The Heaviside Operational Calculus

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SYNOPSIS: The art of electrical communication owes a great and increasingly recognized debt to Oliver Heaviside for his work in developing and emphasizing a correct theory of electrical transmission along wires and in particular for his insistance on the importance of inductance. His operational methods of solving the differential equations which are fundamental of the theory of electric circuits, although not widely known, are important. These methods are peculiarly applicable to many important problems of electrical transmission. The present paper, while theoretical in character, therefore deals with a subject of practical importance to the communication engineer.

Without attempting to give any adequate idea of the striking originality and ingenuity of Heaviside's methods, his operational calculus may be very briefly explained as follows. Problems in electric circuit theory are described by a set of differential equations involving the differential operator $\frac{d}{dt}$. These differential equations may be reduced formally to algebraic equations by replacing the differential operator by the symbol p and by this expedient a purely symbolic solution is obtained. This symbolic solution is called the *operational formula* of the problem.

In order to interpret the purely symbolic operational formula, Heaviside proceeded as follows: By direct comparison of the operational formula of specific problems with their known explicit solutions he was led to assign a definite significance to the operator p. Thereupon, he obtained by induction generalized specific criteria or rules for solving the operational formula.

The present paper, by attacking the problem from a different standpoint, shows that the Heaviside operational formula is a shorthand equivalent of an integral equation from which the methods and rules of his operational calculus are deducible.—*Editor.*

A VERY interesting and by no means the least valuable part of Heaviside's researches relates to operational methods of solving the differential equations of a class of physical problems of which electric circuit theory problems are typical; in fact Volume II of his Electromagnetic Theory is almost entirely devoted to this subject. The methods of solution which he originated and employed are of extraordinary directness and simplicity in a very large class of problems in applied mathematics. In fact it would be difficult to exaggerate the value of his work along this line, and nowhere is it more immediately and usefully applicable than in the theoretical problems of electro-technics.

Heaviside is, however, by no means easy reading and, in spite of the considerable number of published studies relating to his operational calculus, it is less generally understood and applied than its value warrants. The writer has had occasion to apply Heaviside's methods quite extensively in electrical problems and in the course of his study was led to a general formula which to him at least, has proved useful in interpreting and rationalizing the operational calculus. This formula is presented and discussed in the present paper with the hope that it may be of service to students of Heaviside in understanding and applying his methods. The paper may also serve as a brief recapitulation of some of the outstanding methods applicable to the solution of problems in electric circuit theory.

The problems to which Heaviside applied his operational calculus relate to the oscillations of electrical and mechanical systems which can be described by a system of linear differential equations with constant coefficients or to such partial differential equations as *the telegraph equation.* The system is supposed to be set into oscillation *from a state of equilibrium* by suddenly applied forces and the operational formula gives the resulting behavior of the system.

The type of problem to which the operational calculus is applicable and Heaviside's method of solution may be illustrated in a sufficiently general manner for didactic purposes in connection with the solution of the system of equations

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= F_1(t) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n &= F_n(t) \end{aligned} \tag{1}$$

where the coefficients ajk are in general polynominals of the form

$$a_{jk} = \alpha_{jk} + \beta_{jk}\frac{d}{dt} + \gamma_{jk}\frac{d^2}{dt^2} + \ldots$$

and the α, β, γ coefficients are constants.

We are concerned with the determination of the variables $x_1 \ldots x_n$ as functions of the independent variable t for the following boundary conditions; the known functions $F_1 \ldots F_n$ *and the variables* $x_1 \ldots x_n$ *are identically zero for values of the independent variable* $t \angle 0$. In other words, the system is initially in a state of equilibrium when the "forces" $F_1 \ldots F_n$ are applied. These boundary conditions are extremely important in physical problems.

