipled together so that *one* group of 35 first selectors must handle the calls originating from these 591 lines. The 35 first selectors are multipled together so that *one* group of 8 second selectors must handle the calls originating from the 591 lines for a particular level. It is assumed that the 591 calls are distributed at random with reference to the 10 levels of the first selectors.

The probability that X should fail to obtain immediately a first selector can be determined as in the first problem, but now let us determine what is the probability that subscriber X (having obtained immediately a first selector) fails to obtain immediately one of the 8 trunks of a particular one of the 10 levels on the first selectors.

For subscriber Y to interfere with X it is necessary that Y originate his call in the two minutes preceding the instant at which X calls and also that Y call for the particular one of the 10 levels in which X is interested.

The probability of Y fulfilling the first condition is equal to the probability of throwing the ace with a 30 face die. The probability of Y fulfilling the second condition is equal to the probability of throwing the ace with a 10 face die.

The question may then be stated in the form of a dice problem as follows: 591 throws are made with a 30 face die giving C aces. C throws are made with a 10 face die giving D aces, and the question is the probability that D is not less than 8. Assuming no restriction [7] on the value of C this probability is the same as that of throwing at least 8 aces in 591 throws with a die having (30) (10) = 300 faces.

The average number of aces to be expected is (591/300) = 1.97 and with this average the tables tell us that once in a thousand times we may expect at least 8 aces.

[7] Since it is assumed that X obtained a first selector it follows that in the 2 minutes preceding the instant when X called the number of calls must have been less than the number of first selectors and we should, therefore, not count the throws giving values of C which are not less than the total number of first selectors. This restriction becomes of practical importance only where a large proportion of the calls from the first selectors go to one level. To take an extreme case, assume that all the calls went to one level, and that therefore each 10 first selectors would require 10 second selectors to handle the traffic. Placing no restriction on the value of C, since C exceeds the number of first selectors occasionally, we would get the result that 10 second selectors were not enough to handle all the calls from 10 first selectors, which is of course absurd. Where, however, the values of C exceeding the number of first selectors are assumed to be distributed over all 10 levels of the first selectors their effect on the number of second selectors is negligible.

Problem III

In practice, a modification of Problem II frequently arises. Assume an arrangement similar to that of Problem II except that the number of lines is multiplied by a factor of perhaps 3 or more, each line switch, however, still having access to all first selectors. The required number of first selectors will also be larger but not in exactly the same ratio because the margin of idle selectors need not be relatively as great in the large system as in the small. An enlarged group of trunks running from the first selectors to the second selectors will now be required, and it will be assumed that there are four times

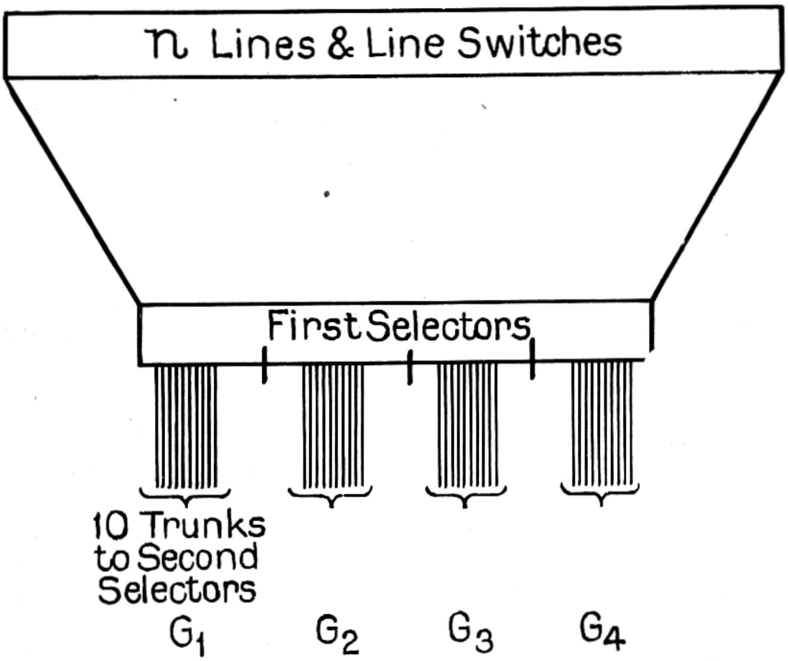

Fig. 4

as many trunks coming from each level of the first selectors as there are points of contact on each level. To meet this situation, the first selectors and their outgoing trunks are divided into four sub-groups as shown in Fig. 4. The corresponding sub-groups of second selectors are designated by G_1, G_2, G_3, G_4, the number of trunks to each sub-group being 10. The solution of this problem depends primarily on the manner in which the line switches distribute calls to the first selectors. Three cases will be considered.