Owing to the linear character of the equations we may without loss of generality set all the F functions equal to zero except one, say $F_1(t)$, and write

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= F(t) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n &= 0. \end{aligned} \tag{3}$$

The solution of equations (3) for the prescribed boundary conditions may be made to depend on the auxiliary equations in the auxiliary variables $h_1 \dots h_n$,—

$$\begin{aligned} a_{11}h_1 + \dots + a_{1n}h_n &= 1 \\ \dots\dots\dots\dots & \quad (t \geq 0) \\ a_{n1}h_1 + \dots + a_{nn}h_n &= 0. \end{aligned} \tag{4}$$

The function on the right hand side, written, in accordance with the Heaviside notation, as unity is identically zero for $t<0$ and unity for $t\geq 0$ and $h_1 \dots h_n$ are identically zero for $t<0$.

It may then be shown that [1]

$$x_j = \frac{d}{dt}\int_0^t F(t-y)\, h_j(y)\, dy, \qquad (j = 1, 2, \dots n) \tag{5}$$

so that the solution of (3) depends entirely on (4).

Equations (4) formulate the problem actually dealt with by Heaviside who did not explicitly consider the more general equations (3). His method of attack was as follows; Writing p^n for the differential operator d^n/dt^n equations (4) become formally algebraic and yield a purely symbloic solution

$$h_j = \frac{1}{H_j(p)}. \tag{6}$$

Equation (6) is the Heaviside operational formula; as it stands, however, it is purely symbolic and the problem remains to find the significance of the equation and to deduce therefrom the value of $h = h(t)$ as a function of t.

Heaviside's method from this point on was one of pure induction. From the known solution of specific problems he inferred general rules for expanding and interpreting the operational formula: the body of rules thus developed for solving the operational equation may be appropriately termed the Heaviside Operational Calculus.

The contribution of the present paper to the theory of the Heaviside operational calculus depends on the following proposition and its immediate corollary.[2]

The differential equations (4), *subject to the prescribed boundary conditions, may be written as:*

$$h_j = 0, \qquad \text{for } t<0 \text{ and } j = 1, \dots n,$$

$$\frac{1}{pH_j(p)} = \int_0^\infty e^{-pt} h_j(t)\, dt, \qquad \text{for } t \geq 0. \tag{7}$$

The integral equation is an identity for all positive real values of p and consequently determines $h_j(t)$ uniquely.

[1] This formula has been established in previous papers. It is briefly discussed in Appendix I.

[2] See Appendix I.

It follows as an immediate corollary that the Heaviside operational equation

$$h = 1/H(p) \tag{8}$$

is merely a short-hand or symbolic equivalent of the integral equation

$$\frac{1}{pH(p)} = \int_0^\infty e^{-pt} h(t)\, dt. \tag{9}$$

The significance of the operational equation and the rules of the Heaviside operational calculus are therefore deducible from the latter equation. The whole problem is thus reduced to the purely mathematical problem of solving the integral equation.

It should be remarked in passing that, while the Heaviside operational calculus has been elucidated in connection with the solution of a set of differential equations involving a finite number of variables, it is not so limited in its applications. It is applicable also when the number of variables is infinite and to such partial differential equtions as *the telegraph equation.* The foregoing theorem applies also to all such physical problems where an operational formula $h = 1/H(p)$ is derivable.

Before discussing the solution of the integral equation (9) and deducing therefrom some of the rules of the operational calculus, a simple but interesting and instructive example of the way the opertional formula is set up will be given.

Consider a transmission line of infinite length along the positive x axis and let it have a distributed inductance L and capacity C per unit length. Let a unit voltage be applied to the line at the origin $x = 0$ at time $t = 0$; required the line current I and voltage V at any point x at any subsequent time t.

The differential equations of the problems are

$$L \frac{\partial}{\partial t} I = - \frac{\partial}{\partial x} V,$$

$$C \frac{\partial}{\partial t} V = - \frac{\partial}{\partial x} I.$$

Replacing $\frac{\partial}{\partial t}$ by p, we get

$$I = \sqrt{\frac{C}{L}}\, e^{-\frac{px}{v}} V_0,$$

$$V = e^{-\frac{px}{v}} V_0,$$

where $v = 1/\sqrt{LC}$ and V_0 is the line voltage at $x = 0$.

Now by the conditions of the problem V_0 is zero before, unity after time $t = 0$; hence the foregoing equations are operational formulas and by (9)

$$\frac{1}{p}\sqrt{\frac{C}{L}}\, e^{-\frac{px}{v}} = \int_0^{\infty} e^{-pt} I_x(t)\, dt,$$

$$\frac{1}{p}\, e^{-\frac{px}{v}} = \int_0^{\infty} e^{-pt} V_x(t)\, dt.$$

The solutions of these equations are obviously

$$\begin{aligned} I_x &= 0 && \text{for } t<x/v, \\ &= \sqrt{\frac{C}{L}} && \text{for } t \geq x/v, \\ V_x &= 0 && \text{for } t<x/v, \\ &= 1 && \text{for } t \geq x/v, \end{aligned}$$

which are, of course, the well known solutions of the problem. The directness and simplicity of the solution from the definite integrals is, however, noteworthy.

By virtue of the foregoing analysis the Heaviside operational calculus becomes identical with the methods and rules for the solution of integral equations of the type

$$1/pH(p) = \int_0^{\infty} e^{-pt} h(t)\, dt \qquad (9)$$

to which brief consideration will now be given.

An integral equation is, of course, one in which the unknown function appears under the sign of integration; the process of determining the unknown function is the solution of the equation. Integral equations of the form of (9) were first employed by Laplace and may be referred to as equations of the Laplace type. More recently they have become of importance in the modern theories of divergent series and summability. The solution of a large number of integral equations of the Laplace type has been worked out; however the procedure is usually peculiar to the particular problem in hand. In this connection it is noteworthy that, from a purely mathematical standpoint, Heaviside's operational calculus is a valuable contribution to the systematic solution of this type of integral equations. That is to say, methods which he developed for the solution of his operational equation suggest systematic procedure in the solution of the integral equation (9), as might be expected from the relationship pointed out in the present paper.

As stated above a large number of infinite integrals of the type appearing in equation (9) have been worked out. Consequently the solution of (9) can frequently be written down by inspection. When this is not the case, however, the appropriate procedure is usually to expand the function $1/pH(p)$ in such a form that the individual terms are recognizable as identical with infinite integrals of the required type.

An interesting expansion of this kind and one which is applicable to a large number of physical problems is as follows:

Expand $1/pH(p)$ asymptotically in the form of the divergent series

$$1/pH(p) \sim \sum a_n/p^{n+1}.$$

This expansion is purely formal and the series is divergent. It is summable, however, in the sense that it may be identified with its generating function $1/pH(p)$. It is also summable in accordance with Borel's definition of the sum of a divergent series by the Borel integral [3]

$$\int_0^\infty dt\ e^{-pt} \sum a_n t^n/n!$$

This suggests that these two series are equal and consequently that

$$1/pH(p) = \int_0^\infty dt\ e^{-pt} \sum a_n t^n/n!$$

The solution is therefore

$$h(t) = \sum a_n t^n/n!$$

provided this series, which is called by Borel the associated function of the divergent expansion, is itself convergent. This is the case in all physical problems to which this form of expansion has been applied.[4]

The foregoing will be recognized as identical with Heaviside's power series solution, obtained by the empirical rule of identifying $1/p^n$ with $t^n/n!$ in the asymptotic expansion of $1/H(p)$.

Another form of solution of very considerable practical value depends on a partial fraction expression which can be carried out in a large number of physical problems. It is

$$1/pH(p) = a + b/p + c/p^2 + \sum A_k/(p - p_k)$$

[3] See Bromwich, Theory of Infinite Series, pp. 267–269.

[4] See Appendix II.

where
$$a = \left(1/pH(p)\right)_{p=\infty},$$
$$b = \left[\frac{d}{dp}\frac{p}{H(p)}\right]_{p=0},$$
$$c = \left[\frac{p}{H(p)}\right]_{p=0},$$
$$A_k = \frac{1}{p_k H'(p_k)},$$

and $p_1 \ldots p_n$ are the roots of $H(p) = 0$.