Case 1

Referring to Fig. 4 consider a group of $n = 1486$ lines, and let the traffic be divided among the ten levels or directions available in such a way that on the average ⅓ of the calls are made for a particular direction or level. Let us suppose the circuit connections between the line switches and first selectors to be such that the calls are distributed individually at random. By this is meant that the first selector seized by a calling line is as likely to be one having access to sub-group G_1 as to sub-group G_2, G_3 or G_4. Note carefully that this distribution is assumed whether or not the calling line wants the particular level under consideration. One way of securing this random distribution by sub-groups would be to allow the line switches first to choose by chance one of the four sub-groups of first selectors and then to choose an idle first selector in the sub-group.

Question—What is the probability that when subscriber X calls he fails to obtain immediately a trunk to a second selector? It is assumed that X obtained a first selector and that his call is for the level under consideration.

As before we are interested in the calls made during the two minutes preceding the instant at which X calls. Let the number of these calls be C. Of these C calls a certain number D want the level for which X has called. If at least 10 of these D calls were distributed by the line switches to first selectors having access to the same sub-group as the one to which the first selector seized by X has access, then there will be no idle trunk in the sub-group for X. Our telephone trunking problem evidently transforms to the following series of dice problems.

1st. 1486 throws are made with a 30 face die giving C aces.
2nd. C throws are made with a 3 face die giving D aces.
3rd. D throws are made with a 4 face die giving x aces, and the question is the probability that x is not less than 10.

By the theory of dice (assuming no restriction [6] on the value of C) the probability is the same as that of throwing at least 10 aces in 1486 throws with a die having $30 \times 3 \times 4 = 360$ faces. The average number of aces to be expected being $1486/360 = 4.13$ the probability tables give .01 as the answer.

Case 2

As in Case 1 assume that on the average ⅓ of the calls are for the level under consideration, but take $n = 1725$ for the number of lines. Now suppose the circuit connections between line switches and first selectors to be such that the calls are distributed uniformly to the first selectors, meaning that if at any instant C calls exist, $C/4$ of them are on first selectors having access to the 10 trunks of sub-group G_1, $C/4$ are on first selectors having access to the 10 trunks of sub-group G_2 and so on. With a constant holding time such as assumed this result could be secured by a device common to all line switches which would route the first call to the first sub-group, the second call to the second sub-group, etc.

X will, as before, be interested in the calls falling in the two minutes preceding him. By hypothesis ¼ of these will have been distributed to first selectors having access to the same sub-group of second selectors as the first selector seized by X. Finally, the probability is ⅓ that one of these calls wants the level in which X is interested. The equivalent dice problem is therefore:

1st. 1725 throws are made with a 30 face die and the number of aces which turn up are noted. Let this number be C.

2nd. $C/4$ throws are made with a 3 face die.

What is the probability that this sequence of throws results in at least 10 aces? This probability is not that of getting at least 10 aces if 1725 throws are made with a die having $30 \times 3 = 90$ faces. We must write separately the formula for each of the two steps of the problem, then multiply them together and finally sum the product for all values of $C/4$ from 10 up. If this is done, again ignoring the restriction on the upper limit of C, the answer will come out 0.01. Note that whereas in Case 1 the average volume of traffic carried by a sub-group of 10 trunks was 4.13, in this case, with the same probability of failure, it is $1725\ (1/30)\ (1/4)\ (1/3) = 4.79$.

Case 3

In conclusion, a third and very interesting case will be mentioned. A distribution of calls *collectively at random* would be an appropriate name, and its nature may be described as follows:

Number each first selector and a corresponding card; shuffle the cards and deal out, for example, 37 of them. The distribution under consideration is such that when 37 calls exist the probability that they occupy a specified set of 37 selectors is equal to the probability that the cards dealt have the corresponding numbers. This distribution of calls would be measurably secured by arranging the line switch multiple so that the trunks to the first selectors appear so far as possible in a different order before every line switch. This case of distribution differs from that of Case 1. In Case 1, if the first call falls on a first selector having access to sub-group G_1, for example, the second call still has the same chance of falling on a first selector having access to sub-group G_1 as on one having access to any one of the other three sub-groups. In Case 3, however, the busy first selectors tend to be distributed uniformly between the 4 sub-groups, so that if any sub-group should have a preponderance of busy first selectors the probability of its receiving another call is less than the probability that one of the other sub-groups, with more idle first selectors, should receive it. The full discussion of this case is reserved for the future.

APPENDIX

Introduction to the Mathematical Theory of Probabilities

If it is known that one of two events must occur in any trial or instance, and that the first can occur in u ways and the second in v ways, all of which are equally likely to happen, then the probability that the first will happen is mathematically expressed by the fraction

$$\frac{u}{u+v},$$

while the probability that the second will happen is

$$\frac{v}{u+v}.$$

Denote these probabilities by p and q respectively; then we have:

$$p = \frac{u}{u+v}, \qquad q = \frac{v}{u+v}, \qquad p + q = 1,$$

the last equation following from the first two, and being the mathematical expression for the certainty that one of the two events must happen.