By virtue of this expansion [5] the solution is

$$h(t) = aP + b + ct + \sum \frac{e^{p_k t}}{p_k H'(p_k)},$$

where P denotes a "pulse" at the origin $t = 0$; that is,

$$\begin{aligned} P &= \infty \quad \text{at } t = 0, \\ &= 0 \quad \text{for } t > 0, \end{aligned}$$
$$\int_0^\infty P dt = 1.$$

In the usual case where $a = c = 0$ and $b = 1/H(0)$, this reduces to

$$h(t) = 1/H(0) + \sum \frac{e^{p_k t}}{p_k H'(p_k)},$$

which will be recognized as the celebrated Heaviside Expansion Solution.

As illustrating the flexibility of the integral identity (9), another form of solution will be given which is often of value in practical problems where an explicit solution cannot be obtained. Suppose that $1/pH(p)$ can be written as

$$\frac{1}{pH(p)} = \frac{1}{pH_1(p)} \cdot \frac{1}{pH_2(p)}$$

and that functions $h_1(t)$ and $h_2(t)$ can be found which satisfy the equations

$$\frac{1}{pH_1(p)} = \int_0^\infty e^{-pt} h_1(t)\, dt$$
$$\frac{1}{pH_2(p)} = \int_0^\infty e^{-pt} h_2(t)\, dt$$

[5] The terms $a + c/p^2$ in this expansion were suggested by Dr. O. J. Zobel and must be included in a number of important problems in electric circuit theory.

Then the required function $h(t)$ is given by

$$h(t) = \int_0^t h_1(t-y)\, h_2(y)\, dy \tag{10}$$

by Borel's Theorem (Bromwich, Theory of Infinite Series, p. 280).[6]

As a final example of the foregoing discussion we shall consider a specific problem of some practical interest in itself and which involves Heaviside's so-called "fractional differentiation" and his resulting asymptotic solutions. The physical problem is as follows: a "unit-voltage" (zero before, unity after time $t = 0$) is applied through a terminal condenser C_0 to an infinitely long cable of resistance R and capacity C per unit length. Required the Voltage V at the cable terminals.

The operational formula of this problem is easily deduced; it is

$$V = \frac{\sqrt{p/a}}{1 + \sqrt{p/a}} \qquad \text{where } 1/\sqrt{a} = C_0 \sqrt{R/C}.$$

Consequently the integral equation can be written

$$\int_0^\infty e^{-pt} V(t)\, dt = \frac{1}{p} \frac{\sqrt{p/a}}{1 + \sqrt{p/a}}$$

$$= \frac{1}{p} \frac{1}{1 + \sqrt{a/p}}$$

Taking the last form of $1/pH(p)$, expanding asymptotically and recognizing that

$$1/p^{n+1} = \int_0^\infty e^{-pt} t^n / n!\, dt$$

$$1/p^n \sqrt{p} = \int_0^\infty e^{-pt} \frac{(2t)^n}{(2n-1)(2n-3)\ldots 1} \frac{dt}{\sqrt{\pi t}}$$

the resulting series solution can be recognized and summed as

$$V(t) = e^{at} - \sqrt{\frac{a}{\pi}}\, e^{at} \int_0^t \frac{e^{-ay}}{\sqrt{y}}\, dy$$

$$= \sqrt{\frac{a}{\pi}}\, e^{at} \int_t^\infty \frac{e^{-ay}}{\sqrt{y}}\, dy.$$

The last expansion by repeated integration by parts leads to the asymptotic series given by Heaviside. It is easy to show, also, that

[6] This formula is quite useful; it is applied in the solution of the last example of this present paper.

the series is truly asymptotic in the sense that the error is less than the last term included.

Another mode of procedure, however, suggests itself, which, by the aid of equation (10) gives the solution directly without series expansion. We have

$$\frac{1}{pH(p)} = \frac{1}{p-a} - \frac{1}{p-a}\sqrt{\frac{a}{p}},$$
$$= \int_0^\infty e^{-pt}\Big[h_1(t) - h_2(t)\Big]\,dt,$$

where

$$\frac{1}{p-a} = \int_0^\infty e^{-pt} h_1(t)\,dt$$

and

$$\frac{1}{p-a}\sqrt{\frac{a}{p}} = \int_0^\infty e^{-pt} h_2(t)\,dt.$$

Consequently $h_1(t) = e^{at}$, and since

$$\frac{1}{\sqrt{p}} = \int_0^\infty e^{-pt}\sqrt{1/\pi t}\;dt$$

it follows at once from (10) that

$$h_2(t) = \sqrt{\frac{a}{\pi}}\,e^{at}\int_0^t e^{-ay}\frac{dy}{\sqrt{y}}.$$

The solution $h(t) = h_1(t) + h_2(t)$ agrees with the preceding derived from the asymptotic expansion, and is considerably more direct and simple.

It is interesting to compare this solution with Heaviside's own operational solution (Electromagnetic Theory Vol. II, p. 40) which amounts to the following. The operational formula is written

$$V = \frac{p}{p-a} - \frac{1}{p-a}\sqrt{ap}.$$

The first term is discarded altogether and the second written as

$$V = \left(1 - \frac{p}{a}\right)^{-1}\sqrt{p/a}$$
$$= \left(1 + \frac{p}{a} + \left(\frac{p}{a}\right)^2 + \ldots\right)\sqrt{p/a}.$$

Identifying $\sqrt{p}$ with $1/\sqrt{\pi t}$ and p^n with d^n/dt^n the expansion becomes

$$V = \left(1 - \frac{1}{2}\left(\frac{1}{at}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{at}\right) - \ldots\right)\sqrt{\frac{1}{\pi at}},$$

which agrees with the foregoing and is the actual asymptotic expansion.[7]

The foregoing discussion is sufficient, it is hoped, to show the place of the integral formula (9) in relation to the Heaviside operational calculus. It is believed to be particularly applicable in connection with a number of questions relating to divergent series and solutions which Heaviside's work has raised and which have received too little attention from mathematicians.

APPENDIX I

A proof of the integral formula

$$1/pH(p) = \int_0^\infty e^{-pt}\, h(t)\, dt \qquad (9)$$

can be made to depend very simply on the formula

$$x(t) = \frac{d}{dt}\int_0^t F(t-y)\, h(y)\, dy. \qquad (5)$$

This equation may be regarded as well established and can in fact be deduced in a quite general manner by synthetic arguments. It is derived and employed in papers by the writer (Trans. A. I. E. E., 1911, pp. 345–427, and Phys. Rev. Feb. 1921, pp. 116–134) and is deducible at once from the work of Fry (Phys. Rev. Aug. 1919, pp. 115–136).

On the basis of equation (5) the deduction of formula (9), in which, however, no pretense to rigor is made, proceeds as follows;

If the function $F(t)$ in equations (3) is set equal to e^{pt}, the complete solution (5) includes the particular solution [8]

$$e^{pt}/H(p)$$

which involves t only through the exponential term. The complete solution must, therefore, admit of reduction to the form

$$x(t) = e^{pt}/H(p) + y(t) \qquad (a)$$

where $y(t)$ is the complementary solution.

[7] The procedure by which Heaviside arrived at the foregoing asymptotic solution is not, however, always so fortunate. For example if a terminal inductance is substituted for the terminal condenser of the preceding problem, precisely the same procedure gives an incomplete result. Heaviside recognized this and added an extra term without explanation (Elm. Th. Vol. II, p. 42) but his solution appears to be doubtful in the light of some recent work by the writer in applying the formula of the present paper to the same problem.

[8] Provided $H(p) \neq o$. This restriction is of no consequence in physical problems, where the roots of $H(p)$ are in general complex with ***real part negative.***

Now equation (5) may be written, when $F(t) = e^{pt}$, as

$$x(t) = \frac{d}{dt} e^{pt} \int_0^t e^{-py} h(y)\, dy$$

$$= \frac{d}{dt} \left\{ e^{pt} \int_0^\infty e^{-py} h(y)\, dy - e^{pt} \int_t^\infty e^{-py} h(y)\, dy \right\}. \qquad \textbf{(b)}$$

Now the first term of the expression involves t only through the exponential term while the second term involves t through the lower limit of the integral which ultimately vanishes and therefore includes no term involving t only through the exponential. Consequently the first term of (b) is identifiable as the particular solution of (a) and by direct equation it follows that

$$1/pH(p) = \int_0^\infty e^{-py} h(y)\, dy \qquad \textbf{(9)}$$

which is the required formula.