If the probabilities of two independent events are p_1 and p_2 respectively, the probability of their concurrence in any single instance is $p_1 p_2$, and in general if $p_1, p_2, p_3, \ldots \ldots p_n$ denote the probabilities of several independent events, and P the probability of their concurrence, then

$$P = p_1 p_2 p_3 \ldots \ldots \ldots p_n.$$

Consider, now, what may happen in n trials of an event, for which the probability is p and against which the probability is q. The probability that the event will happen every time is $pppp \ldots\ldots p$, where the factor p appears n times; that is the probability is p^n. The probability that the event will occur $(n-1)$ times in succession and then fail is $p^{n-1}q$.

But if the order of occurrence is disregarded, this last combination may arrive in n different ways; so that the probability that the event will occur $(n-1)$ times and fail once is $np^{n-1}q$. Similarly, the probability that the event will happen $(n-2)$ times and fail twice is $p^{n-2}q^2$ multiplied by $n(n-1)/2$, etc. That is, the probabilities of the several possible occurrences are given by the corresponding terms of the binominal expansion of $(p+q)^n$. Let

$$P = p^n + \binom{n}{1}p^{n-1}q + \binom{n}{2}p^{n-2}q^2 + \ldots\ldots + \binom{n}{c+1}p^{c+1}q^{n-c-1} + \binom{n}{c}p^c q^{n-c}, \tag{1}$$

where $\binom{n}{x}$ means $n(n-1)(n-2)\ldots\ldots(n-x+1)/(1)(2)\ldots\ldots(x)$.

Then P = probability that the event happens exactly n times, plus the probability that it happens exactly $(n-1)$ times . . . plus the probability that it happen exactly c times: in other words, the probability that the event happens at least c times in n trials.

If the series for P contains few terms it may be computed easily. In general, however, it is impracticable to compute P by means of the above binomial expansion. Other forms for the value of P must, therefore, be developed.

One of the most convenient approximations for P when p is small has been developed by Poisson. It is known as Poisson's Exponential Binomial Limit and gives the value of P by the following expansion

$$P = e^{-a}x^c/(c)! + e^{-a}a^{c+1}/(c+1)! + e^{-a}a^{c+2}/(c+2)! \ldots\ldots\ldots\ldots ad\ inf. \tag{2}$$

where e = base of natural logarithms = 2.718 . . . . . , $a = (np)$ and $(c)! = c(c-1)(c-2)(c-3)\ldots\ldots(3)(2)(1)$.

The following Table gives corresponding values of P, a, c satisfying equation (2).

TABLE I.

Averages (a) Corresponding to Deviation (d) plus Average (a) to be Expected with Different Probabilities

Deviation Plus Average, $c = a + d$	Probabilities .001	.002	.004	.006	.008	.010	Deviation Plus Average, $c = a + d$
	Average = a						
1	.001	.002	.004	.006	.008	.010	1
2	.045	.065	.092	.114	.133	.149	2
3	.191	.243	.312	.361	.402	.436	3
4	.429	.518	.630	.709	.771	.823	4
5	.739	.867	1.02	1.13	1.21	1.28	5
6	1.11	1.27	1.47	1.60	1.70	1.79	6
7	1.52	1.72	1.95	2.11	2.23	2.33	7
8	1.97	2.20	2.47	2.65	2.79	2.91	8
9	2.45	2.72	3.02	3.22	3.38	3.51	9
10	2.96	3.26	3.60	3.82	3.99	4.13	10
11	3.49	3.82	4.19	4.43	4.62	4.77	11
12	4.04	4.40	4.80	5.06	5.26	5.43	12
13	4.61	5.00	5.43	5.71	5.92	6.10	13
14	5.20	5.61	6.07	6.37	6.60	6.78	14
15	5.79	6.23	6.72	7.04	7.28	7.48	15
16	6.41	6.87	7.39	7.72	7.97	8.18	16
17	7.03	7.52	8.06	8.41	8.68	8.90	17
18	7.66	8.17	8.75	9.11	9.39	9.62	18
19	8.31	8.84	9.44	9.82	10.11	10.35	19
20	8.96	9.52	10.14	10.54	10.84	11.08	20
21	9.62	10.20	10.84	11.26	11.57	11.83	21
22	10.29	10.89	11.56	11.99	12.31	12.57	22
23	10.97	11.59	12.28	12.73	13.06	13.33	23
24	11.65	12.29	13.01	13.47	13.81	14.09	24
25	12.34	13.00	13.74	14.21	14.57	14.85	25