The most important restriction which is implicit in the foregoing is that in splitting up the definite integral of (5) we have tacitly assumed that $h(t)$ is finite for all values of t; a restriction which is necessary in order that the infinite integral shall be convergent for all positive real values of p. This condition is satisfied in all physical problems and therefore introduces no practical limitation of importance.

However, even when this restriction does not hold formula (9) may be valid and uniquely determine $h(t)$ if p is restricted to values which make the infinite integral convergent, or when the problem is such that $e^{-pt}h(t)$ is an exact derivative. As an example, suppose that

$$1/H(p) = \frac{p}{p - a}$$

where a is a real positive quantity. It may be otherwise shown that $h(t) = e^{at}$ and formula (9) becomes

$$\frac{1}{p - a} = \int_0^\infty e^{-(p-a)t}\, dt$$

which is valid when $p > a$.

APPENDIX II

The discussion in the text does not pretend to be a proof of the power series expansion in any strict sense. A more satisfactory discussion proceeds as follows:

We assume that $1/H(p)$ can be formally expanded in the series

$$\sum_{0}^{\infty} a_n/p^n$$

We shall here introduce a necessary restriction on the function $1/H(p)$. It must include no function which is represented asymptotically by a series all of whose terms are zero; that is a function $\phi(p)$ such that the limit, as p approaches ∞, of $p^n\phi(p)$ is zero for every value of n. The function e^{-p} is such a function. (See Whittaker & Watson, p. 154.)

With this restriction understood, start with the integral (9) and integrate by parts; we get

$$\frac{1}{H(p)} = h(o) + \int_0^\infty e^{-pt}h^{(1)}(t)dt$$

where $h^{(n)}(t) = d^n/dt^n h(t)$.

Now let p approach infinity; in the limit the integral vanishes and

$$h(o) = 1/H(\infty) = a_0$$

from the asymptotic expansion.

Integrate again by parts; we get

$$p(1/H(p) - a_0) = h^{(1)}(o) + \int_0^\infty e^{-pt}h^{(2)}(t)dt.$$

Now let p again approach infinity; in the limit the integral vanishes, and the right hand side, by virtue of the asymptotic expansion, approaches the limit a_1, whence $h^{(1)}(o) = a_1$. Proceeding in this manner, repeated integrations by parts establish the relation $h^{(n)}(o) = a_n$. But provided the series is absolutely convergent, then

$$h(t) = \sum h^{(n)}(o)t^n/n!$$

$$= \sum a_n t^n/n!$$

which establishes the formula.

The power series solution is applicable to a large class of physical problems and has been rigorously established under certain restrictions by other methods than that employed above (see papers by Bromwich, *Phil. Mag*, May 1920, p. 407; Fry, *Phys. Rev.* Aug. 1919, p. 115; and the writer, Trans. A. I. E. E. 1919, p. 345).

On the basis of the preceding and with the aid of formula (10), expansions of the type

$$1/pH(p) \sim \frac{1}{\sqrt{p}} \sum b_n/p^{n+1} = \int_0^\infty e^{-pt}h(t)dt$$

which occur in physical problems, can be dealt with. For since

$$\sum b_n/p^{n+1} = \int_0^\infty dt\, e^{-pt} \sum b_n t^n/n!$$

and

$$\frac{1}{\sqrt{p}} = \int_0^\infty e^{-pt}\, dt/\sqrt{\pi t},$$

it follows from (10) that

$$h(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{dy}{\sqrt{t-y}} \sum b_n y^n/n!$$

$$= \frac{1}{\sqrt{\pi t}} \sum \frac{b_n (2t)^n}{(2n-1)(2n-3)\ldots 1}.$$